EXAMPLE CLASS

22 OCTOBER

12:00-13:00PM

MATH3/4/68181

Q1

$X_1, \dots, X_\alpha$ IID $\text{Exp}(\lambda)$.

Let $X = \max(X_1, \dots, X_\alpha)$.

1. The CDF of X is

$$F_X(x) = P[\max(X_1, \dots, X_\alpha) \leq x]$$

$$= P[X_1 \leq x, \dots, X_\alpha \leq x]$$

$$= P[X_1 \leq x] \cdots P[X_\alpha \leq x] \quad \text{by indep}$$

$$= (1 - e^{-\lambda x}) \cdots (1 - e^{-\lambda x})$$

$$= (1 - e^{-\lambda x})^\alpha$$

2. The PDF of X is

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$= \alpha (1 - e^{-\lambda x})^{\alpha - 1} \lambda e^{-\lambda x}$$

$$= \alpha \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha - 1}$$

3.

$$E(X^n) = \int_0^\infty x^n \alpha\lambda e^{-\lambda x}\left[1-e^{-\lambda x}\right]^{\alpha-1} dx$$

$$= \alpha\lambda \int_0^\infty x^n e^{-\lambda x}\left[1-e^{-\lambda x}\right]^{\alpha-1} dx$$

$$\boxed{\begin{aligned} \text{Set} \quad y &= e^{-\lambda x} \\ x &= -\frac{\log y}{\lambda} \\ \frac{dx}{dy} &= -\frac{1}{y\lambda} \end{aligned}}$$

$$= \alpha\cancel{\lambda} \int_1^0 \left(-\frac{\log y}{\lambda}\right)^n \cancel{y}\,[1-y]^{\alpha-1}\left(-\frac{1}{\cancel{y\lambda}}\right) dy$$

$$= \alpha\left(-\frac{1}{\lambda}\right)^n \int_0^1 (\log y)^n [1-y]^{\alpha-1} dy$$

$$\boxed{\begin{aligned} &\frac{d^n y^\beta}{d\beta^n} = y^\beta (\log y)^n \\ &\Rightarrow \frac{d^n}{d\beta^n} y^\beta \Big|_{\beta=0} = y^0 (\log y)^n = (\log y)^n \cdots (*) \end{aligned}}$$

$$\overset{(*)}{=} \alpha\left(-\frac{1}{\lambda}\right)^n \int_0^1 \frac{d^n}{d\beta^n} y^\beta \Big|_{\beta=0} [1-y]^{\alpha-1} dy$$

$$= \alpha\left(-\frac{1}{\lambda}\right)^n \frac{d^n}{d\beta^n}\left[\int_0^1 y^\beta [1-y]^{\alpha-1} dy\right]\Big|_{\beta=0}$$

$$= \alpha\left(-\frac{1}{\lambda}\right)^n \frac{d^n}{d\beta^n} B(\beta+1, \alpha)\Big|_{\beta=0}$$

4. Mean $= \dfrac{-\alpha}{\lambda} \dfrac{d}{d\beta} B(\beta+1, \alpha) \Big|_{\beta=0}$

$$\boxed{B(a, b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}}$$

$$= -\frac{\alpha}{\lambda} \frac{d}{d\beta} \frac{\Gamma(\beta+1)\boxed{\Gamma(\alpha)}}{\Gamma(\beta+1+\alpha)} \Bigg|_{\beta=0}$$

$$= -\frac{\alpha\,\Gamma(\alpha)}{\lambda} \frac{\Gamma(\beta+1+\alpha)\,\Gamma'(\beta+1) - \Gamma(\beta+1)\,\Gamma'(\beta+1+\alpha)}{[\Gamma(\beta+1+\alpha)]^2} \Bigg|_{\beta=0}$$

$$= -\frac{\alpha\,\Gamma(\alpha)}{\lambda} \frac{\Gamma(1+\alpha)\,\Gamma'(1) - \Gamma'(1+\alpha)}{[\Gamma(1+\alpha)]^2} .$$

5. Variance $= E(X^2) - [E(X)]^2$.

$$E(X^2) = +\frac{\alpha}{\lambda^2} \frac{d^2}{d\beta^2} B(\beta+1, \alpha) \Big|_{\beta=0}$$

$$= \frac{\alpha}{\lambda^2} \frac{d^2}{d\beta^2} \frac{\Gamma(\beta+1)\,\Gamma(\overset{\alpha}{\cancel{\beta+1+\alpha}})}{\Gamma(\beta+1+\alpha)} \Bigg|_{\beta=0}$$

$$= \frac{\alpha\,\Gamma(\alpha)}{\lambda^2} \frac{d^2}{d\beta^2} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)} \Bigg|_{\beta=0}$$

6 & 7. Need to wait until after the Reading week.

8. Suppose there is <u>only 1 observation</u> on x. The likelihood function is

$$L(\alpha, \lambda) = \alpha\lambda\, e^{-\lambda x}\left[1-e^{-\lambda x}\right]^{\alpha-1}$$

$$\Rightarrow \quad \log L(\alpha,\lambda) = \log(\alpha\lambda) - \lambda x$$
$$+ (\alpha-1)\log\left[1-e^{-\lambda x}\right]$$

$$\Rightarrow \quad \frac{\partial \log L}{\partial \alpha} = \frac{1}{\alpha} + \log\left[1-e^{-\lambda x}\right], \qquad = 0 \quad \text{— (1)}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{1}{\lambda} - x + (\alpha-1)\frac{x\, e^{-\lambda x}}{1-e^{-\lambda x}} = 0 \quad \text{— (2)}$$

$$(1) \Rightarrow \quad 1-e^{-\lambda x} = e^{-\frac{1}{\alpha}}$$

$$\Rightarrow \quad e^{-\lambda x} = 1 - e^{-\frac{1}{\alpha}}$$

$$\Rightarrow \quad \lambda = -\frac{1}{x}\log\left(1-e^{-\frac{1}{\alpha}}\right) \quad \text{— (4)}$$

Sub these into (2) gives

$$-\frac{x}{\log\left(1-e^{-\frac{1}{\alpha}}\right)} - x + (\alpha-1)\frac{x\left(1-e^{-\frac{1}{\alpha}}\right)}{e^{-\frac{1}{\alpha}}} = 0 \quad \text{— (3)}$$

(3) depends ONLY on α. Solve it for α. (4) will give the MLE for λ.