LECTURE

9 OCTOBER

9:00-10:00AM

MATH3/4/68181

ML Estimation

Suppose $x_1, x_2, \ldots, x_n$ are IID from the GP model.

$$L(\sigma, \xi) = \prod_{i=1}^{n} \frac{q}{\sigma} \left[1 + \xi \frac{x_i - \mu}{\sigma}\right]^{-\frac{1}{\xi} - 1}$$

$$= \frac{q^n}{\sigma^n} \left\{\prod_{i=1}^{n} \left[1 + \xi \frac{x_i - \mu}{\sigma}\right]\right\}^{-\frac{1}{\xi} - 1}$$

$$\Rightarrow \log L(\sigma, \xi) = n \log q - n \log \sigma$$

$$- \left(\frac{1}{\xi} + 1\right) \sum_{i=1}^{n} \log\left[1 + \xi \frac{x_i - \mu}{\sigma}\right]$$

MLE equations for the GP distribution

The MLEs of σ and ξ are the simultaneous solutions of

$$\begin{aligned}\frac{\partial \log L}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{1+\xi}{\sigma^2} \sum_{i=1}^{n} (x_i - t)\left(1 + \xi \frac{x_i - t}{\sigma}\right)^{-1} \\ &= 0, \qquad \cdots \quad (1)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \xi} &= \frac{1}{\xi^2} \sum_{i=1}^{n} \log\left(1 + \xi \frac{x_i - t}{\sigma}\right) \\ &\quad - \frac{1+\xi}{\xi\sigma} \sum_{i=1}^{n} (x_i - t)\left(1 + \xi \frac{x_i - t}{\sigma}\right)^{-1} \\ &= 0. \qquad \cdots\cdots \quad (2)\end{aligned}$$

The MLEs of σ and ξ are the simultaneous solutions of (1) and (2).

In R, $\boxed{\text{fpot}}$ will compute the MLEs of σ and ξ.

Is there a way to check that ETT will hold without checking conditions (I) – (III)? Yes.

Suppose X is a discrete RV. Let $F(\cdot)$ denote the CDF of X. Then ETT will <u>not hold</u> if

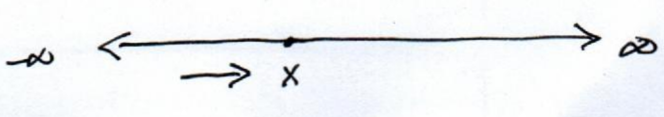
$$\frac{P(X=k)}{1-F(k-1)} \not\rightarrow 0 \qquad (*)$$

as $k \longrightarrow w(F)$.

Suppose X is a continuous RV with CDF $F(\cdot)$. Then ETT will <u>not hold</u> if

$$\frac{f(x)}{1-F(x^-)} \not\rightarrow 0$$

as $x \longrightarrow w(F)$.



Ex 3 $\quad X \sim \text{Pois}(\lambda).$

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, \dots$$

$$\omega(F) = \infty$$

$$(*) = \lim_{k \to \infty} \frac{\frac{e^{-\lambda}\lambda^k}{k!}}{1 - \sum_{i=0}^{k-1} \frac{e^{-\lambda}\lambda^i}{i!}}$$

$$= 1 - P(X \leq k-1)$$

$$= P(X > k-1) = P(X \geq k)$$

$$= \lim_{k \to \infty} \frac{\frac{\cancel{e^{-\lambda}}\lambda^k}{k!}}{\sum_{i=k}^{\infty} \frac{\cancel{e^{-\lambda}}\lambda^i}{i!}}$$

$$= \lim_{k \to \infty} \frac{\frac{\lambda^k}{k!}}{\sum_{i=k}^{\infty} \frac{\lambda^i}{i!}}$$

$$= \lim_{k \to \infty} \frac{\frac{\lambda^k}{k!}}{\frac{\lambda^k}{k!} + \sum_{i=k+1}^{\infty} \frac{\lambda^i}{i!}}$$

$$= \lim_{k \to \infty} \frac{1}{1 + \sum_{i=k+1}^{\infty} \frac{k!\,\lambda^{i-k}}{i!}}$$

$$\frac{k!}{i!} = \frac{1}{(k+1)(k+2)\cdots(k+i-k)}$$

$$(k+1) > k, \quad (k+2) > k, \quad (k+i-k) > k$$

$$< \frac{1}{k^{i-k}}$$

$$> \lim_{k \to \infty} \frac{1}{1 + \sum_{i=k+1}^{\infty} \frac{\lambda^{i-k}}{k^{i-k}}}$$

$$\boxed{j = i - k}$$

$$= \lim_{k \to \infty} \frac{1}{1 + \sum_{j=1}^{\infty} \left(\frac{\lambda}{k}\right)^{j}}$$

$$= \lim_{k \to \infty} \frac{1}{1 + \frac{1}{1 - \frac{\lambda}{k}} - 1} = 1$$

$$\frac{\lambda}{k} \to 0$$

$\Rightarrow \quad (*) > 1$

$\Rightarrow \quad (*) \neq 0$

$\Rightarrow$ ETT will not hold.

$$\boxed{\sum_{n=1}^{\infty} \theta^{n} = \frac{1}{1-\theta} - 1}$$