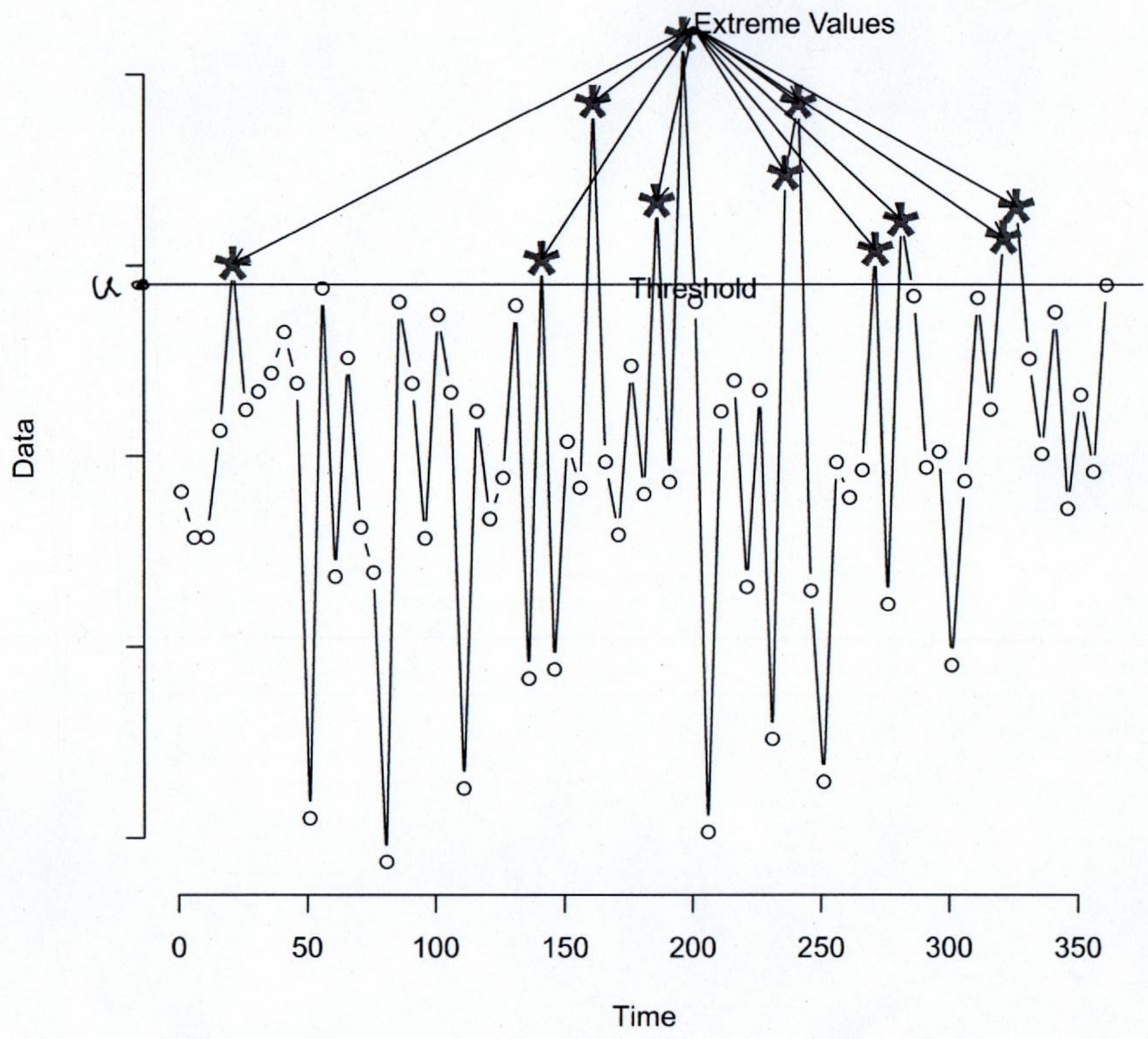
LECTURE

8 OCTOBER

9:00-10:00AM

MATH3/4/68181

Definition 2



Definition 2

X = variable of interest (eg stock returns)

u = Threshold (considered high enough)

X is an extreme value

if $X > u$

$X - u$ = <u>Excess amount</u>

What is the distribution of

$X - u$ (the excess) if $X > u$

(that is X is an extreme value)?

$$P(X - u < x \mid X > u) = ?$$

$$= \frac{P(X - u < x, X > u)}{P(X > u)}$$

$$= \frac{P(u < X < u + x)}{P(X > u)}$$

$$= \frac{F(u+x) - F(u)}{1 - F(u)}$$

$\boxed{\text{F denotes the CDF of X}}$

$$\Big\downarrow$$

$$1 - \left[1 + \xi \frac{x}{\sigma}\right]^{-\frac{1}{\xi}}$$

$\sigma > 0$ "scale", $-\infty < \xi < \infty$ "shape"

as $u \longrightarrow \omega(F)$

This result is due to Pickands (1975).

For u large enough,

$$\frac{F(u+x)-F(u)}{1-F(u)} \approx 1-\left[1+\xi\frac{x}{\sigma}\right]^{-\frac{1}{\xi}}$$

$$\Leftrightarrow \quad \frac{1-F(u+x)}{1-F(u)} \approx \left[1+\xi\frac{x}{\sigma}\right]^{-\frac{1}{\xi}}$$

$$\Leftrightarrow \quad 1-F(u+x) \approx \left[1-F(u)\right]\left[1+\xi\frac{x}{\sigma}\right]^{-\frac{1}{\xi}}$$

$$\Leftrightarrow \quad F(\underbrace{u+x}_{=y}) \approx 1-\left[1-F(u)\right]\left[1+\xi\frac{x}{\sigma}\right]^{-\frac{1}{\xi}}$$

$$\Leftrightarrow \quad F(y) \approx 1-\left[1-F(u)\right]\left[1+\xi\frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}}$$

$\boxed{\text{Let } q = 1-F(u)}$

$$\Leftrightarrow \quad \boxed{F(y) \approx 1-q\left[1+\xi\frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}}}$$

<u>Generalized</u> <u>Pareto (GP)</u> <u>Model</u>

$-\infty < \xi < \infty$ "shape parameter"

$\sigma > 0$ "scale " "

u = threshold

q = prob that $X > u$.

Special Cases

i) $\boxed{\xi = 0}$

$$F(y) = 1 - \lim_{\xi \to 0} q\left[1 + \xi\,\frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}}$$

$$= 1 - q \lim_{\xi \to 0}\left[1 + \xi\,\frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}}$$

$\boxed{\text{Set } m = \frac{1}{\xi}}$

$$= 1 - q \lim_{m \to \infty}\left[1 + \frac{\frac{y-u}{\sigma}}{m}\right]^{-m}$$

$$= 1 - q \lim_{m \to \infty}\left\{\left[1 + \frac{\frac{y-u}{\sigma}}{m}\right]^{m}\right\}^{-1}$$

$$\boxed{\lim_{m \to \infty}\left(1 + \frac{z}{m}\right)^{m} = e^{z}}$$

$$= 1 - q\left[e^{\frac{y-u}{\sigma}}\right]^{-1}$$

$$= 1 - q\,e^{-\frac{y-u}{\sigma}}, \quad y > u$$

Truncated exponential distribution

The domain is (u, ∞).

ii) $\boxed{\xi > 0}$

$$F(y) \approx 1 - q\left[1 + \xi \frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}}, \quad y > u$$

provided

$$1 + \xi \frac{y-u}{\sigma} > 0$$

$$\Longleftrightarrow \quad \frac{y-u}{\sigma} > -\frac{1}{\xi}$$

$$\Longleftrightarrow \quad y > u - \frac{\sigma}{\xi}$$

So the domain of $F(y)$ is

$$\cancel{\left(u - \frac{\sigma}{\xi}, \infty\right)}.$$

$$(u, \infty).$$

iii) $\boxed{\xi < 0}$

$$F(y) \approx 1 - q\left[1 + \xi \frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}}, \quad y > u$$

provided

$$1 + \xi \frac{y-u}{\sigma} > 0$$

$$\Leftrightarrow \quad \xi \frac{y-u}{\sigma} > -1$$

$$\Leftrightarrow \quad y - u < -\frac{\sigma}{\xi}$$

$$\Leftrightarrow \quad y < u - \frac{\sigma}{\xi}$$

So the domain of $F(y)$ is

$$\left(u, u - \frac{\sigma}{\xi}\right)$$

$F(y)$ has a finite upper end point,

$w(F) = u - \frac{\sigma}{\xi}$.

CDF

$$F(y) \approx 1 - q \left[1 + \xi \frac{y-u}{\sigma} \right]^{-\frac{1}{\xi}}$$

PDF

$$f(y) \approx \frac{q}{\sigma} \left[1 + \xi \frac{y-u}{\sigma} \right]^{-\frac{1}{\xi}-1}$$

Quantile

Set $F(y) = p$, $0 < p < 1$

$$\Leftrightarrow 1 - q \left[1 + \xi \frac{y-u}{\sigma} \right]^{-\frac{1}{\xi}} = p$$

$$\Leftrightarrow \left[1 + \xi \frac{y-u}{\sigma} \right]^{-\frac{1}{\xi}} = \frac{1-p}{q}$$

$$\Leftrightarrow 1 + \xi \frac{y-u}{\sigma} = \left(\frac{1-p}{q} \right)^{-\xi}$$

$$\Leftrightarrow y = u + \frac{\sigma}{\xi} \left[\left(\frac{1-p}{q} \right)^{-\xi} - 1 \right].$$

$$\text{Median} = u + \frac{\sigma}{\xi} \left[(2q)^{\xi} - 1 \right].$$

Return Level

is the extreme value occuring on average every T years.

This corresponds to taking

$$p = 1 - \frac{1}{mT}$$

average number of exceedances per year.

$\Rightarrow$ Return level

$$= u + \frac{\sigma}{\xi}\left[(mTq)^{\xi} - 1\right].$$

How to choose u ?

For the GP model,

$$\underbrace{\boxed{E(Y-u \mid Y > u)}}_{\text{Mean Excess}} = a \cdot \underbrace{u}_{\text{Threshold}} + b$$

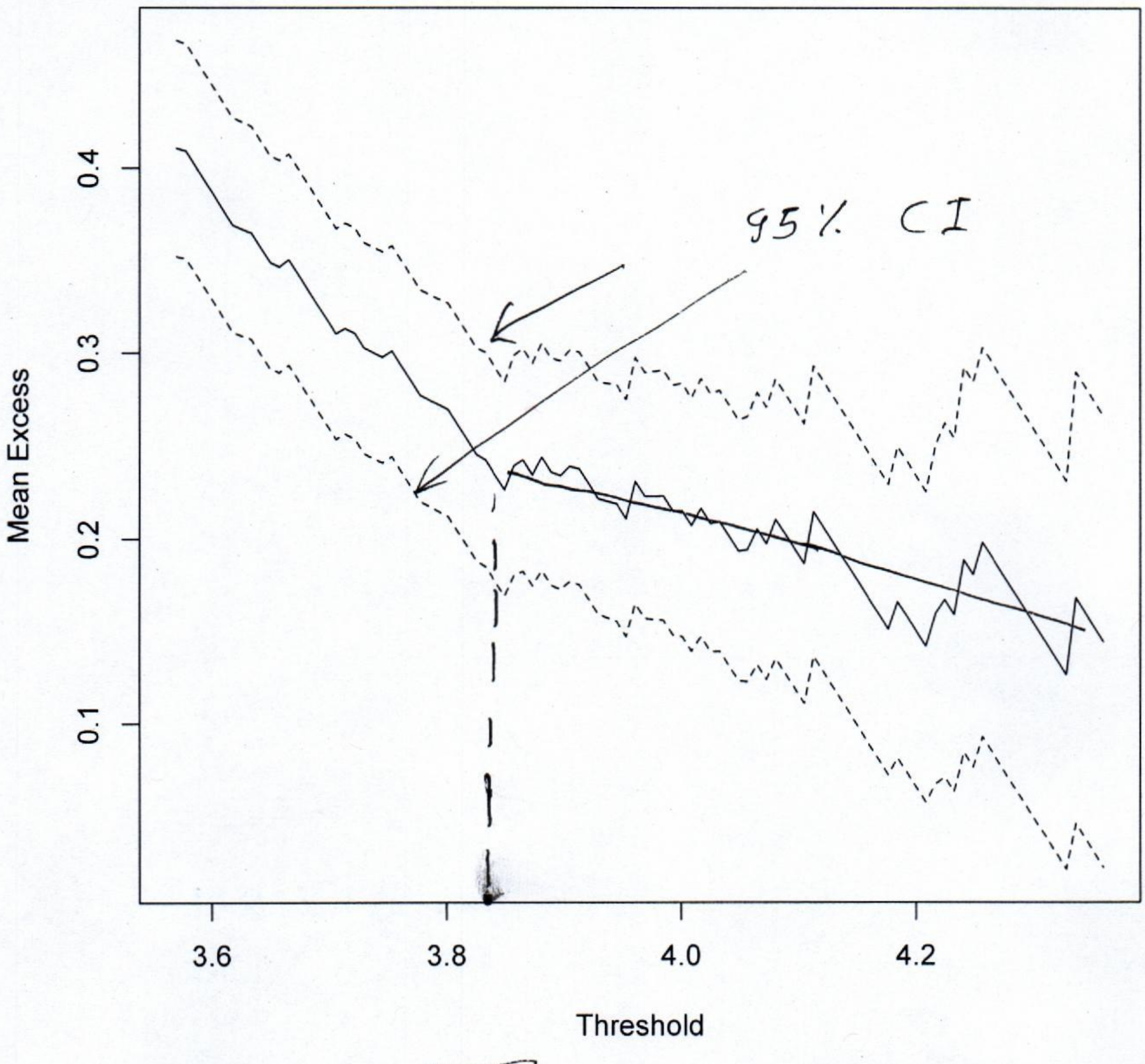
constants

- Plot mean excess vs u.
- Choose u so that the plot is approximately linear.

MRLplot

In R, `mrlplot` to draw the plot of mean excess vs u.

Mean ~~Residual~~ Excess Life Plot



$\boxed{u = 3.9}$