LECTURE

16 OCTOBER

9:00-10:00AM

MATH3/4/68181

QUIZ 2 DUE 12:00 NOON
TUESDAY
23 OCTOBER

N is fixed
(<u>not</u> a RV)

Ex 2

Suppose $X_1, \ldots, X_n$ IID $Exp(a)$.

$$F_U(u) = [1-e^{-au}]^n$$

$$f_U(u) = n[1-e^{-au}]^{n-1} a e^{-au}$$

$$E(U^m) = na \int_0^\infty u^m e^{-au} [1-e^{-au}]^{n-1} du$$

Set $y = e^{-au}$

$\Rightarrow u = -\frac{1}{a}\log y \Rightarrow \frac{du}{dy} = -\frac{1}{ay}$

$$= n\cancel{a} \int_1^0 \left(-\frac{1}{a}\log y\right)^m \cancel{y} [1-y]^{n-1} \left(-\frac{1}{\cancel{a}\cancel{y}}\right) dy$$

$$= n\left(-\frac{1}{a}\right)^m \int_0^1 (\log y)^m [1-y]^{n-1} dy$$

$$\frac{d^m y^\alpha}{d\alpha^m} = y^\alpha (\log y)^m$$

$$\Rightarrow \frac{d^m}{d\alpha^m} y^\alpha \Big|_{\alpha=0} = (\log y)^m \qquad (*)$$

$$\overset{(*)}{=} n\left(-\frac{1}{a}\right)^m \int_0^1 \boxed{\frac{d^m}{d\alpha^m}} y^\alpha \Big|_{\alpha=0} [1-y]^{n-1} dy$$

$$= n\left(-\frac{1}{a}\right)^m \frac{d^m}{d\alpha^m} \boxed{\int_0^1 y^\alpha [1-y]^{n-1} dy} \Big|_{\alpha=0}$$

$$= n\left(-\frac{1}{a}\right)^m \frac{d^m}{d\alpha^m} B(\alpha+1, n) \Big|_{\alpha=0}$$

Ex 3 Suppose $X_1, \ldots, X_n$ IID $\text{Exp}(a)$.

$$F_V(v) = 1 - [1 - [1 - e^{-av}]]^n$$

$$= 1 - e^{-nav}$$

$$\Rightarrow \quad V \sim \text{Exp}(na)$$

$$f_V(v) = na\, e^{-nav}$$

$$E(V) = \frac{1}{na}$$

$$\text{Var}(V) = \frac{1}{(na)^2}.$$

$\boxed{\text{Case ii}}$ $X_1, \ldots, X_N$ IID

N is a RV indep of $X_1, X_2, \ldots$

The CDF of $S = X_1 + \cdots + X_N$ is

$$F_S(s) = P[X_1 + \cdots + X_N \leq s]$$

$$= \sum_{n=1}^{\infty} P[X_1 + \cdots + X_n \leq s \mid N = n] \; P(N = n)$$

$\boxed{\text{Total Probability Rule}}$

$$= \sum_{n=1}^{\infty} \left[\int \cdots \int_{x_1 + \cdots + x_n \leq s} F(s - x_1 - \cdots - x_{n-1}) \prod_{j=1}^{n-1} f(x_j) \; dx_{n-1} \, dx_{n-2} \cdots dx_1 \right] P(N=n)$$

The PDF of S is

$$f_S(s) = \sum_{n=1}^{\infty} \left[\int \cdots \int_{x_1 + \cdots + x_n = s} f(s - x_1 - \cdots - x_{n-1}) \prod_{j=1}^{n-1} f(x_j) \; dx_{n-1} \, dx_{n-2} \cdots dx_1 \right] P(N = n)$$

The mth moment of S is

$$E[S^m] = \sum_{n=1}^{\infty} E[S^m \mid N = n] \; P(N = n)$$

$$= \sum_{n=1}^{\infty} \sum_{p_1 + \cdots + p_n = m} \binom{m}{p_1 \, p_2 \cdots p_n} E[X_1^{p_1}] \cdots E[X_n^{p_n}] \; P(N = n)$$

The CDF of U is

$$F_U(u) = P\left[\max(X_1, \dots, X_N) \leq u\right]$$

$$= \sum_{n=1}^{\infty} P\left[\max(X_1, \dots, X_N) \leq u \,\middle|\, N = n\right] P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[F(u)\right]^n P(N=n)$$

The PDF of U is

$$f_U(u) = \sum_{n=1}^{\infty} n\left[F(u)\right]^{n-1} f(u)\, P(N=n)$$

$$= f(u) \sum_{n=1}^{\infty} n\left[F(u)\right]^{n-1} P(N=n)$$

The mth moment of U is

$$E\left[U^m\right] = \sum_{n=1}^{\infty} n\, P(N=n) \int_{-\infty}^{\infty} u^m f(u) \left[F(u)\right]^{n-1} du$$

The CDF of V is

$$F_V(v) = P[\min(X_1, \ldots, X_N) \leq v]$$

$$= \sum_{n=1}^{\infty} P[\min(X_1, \ldots, X_n) \leq v \mid N = n] \, P(N = n)$$

$$= \sum_{n=1}^{\infty} \{1 - [1 - F(v)]^n\} \, P(N = n)$$

$$= 1 - \sum_{n=1}^{\infty} [1 - F(v)]^n \, P(N = n)$$

The PDF of V is

$$f_V(v) = f(v) \sum_{n=1}^{\infty} n [1 - F(v)]^{n-1} P(N = n)$$

The mth moment of V is

$$E(V^m) = \sum_{n=1}^{\infty} n \, P(N = n) \int_{-\infty}^{\infty} v^m f(v) [1 - F(v)]^{n-1} dv$$

Ex 1

Suppose $X_1, X_2, \ldots$ IID $N(\mu, \sigma^2)$

Suppose also $N \sim Geom(p)$ indep of $X_1, X_2, \ldots$

The CDF of S is

$$F_S(s) = \sum_{n=1}^{\infty} P[X_1 + \cdots + X_n \leq s \mid N=n] \, P(N=n)$$

$$= \sum_{n=1}^{\infty} \Phi\left(\frac{s - n\mu}{\sqrt{n}\,\sigma}\right) p(1-p)^{n-1}$$

$$= p \sum_{n=1}^{\infty} \Phi\left(\frac{s - n\mu}{\sqrt{n}\,\sigma}\right) (1-p)^{n-1}$$

The PDF of S is

$$f_S(s) = p \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{n}\,\sigma} e^{-\frac{(s-n\mu)^2}{2n\sigma^2}} (1-p)^{n-1}$$

$$= \frac{p}{\sqrt{2\pi}\,\sigma} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} e^{-\frac{(s-n\mu)^2}{2n\sigma^2}} (1-p)^{n-1}$$

The mean of S is

$$E[S] = E[E[S \mid N]]$$

$$= E[N\mu] = \mu E[N] = \frac{\mu}{p}$$

The variance of S' is

$$Var[S] = E[S'^2] - (E[S'])^2$$

$$= E[S'^2] - \left(\frac{\mu}{p}\right)^2$$

$$= E[E[S'^2 \mid N]] - \left(\frac{\mu}{p}\right)^2$$

$$= E[N\sigma^2 + (N\mu)^2] - \left(\frac{\mu}{p}\right)^2$$

$$= \sigma^2 E(N) + \mu^2 E(N^2) - \left(\frac{\mu}{p}\right)^2$$

$$= \frac{\sigma^2}{p} + \mu^2 \cdot \left(\frac{2}{p^2} - \frac{1}{p}\right) - \left(\frac{\mu}{p}\right)^2$$

$$= \frac{\sigma^2}{p} + \frac{\mu^2}{p^2} - \frac{\mu^2}{p}$$