LECTURE

19 OCTOBER

10:00-11:00AM

MATH4/68181

Six Cases

i) $X_1, \ldots, X_n$ IID, n fixed

ii) $X_1, \ldots, X_N$ IID, N is a RV

iii) $X_1, \ldots, X_n$ INID, n fixed
(Independent but not identically distributed)

iv) $X_1, \ldots, X_N$ INID, N RV

Level 4 & Level 6 ONLY

v) $X_1, \ldots, X_n$ dependent RVs, n fixed

vi) $X_1, \ldots, X_n$ " ", N is a RV

Case iii $\quad X_1, \dots, X_n$ are INID, n fixed

1) $S = X_1 + \dots + X_n$

$= \text{total portfolio loss}$

The CDF of S is

$$F_S(s) = P(X_1 + \dots + X_n \leq s)$$

$$= \int \cdots \int_{x_1 + \dots + x_n \leq s} F_n(s - x_1 - \dots - x_{n-1}) \prod_{j=1}^{n-1} f_j(x_j) \, dx_{n-1} \, dx_{n-2} \cdots dx_1$$

$$\boxed{\begin{aligned} F_j(\cdot) &= \text{CDF of } X_j \\ f_j(\cdot) &= \text{PDF of } X_j \end{aligned}}$$

The PDF of S is

$$f_S(s) = \int \cdots \int_{x_1 + \dots + x_n = s} f_n(s - x_1 - \dots - x_{n-1}) \prod_{j=1}^{n-1} f_j(x_j) \, dx_{n-1} \, dx_{n-2} \cdots dx_1$$

The mth moment of S is

$$E[S^m] = E[(X_1 + \dots + X_n)^m]$$

$$= \sum_{p_1 + \dots + p_n = m} \binom{m}{p_1 \, p_2 \cdots p_n} E[X_1^{p_1} X_2^{p_2} \cdots X_n^{p_n}]$$

$$= \sum_{p_1 + \dots + p_n = m} \binom{m}{p_1 \, p_2 \cdots p_n} E[X_1^{p_1}] E[X_2^{p_2}] \cdots E[X_n^{p_n}]$$

$$E(S) = E(X_1 + \cdots + X_n)$$
$$= E(X_1) + \cdots + E(X_n)$$
$$= \sum_{j=1}^{n} E(X_j)$$

$$Var(S) = Var(X_1 + \cdots + X_n)$$
$$= Var(X_1) + \cdots + Var(X_n) \quad \text{by indep}$$
$$= \sum_{j=1}^{n} Var(X_j)$$

Ex 3

Suppose $X_i \sim N(\mu_i, \sigma_i^2)$ are indep RVs.

Let $S = X_1 + \cdots + X_n$.

Then $S \sim N(\mu_1 + \cdots + \mu_n, \sigma_1^2 + \cdots + \sigma_n^2)$

The CDF of S is

$$F_S(s) = \Phi\left(\frac{s - (\mu_1 + \cdots + \mu_n)}{\sqrt{\sigma_1^2 + \cdots + \sigma_n^2}}\right)$$

CDF of $N(0,1)$

The PDF of S is

$$f_S(s) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2 + \cdots + \sigma_n^2}} e^{-\frac{[s - (\mu_1 + \cdots + \mu_n)]^2}{2(\sigma_1^2 + \cdots + \sigma_n^2)}}$$

Mean of S = $\mu_1 + \cdots + \mu_n$

Variance of S = $\sigma_1^2 + \cdots + \sigma_n^2$

2) $U = \max(X_1, \dots, X_n)$

The CDF of U is

$$F_U(u) = P[\max(X_1, \dots, X_n) \leq u]$$

$$= P[X_1 \leq u, \dots, X_n \leq u]$$

$$= P[X_1 \leq u] \cdots P[X_n \leq u] \quad \text{by independence}$$

$$= F_1(u) \cdots F_n(u)$$

$$= \prod_{j=1}^{n} F_j(u)$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \sum_{k=1}^{n} f_k(u) \prod_{\substack{j=1 \\ j \neq k}}^{n} F_j(u)$$

PRODUCT RULE OF DIFFERENTIATION FOR n FUNCTION

The mth moment of U is

$$E[U^m] = \int_{-\infty}^{\infty} u^m f_U(u)\, du$$

$$= \sum_{k=1}^{n} \int_{-\infty}^{\infty} u^m f_k(u) \prod_{\substack{j=1 \\ j \neq k}}^{n} F_j(u)\, du$$

Ex 1

$\boxed{n=2}$

$$F_U(u) = F_1(u)\,F_2(u)$$

$$f_U(u) = f_1(u)\,F_2(u) + F_1(u)\,f_2(u)$$

$\boxed{n=3}$

$$F_U(u) = F_1(u)\,F_2(u)\,F_3(u)$$

$$f_U(u) = f_1(u)\,F_2(u)\,F_3(u)$$

$$+ F_1(u)\,f_2(u)\,F_3(u)$$

$$+ F_2(u)\,F_2(u)\,f_3(u)$$

3) $V = \min(X_1, \ldots, X_n)$

The CDF of V is

$$F_V(v) = P[\min(X_1, \ldots, X_n) \leq v]$$

$$= 1 - P[\min(X_1, \ldots, X_n) > v]$$

$$= 1 - P[X_1 > v, \ldots, X_n > v]$$

$$= 1 - P[X_1 > v] \cdots P[X_n > v] \quad \text{by independence}$$

$$= 1 - (1 - P(X_1 \leq v)) \cdots (1 - P(X_n \leq v))$$

$$= 1 - (1 - F_1(v)) \cdots (1 - F_n(v))$$

$$= 1 - \prod_{j=1}^{n} [1 - F_j(v)]$$

The PDF of V is

$$f_V(v) = \sum_{k=1}^{n} f_k(v) \prod_{\substack{j=1 \\ j \neq k}}^{n} [1 - F_j(v)]$$

The m^{th} moment of V is

$$E[V^m] = \int_{-\infty}^{\infty} v^m f_V(v)\, dv$$

$$= \sum_{k=1}^{n} \int_{-\infty}^{\infty} v^m f_k(v) \prod_{\substack{j=1 \\ j \neq k}}^{n} [1 - F_j(v)]\, dv$$

Ex 2

$\boxed{n=2}$

$$F_V(v) = 1 - [1 - F_1(v)][1 - F_2(v)]$$

$$= F_1(v) + F_2(v) - F_1(v) F_2(v)$$

$$f_V(v) = f_1(v) + f_2(v) - f_1(v) F_2(v) - F_1(v) f_2(v)$$

$\boxed{n=3}$ $$F_V(v) = 1 - [1 - F_1(v)][1 - F_2(v)][1 - F_3(v)]$$

$$= F_1(v) + F_2(v) + F_3(v) - F_1(v) F_2(v) - F_1(v) F_3(v) - F_2(v) F_3(v) + F_1(v) F_2(v) F_3(v)$$

$$f_V(v) = f_1(v) + f_2(v) + f_3(v)$$
$$- f_1(v) F_2(v) - F_1(v) f_2(v)$$
$$- f_1(v) F_3(v) - F_1(v) f_3(v)$$
$$- f_2(v) F_3(v) - F_2(v) f_3(v)$$
$$+ f_1(v) F_2(v) F_3(v)$$
$$+ F_1(v) f_2(v) F_3(v)$$
$$+ F_1(v) F_2(v) f_3(v).$$

Case iv $X_1, \dots, X_N$ are INID, N is a RV independent of $X_1, X_2, \dots$

1) $S = X_1 + \dots + X_N$

The CDF of S is

$$F_S(s) = P[X_1 + \dots + X_N \leq s]$$

$$= \sum_{n=1}^{\infty} P[X_1 + \dots + X_N \leq s \mid N=n] \, P(N=n)$$

$\boxed{\text{Total}}$ $\boxed{\text{Probability}}$ $\boxed{\text{Rule}}$

$$= \sum_{n=1}^{\infty} \left[\int \cdots \int_{x_1 + \dots + x_n \leq s} F_n(s - x_1 - \dots - x_{n-1}) \prod_{j=1}^{n-1} f_j(x_j) \, dx_{n-1} \, dx_{n-2} \cdots dx_1 \right] P(N=n)$$

The PDF of S is

$$f_S(s) = \sum_{n=1}^{\infty} \left[\int \cdots \int_{x_1 + \dots + x_n = s} f_n(s - x_1 - \dots - x_{n-1}) \prod_{j=1}^{n-1} f_j(x_j) \, dx_{n-1} \, dx_{n-2} \cdots dx_1 \right] P(N=n)$$

The mth moment of S is

$$E[S^m] = E[(X_1 + \dots + X_N)^m]$$

$$= \sum_{n=1}^{\infty} E[(X_1 + \dots + X_N)^m \mid N=n] \, P(N=n)$$

$$= \sum_{n=1}^{\infty} \sum_{p_1 + \dots + p_n = m} \binom{m}{p_1 \, p_2 \cdots p_n} E[X_1^{p_1}] \cdots E[X_n^{p_n}] \, P(N=n)$$

The mean of S is

$$E[S] = \sum_{n=1}^{\infty} E[S \mid N=n] \; P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[\sum_{j=1}^{n} E(X_j) \right] P(N=n)$$

The variance of S is

$$Var[S] = \sum_{n=1}^{\infty} Var[S \mid N=n] \; P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[\sum_{j=1}^{n} Var(X_j) \right] P(N=n)$$