LECTURE

15 OCTOBER

9:00-10:00AM

MATH3/4/68181

PORTFOLIO THEORY

Portfolio is a collection of investments

<u>Ex</u> gold, silver, copper

Suppose there are n investments.

Let

X_1 = Loss on investment 1

X_2 = " " " 2

$\vdots$

X_n = " " " n

$X_1, X_2, \ldots, X_n$ are random variables

Variables of Interest

1) $S = X_1 + \cdots + X_n$

$=$ "total portfolio loss"

2) $U = \max(X_1, \ldots, X_n)$

$=$ "maximum portfolio loss"

3) $V = \min(X_1, \ldots, X_n)$

$=$ "minimum portfolio loss"

Six Cases

i) $X_1, \ldots, X_n$ IID, n fixed

ii) $X_1, \ldots, X_N$ IID, N is a RV

iii) $X_1, \ldots, X_n$ INID, n fixed
(Independent but not identically distributed)

iv) $X_1, \ldots, X_N$ INID, N RV

Level 4 & Level 6 ONLY

v) $X_1, \ldots, X_n$ dependent RVs, n fixed

vi) $X_1, \ldots, X_n$ " ", N is a RV

$\boxed{\text{Case i}}$ $X_1, \ldots, X_n$ IID, n fixed

1) $S = X_1 + \cdots + X_n$

The CDF of S is

$$F_S(s) = P(S \leqslant s)$$

$$= P(X_1 + \cdots + X_n \leqslant s)$$

$$= \int \cdots \int_{x_1 + \cdots + x_n \leqslant s} F(s - x_1 - \cdots - x_{n-1}) \prod_{j=1}^{n-1} f(x_j) \, dx_{n-1} \, dx_{n-2} \cdots dx_1$$

$$\boxed{\begin{array}{l} F = \text{CDF of } X_1, \ldots, X_n \\ f = \text{PDF of } X_1, \ldots, X_n \end{array}}$$

The PDF of S is

$$f_S(s) = \frac{d}{ds} F_S(s)$$

$$= \int \cdots \int_{x_1 + \cdots + x_n = s} f(s - x_1 - \cdots - x_{n-1}) \prod_{j=1}^{n-1} f(x_j) \, dx_{n-1} \, dx_{n-2} \cdots dx_1$$

The m^{th} moment of S

$$E[S^m] = E[(X_1 + \cdots + X_n)^m]$$

$$= \sum_{p_1 + p_2 + \cdots + p_n = m} \binom{m}{p_1 \, p_2 \cdots p_n} E[X_1^{p_1} X_2^{p_2} \cdots X_n^{p_n}]$$

$$\boxed{\binom{m}{p_1 \, p_2 \cdots p_n} = \frac{m!}{p_1! \, p_2! \cdots p_n!}}$$

$$= \sum \binom{m}{p_1 \, p_2 \cdots p_n} E[X_1^{p_1}] \, E[X_2^{p_2}] \cdots E[X_n^{p_n}]$$

Mean of S

$= E[X_1 + \cdots + X_n]$

$= E[X_1] + \cdots + E[X_n]$

$= n \cdot E(X)$ because of the identical assumption

Variance of S

$= Var[X_1 + \cdots + X_n]$

$= Var[X_1] + \cdots + Var[X_n]$ because of independence

$= n \cdot Var[X]$ because of the identical assumption

2) $U = \max(X_1, \ldots, X_n)$

The CDF of U is

$$F_U(u) = P(U \leq u)$$

$$= P[\max(X_1, \ldots, X_n) \leq u]$$

$$= P[X_1 \leq u, \ldots, X_n \leq u]$$

$$= P[X_1 \leq u] \cdots P[X_n \leq u] \quad \text{by independence}$$

$$= F(u) \cdots F(u) \quad \text{by the identical assumption}$$

$$= [F(u)]^n$$

The PDF of U is

$$f_U(u) = n[F(u)]^{n-1} f(u).$$

The mth moment of U is

$$E[U^m] = \int_{-\infty}^{\infty} u^m f_U(u)\, du$$

$$= n \int_{-\infty}^{\infty} u^m [F(u)]^{n-1} f(u)\, du.$$

3) $V = \min(X_1, \dots, X_n)$.

The CDF of V is

$$F_V(v) = P(V \leqslant v) = 1 - P(V > v)$$

$$= 1 - P[\min(X_1, \dots, X_n) > v]$$

$$= 1 - P[X_1 > v, \dots, X_n > v]$$

$$= 1 - P[X_1 > v] \cdots P[X_n > v] \quad \text{by independence}$$

$$= 1 - (1 - P(X_1 \leqslant v)) \cdots (1 - P(X_n \leqslant v))$$

$$= 1 - (1 - F(v)) \cdots (1 - F(v)) \quad \text{by the identical assumption}$$

$$= 1 - [1 - F(v)]^n$$

The PDF of V is

$$f_V(v) = n[1 - F(v)]^{n-1} f(v)$$

The m th moment of V is

$$E(V^m) = \int_{-\infty}^{\infty} v^m f_V(v)\, dv$$

$$= n \int_{-\infty}^{\infty} v^m [1 - F(v)]^{n-1} f(v)\, dv.$$

Ex 1

Suppose $X_1, \ldots, X_n$ IID $N(\mu, \sigma^2)$

$$S = X_1 + \cdots + X_n$$

$$\sim N(n\mu, n\sigma^2)$$

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The CDF of S is

$$F_S(s) = \Phi\left(\frac{s - n\mu}{\sqrt{n}\,\sigma}\right)$$

CDF of $N(0,1)$

The PDF of S is

$$f_S(s) = \frac{1}{\sqrt{2\pi}\sqrt{n}\,\sigma} e^{-\frac{(s-n\mu)^2}{2n\sigma^2}}$$

The mean of $S = n\mu$

The variance $S = n\sigma^2$.