EXAMPLE CLASS

23 OCTOBER

16:00-17:00PM

MATH3/4/68181

$X_1, X_2, \ldots, X_\alpha$ IID $\text{Exp}(\lambda)$.

$X = \max(X_1, X_2, \ldots, X_\alpha)$

1) The CDF of X is

$$F_X(x) = P(X \leq x)$$

$$= P[\max(X_1, \ldots, X_\alpha) \leq x]$$

$$= P[X_1 \leq x, \ldots, X_\alpha \leq x]$$

$$= P[X_1 \leq x] \cdots P[X_\alpha \leq x] \quad \text{independence}$$

$$= (1 - e^{-\lambda x}) \cdots (1 - e^{-\lambda x})$$

$$= (1 - e^{-\lambda x})^\alpha$$

2) $$f_X(x) = \frac{d}{dx} F_X(x)$$

$$= \alpha (1 - e^{-\lambda x})^{\alpha - 1} \lambda e^{-\lambda x}$$

$$= \alpha \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha - 1}$$

3) $$E(X^n) = \int_0^\infty x^n \alpha\lambda e^{-\lambda x}[1-e^{-\lambda x}]^{\alpha-1} dx$$

$$= \alpha\lambda \int_0^\infty x^n e^{-\lambda x}[1-e^{-\lambda x}]^{\alpha-1} dx$$

$$\boxed{\begin{array}{l} \text{Set } y = e^{-\lambda x} \Rightarrow x = -\frac{1}{\lambda}\log y \\ \Rightarrow \frac{dx}{dy} = -\frac{1}{\lambda y} \end{array}}$$

$$= \alpha\cancel{\lambda} \int_1^0 \left(-\frac{1}{\lambda}\log y\right)^n \cancel{y}[1-y]^{\alpha-1}\left(-\frac{1}{\cancel{\lambda}\cancel{y}}\right) dy$$

$$= \alpha \int_0^1 \left(-\frac{1}{\lambda}\log y\right)^n [1-y]^{\alpha-1} dy$$

$$= \alpha\left(-\frac{1}{\lambda}\right)^n \int_0^1 (\log y)^n [1-y]^{\alpha-1} dy$$

$$\boxed{\begin{array}{l} \frac{d^n y^\beta}{d\beta^n} = y^\beta (\log y)^n \\ \Rightarrow \frac{d^n}{d\beta^n} y^\beta \Big|_{\beta=0} = y^0(\log y)^n = (\log y)^n \quad (*) \end{array}}$$

$$\overset{(*)}{=} \alpha\left(-\frac{1}{\lambda}\right)^n \int_0^1 \boxed{\frac{d^n}{d\beta^n}} y^\beta \Big|_{\beta=0} [1-y]^{\alpha-1} dy$$

$$= \alpha\left(-\frac{1}{\lambda}\right)^n \frac{d^n}{d\beta^n}\left[\int_0^1 y^\beta [1-y]^{\alpha-1} dy\right]\Bigg|_{\beta=0}$$

$$= \alpha\left(-\frac{1}{\lambda}\right)^n \frac{d^n}{d\beta^n} B(\beta+1, \alpha)\Big|_{\beta=0}$$

$$\boxed{\begin{array}{l} B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt \\ \text{Beta function} \end{array}}$$

4) $$\text{Mean} = -\frac{\alpha}{\lambda}\frac{d}{d\beta} B(\beta+1,\alpha)\Big|_{\beta=0}$$

$$\boxed{B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}}$$

$$= -\frac{\alpha}{\lambda}\frac{d}{d\beta}\frac{\Gamma(\beta+1)\,\Gamma(\alpha)}{\Gamma(\beta+1+\alpha)}\Big|_{\beta=0}$$

$$= -\frac{\alpha\,\Gamma(\alpha)}{\lambda}\frac{d}{d\beta}\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}\Big|_{\beta=0}$$

$$= -\frac{\alpha\,\Gamma(\alpha)}{\lambda}\frac{\Gamma(\beta+1+\alpha)\,\Gamma'(\beta+1) - \Gamma(\beta+1)\,\Gamma'(\beta+1+\alpha)}{[\Gamma(\beta+1+\alpha)]^2}\Big|_{\beta=0}$$

$$= -\frac{\alpha\,\Gamma(\alpha)}{\lambda}\frac{\Gamma(1+\alpha)\,\Gamma'(1) - \Gamma'(1+\alpha)}{[\Gamma(1+\alpha)]^2}.$$

5) $$\text{Variance} = E[X^2] - [E(X)]^2$$

$$= \frac{\alpha}{\lambda^2}\frac{d^2}{d\beta^2} B(\beta+1,\alpha)\Big|_{\beta=0} - [E(X)]^2$$

$$= \frac{\alpha}{\lambda^2}\frac{d^2}{d\beta^2}\frac{\Gamma(\beta+1)\,\Gamma(\alpha)}{\Gamma(\beta+1+\alpha)}\Big|_{\beta=0} - [E(X)]^2$$

$$= \frac{\alpha\,\Gamma(\alpha)}{\lambda^2}\frac{d^2}{d\beta^2}\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}\Big|_{\beta=0} - [E(X)]^2$$

Simplify this by yourself.

6) & 7) You need to wait till the week after Reading week

8) Suppose there is only 1 observation on X. Then

$$L(\alpha, \lambda) = \alpha \lambda e^{-\lambda x} \left[1 - e^{-\lambda x}\right]^{\alpha - 1}$$

$$\Rightarrow \log L(\alpha, \lambda) = \log \alpha + \log \lambda - \lambda x + (\alpha - 1) \log\left[1 - e^{-\lambda x}\right]$$

$$\Rightarrow \frac{\partial \log L}{\partial \alpha} = \frac{1}{\alpha} + \log\left[1 - e^{-\lambda x}\right] = 0 \quad \text{--- (1)}$$

$$\frac{\partial \log L}{\partial \lambda} = -x + \frac{1}{\lambda} + (\alpha - 1) \frac{x e^{-\lambda x}}{1 - e^{-\lambda x}} = 0 \quad \text{--- (2)}$$

$$(1) \Rightarrow \alpha = -\frac{1}{\log\left[1 - e^{-\lambda x}\right]} \quad \text{--- (3)}$$

Sub (3) into (2) $\Rightarrow$

$$-x + \frac{1}{\lambda} - \left(\frac{1}{\log\left[1 - e^{-\lambda x}\right]} + 1\right) \frac{x e^{-\lambda x}}{1 - e^{-\lambda x}} = 0 \quad \text{--- (4)}$$

Solve (4) for λ, say $\hat{\lambda}$.

$$\hat{\alpha} = -\frac{1}{\log\left[1 - e^{-\hat{\lambda} x}\right]}.$$