LECTURE

22 OCTOBER

9:00-10:00AM

MATH3/4/68181

2 variable case

(X_1, X_2)

The joint CDF of (X_1, X_2) is

$$F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)$$

The joint survival function of (X_1, X_2) is

$$\bar{F}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$$

The marginal CDFs are

$$P(X_1 \leq x_1) = F(x_1, \infty)$$
$$= 1 - \bar{F}(x_1, -\infty),$$

$$P(X_2 \leq x_2) = F(\infty, x_2)$$
$$= 1 - \bar{F}(-\infty, x_2)$$

The relationships between F and $\bar{F}$:

$$\bar{F}(x_1, x_2) = 1 - F(x_1, \infty) - F(\infty, x_2) + F(x_1, x_2),$$

$$F(x_1, x_2) = 1 - \bar{F}(x_1, -\infty) - \bar{F}(-\infty, x_2) + \bar{F}(x_1, x_2)$$

3 variables case

(X_1, X_2, X_3)

The joint CDF of (X_1, X_2, X_3) is

$$F(x_1, x_2, x_3) = P(X_1 \leq x_1, X_2 \leq x_2, X_3 \leq x_3)$$

The joint survival function of (X_1, X_2, X_3) is

$$\bar{F}(x_1, x_2, x_3) = P(X_1 > x_1, X_2 > x_2, X_3 > x_3)$$

The relationships between F and $\bar{F}$ are:

$$\begin{aligned}
\bar{F}(x_1, x_2, x_3) = \; & 1 - F(x_1, \infty, \infty) \\
& - F(\infty, x_2, \infty) \\
& - F(\infty, \infty, x_3) \\
& + F(x_1, x_2, \infty) \\
& + F(x_1, \infty, x_3) \\
& + F(\infty, x_2, x_3) \\
& - F(x_1, x_2, x_3),
\end{aligned}$$

$$\begin{aligned}
F(x_1, x_2, x_3) = \; & 1 - \bar{F}(x_1, -\infty, -\infty) \\
& - \bar{F}(-\infty, x_2, -\infty) \\
& - \bar{F}(-\infty, -\infty, x_3) \\
& + \bar{F}(x_1, x_2, -\infty) \\
& + \bar{F}(x_1, -\infty, x_3) \\
& + \bar{F}(-\infty, x_2, x_3) \\
& - \bar{F}(x_1, x_2, x_3).
\end{aligned}$$

p variables case

$$(X_1, X_2, \ldots, X_p)$$

The joint CDF of $(X_1, \ldots, X_p)$ is

$$F(x_1, \ldots, x_p) = P(X_1 \leq x_1, \ldots, X_p \leq x_p)$$

The joint survival function of $(X_1, \ldots, X_p)$ is

$$\bar{F}(x_1, \ldots, x_p) = P(X_1 > x_1, \ldots, X_p > x_p).$$

The relationships between F and $\bar{F}$ are:

$$\bar{F}(x_1, \ldots, x_p) = 1 - \sum_{i=1}^{p} F(\infty, \ldots, \infty, \underset{\uparrow\ i\text{th position}}{\boxed{x_i}}, \infty, \ldots, \infty)$$

$$+ \sum_{\substack{i=1 \\ i<j}}^{p} \sum_{j=1}^{p} F(\infty, \ldots, \infty, \underset{\uparrow\ i\text{th position}}{\boxed{x_i}}, \infty, \ldots, \infty, \underset{\uparrow\ j\text{th position}}{\boxed{x_j}}, \infty, \ldots, \infty)$$

$$\vdots$$

$$+ (-1)^p F(x_1, \ldots, x_p),$$

$$F(x_1, \ldots, x_p) = 1 - \sum_{i=1}^{p} \bar{F}(-\infty, \ldots, -\infty, \underset{\uparrow\ i\text{th position}}{\boxed{x_i}}, -\infty, \ldots, -\infty)$$

$$+ \sum_{\substack{i=1 \\ i<j}}^{p} \sum_{j=1}^{p} \bar{F}(-\infty, \ldots, -\infty, \underset{\uparrow\ i\text{th position}}{\boxed{x_i}}, -\infty, \ldots, -\infty, \underset{\uparrow\ j\text{th position}}{\boxed{x_j}}, -\infty, \ldots, -\infty)$$

$$\vdots$$

$$+ (-1)^p \bar{F}(x_1, \ldots, x_p).$$

$\boxed{\text{Case V}}$ $X_1, X_2, \dots, X_n$ are dependent, n fixed

Let $F(x_1, x_2, \dots, x_n)$ denote the joint CDF of $(X_1, \dots, X_n)$. Let $\bar{F}(x_1, \dots, x_n)$ denote the joint survival function of $(X_1, \dots, X_n)$.

1) $S = X_1 + \dots + X_n$

The CDF of S is

$$F_S(s) = P(S \leq s) = P(X_1 + \dots + X_n \leq s)$$

$$= \int \cdots \int_{x_1 + \dots + x_n \leq s} \boxed{f(x_1, x_2, \dots, x_n)} \, dx_n \cdots dx_2 \, dx_1$$

Joint PDF of $(X_1, \dots, X_n)$

$$f(x_1, \dots, x_p) = \frac{\partial^p}{\partial x_1 \cdots \partial x_p} F(x_1, \dots, x_p)$$

$$f(x_1, \dots, x_p) = \frac{(-1)^p \partial^p}{\partial x_1 \cdots \partial x_p} \bar{F}(x_1, \dots, x_p)$$

The PDF of S is

$$f_S(s) = \frac{d}{ds} F_S(s)$$

$$= \int \cdots \int_{x_1 + \dots + x_n = s} f(x_1, x_2, \dots, x_n) \, dx_n \cdots dx_2 \, dx_1$$

The mth moment of S is

$$E[S^m]$$

$$= E[(X_1 + \cdots + X_n)^m]$$

$$= \sum_{p_1 + \cdots + p_n = m} \binom{m}{p_1 p_2 \cdots p_n} E[X_1^{p_1} X_2^{p_2} \cdots X_n^{p_n}]$$

The mean of S is

$$E[S] = E(X_1) + \cdots + E(X_n)$$

$$= \sum_{i=1}^{n} E(X_i)$$

The variance of S is

$$Var[S] = Var(X_1 + \cdots + X_n)$$

$$\neq \sum_{i=1}^{n} Var(X_i)$$ because $X_1, \ldots, X_n$ are dependent RVs.

2) $U = \max(X_1, \ldots, X_n)$

The CDF of U is

$$F_U(u) = P[\max(X_1, \ldots, X_n) \leq u]$$

$$= P[X_1 \leq u, \ldots, X_n \leq u]$$

$$= F(u, \ldots, u)$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \frac{d}{du} F(u, \ldots, u)$$

The m^{th} moment of U is

$$E[U^m] = \int_{-\infty}^{\infty} u^m f_U(u)\, du$$

$$= \int_{-\infty}^{\infty} u^m \frac{d}{du} F(u, \ldots, u)\, du.$$

3) $V = \min(X_1, \ldots, X_n)$

The CDF of V is

$$F_V(v) = P[\min(X_1, \ldots, X_n) \leq v]$$

$$= 1 - P[\min(X_1, \ldots, X_n] > v]$$

$$= 1 - P[X_1 > v, \ldots, X_n > v]$$

$$= 1 - \bar{F}(v, \ldots, v)$$

The PDF of V is

$$f_V(v) = \frac{d}{dv} F_V(v)$$

$$= - \frac{d}{dv} \bar{F}(v, \ldots, v)$$

The m^{th} moment of V is

$$E[V^m] = \int_{-\infty}^{\infty} v^m f_V(v)\, dv$$

$$= - \int_{-\infty}^{\infty} v^m \frac{d}{dv} \bar{F}(v, \ldots, v)\, dv.$$