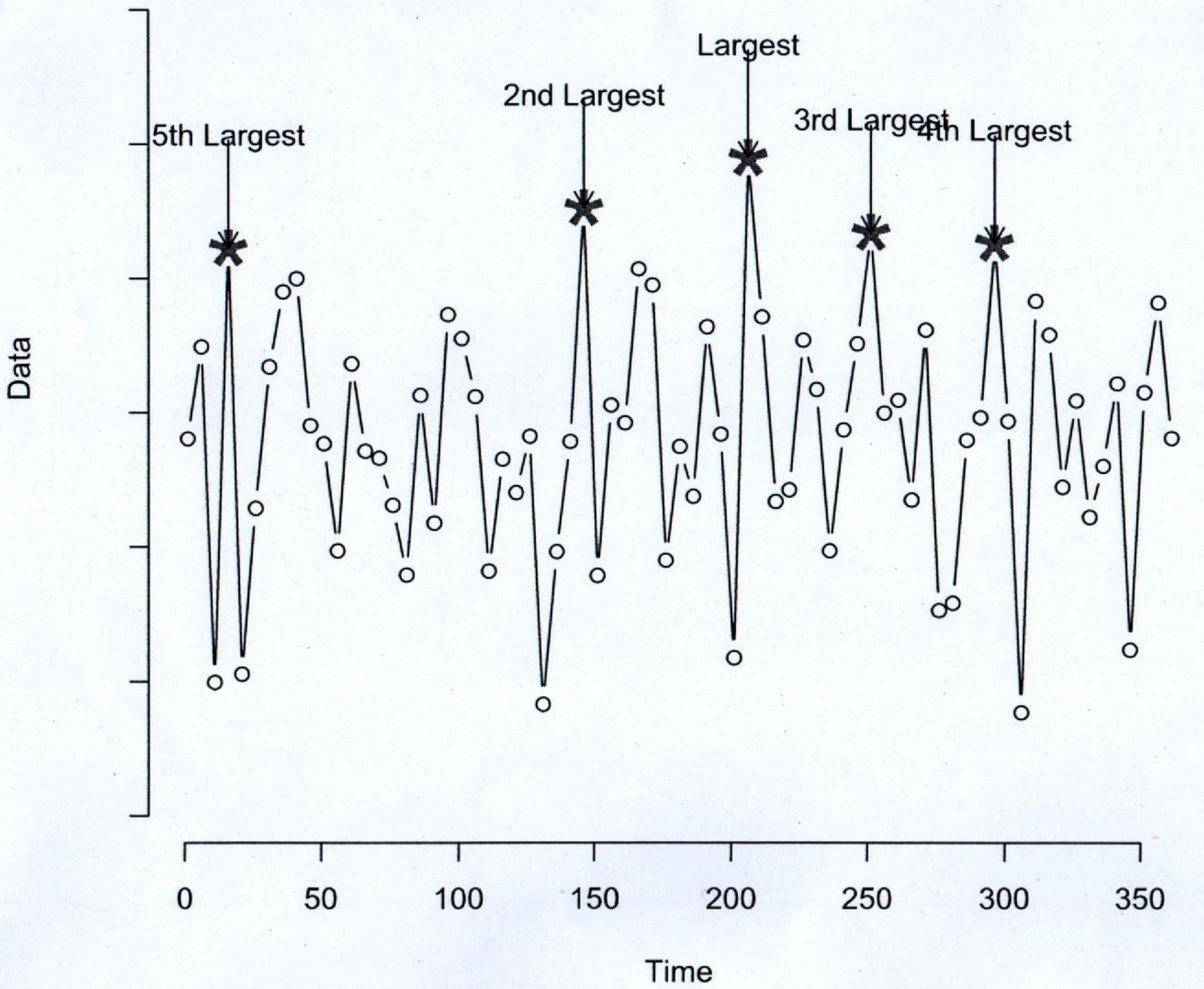
LECTURE

5 OCTOBER

10:00-11:00AM

MATH4/68181

Definition 3 " r largest " method



Extreme Values

Level 4 & 6 students

- Defn 1
- Defn 2
- Defn 3

Level 3 students

ML Estimation

$\cancel{x_{1,1}, x_{1,2}, \ldots, x_{1,n}}$

$x_{1,1}, x_{2,1}, \ldots, x_{n,1}$

(largest obsns for n years)

$x_{1,2}, x_{2,2}, \ldots, x_{n,2}$

(2^{nd} largest obsns for n years)

$\vdots$

$x_{1,r}, x_{2,r}, \ldots, x_{n,r}$

(r^{th} largest obsn for n years)

Definition 3

Suppose $X_1, X_2, \ldots, X_n$ are IID observations with CDF $F(\cdot)$

Let $M_n^{(1)}$ = largest of $X_1, \ldots, X_n$

$M_n^{(2)} = 2^{nd}$ " " "

$\vdots$

$M_n^{(r)} = r^{th}$ " " "

It can be shown under certain conditions that

$$\Pr\left(\frac{M_n^{(1)} - b_n}{a_n} < x_1, \ldots, \frac{M_n^{(r)} - b_n}{a_n} < x_r\right)$$

$$\xrightarrow[\text{as } n \to \infty]{} \sum_{s_1=0}^{1} \sum_{s_2=0}^{2-s_1} \cdots \sum_{s_{r-1}=0}^{r-1-s_1-\cdots-s_{r-2}}$$

$$\frac{(\gamma_2-\gamma_1)^{s_1}}{s_1!} \cdots \frac{(\gamma_r-\gamma_{r-1})^{s_{r-1}}}{s_{r-1}!} e^{-\gamma_r} \qquad (*)$$

where $\gamma_i = \left(1 + \xi \dfrac{x_i - 0}{1}\right)^{-\frac{1}{\xi}}$

$= (1 + \xi x_i)^{-\frac{1}{\xi}}, \quad i = 1, 2, \ldots, r$

It can be deduced from ($*$) that the joint PDF of $(M_n^{(1)}, M_n^{(2)}, \ldots, M_n^{(r)})$ is

$$f(x_1, x_2, \ldots, x_r)$$

$$= \sigma^{-r} e^{-\left(1+\xi \frac{x_r-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$\times e^{-\left(\frac{1}{\xi}+1\right)\sum_{i=1}^{r} \log\left(1+\frac{x_i-\mu}{\sigma}\right)}$$

where $x_1 \geq x_2 \geq \ldots \geq x_r$ and

$$1+\xi\frac{(x_i-\mu)}{\sigma} > 0, \quad i = 1, 2, \ldots, r$$

$$L(\mu, \sigma, \xi) = \prod_{i=1}^{n} f(x_{i,1}, x_{i,2}, \ldots, x_{i,r})$$

$$= \prod_{i=1}^{n} \left[\sigma^{-r} e^{-\left(1+\xi \frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} e^{-\left(1+\frac{1}{\xi}\right)\sum_{j=1}^{r} \log\left(1+\frac{x_{i,j}-\mu}{\sigma}\right)} \right]$$

$$= \sigma^{-nr} e^{-\sum_{i=1}^{n} \left(1+\xi \frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} e^{-\left(1+\frac{1}{\xi}\right)\sum_{i=1}^{n}\sum_{j=1}^{r} \log\left(1+\frac{x_{i,j}-\mu}{\sigma}\right)}.$$

$$\Rightarrow \log L(\mu, \sigma, \xi) = -nr \log \sigma - \sum_{i=1}^{n} \left(1+\xi \frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}} - \left(1+\frac{1}{\xi}\right)\sum_{i=1}^{n}\sum_{j=1}^{r} \log\left(1+\frac{x_{i,j}-\mu}{\sigma}\right).$$

MLE equations for the r largest distribution

The MLEs of μ, σ and ξ are the simultaneous solutions of

$$\begin{aligned}\frac{\partial \log L}{\partial \mu} &= -\frac{1}{\sigma}\sum_{i=1}^{n}\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &\quad -\frac{1+\xi}{\sigma}\sum_{i=1}^{n}\sum_{j=1}^{r}\left(1+\xi\frac{x_{i,j}-\mu}{\sigma}\right)^{-1} \\ &= 0, \qquad \cdot\cdot \quad (1)\end{aligned}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \sigma} &= -\frac{nr}{\sigma}-\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_{i,r}-\mu)\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &\quad +\frac{1+\xi}{\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{r}(x_{i,j}-\mu)\left(1+\xi\frac{x_{i,j}-\mu}{\sigma}\right)^{-1} \\ &= 0, \qquad \cdot\cdot \quad (2)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \xi} &= -\frac{1}{\xi^2}\sum_{i=1}^{n}\log\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}} \\ &\quad +\frac{1}{\xi\sigma}\sum_{i=1}^{n}(x_{i,r}-\mu)\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &\quad +\frac{1}{\xi^2}\sum_{i=1}^{n}\sum_{j=1}^{r}\log\left(1+\xi\frac{x_{i,j}-\mu}{\sigma}\right) \\ &\quad -\frac{1+\xi}{\xi\sigma}\sum_{i=1}^{n}\sum_{j=1}^{r}(x_{i,j}-\mu)\left(1+\xi\frac{x_{i,j}-\mu}{\sigma}\right)^{-1} \\ &= 0. \qquad \cdot\cdot \quad (3)\end{aligned}$$

The MLEs of μ, σ & ξ are the simultaneous solutions of (1), (2) and (3). The R package **evd** can compute the MLEs μ, σ & ξ.

$$e^y \simeq 1 + y \quad \text{small } y$$

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

$$\underline{\underline{\gamma(t)}} > 0 \quad \forall\, t$$