LECTURE

2 OCTOBER

9:00-10:00AM

MATH3/4/68181

For a practitioner, dealing with 3 distributions is not convenient.

Is there a distribution that unifies the 3 distributions into a single form?

Yes.

GEV (Generalized Extreme Value) Distribution has the CDF

$$G(x) = e^{-\left(1 + \xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

where

$$1 + \xi \frac{x-\mu}{\sigma} > 0$$

$-\infty < \xi < \infty$ "shape"

$-\infty \quad \mu < \infty$ "location"

$\sigma > 0$ "scale"

Particular Cases

i) $\boxed{\xi = 0}$

$$\lim_{\xi \to 0} e^{-\left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= \lim_{\xi \to 0} e^{-\left(1+\frac{\frac{x-\mu}{\sigma}}{\frac{1}{\xi}}\right)^{-\frac{1}{\xi}}}$$

$\boxed{\text{Set } m = \frac{1}{\xi}}$

$$= \lim_{m \to \infty} e^{-\left(1+\frac{\frac{x-\mu}{\sigma}}{m}\right)^{-m}}$$

$$= \lim_{m \to \infty} e^{-\left[\left(1+\frac{\frac{x-\mu}{\sigma}}{m}\right)^{m}\right]^{-1}}$$

$$\boxed{\left(1+\frac{y}{m}\right)^{m} \to e^{y} \quad \text{as } m \to \infty}$$

$$= e^{-\left[e^{\frac{x-\mu}{\sigma}}\right]^{-1}}$$

$$= e^{-e^{-\frac{x-\mu}{\sigma}}}$$

the <u>same type</u> as $\Lambda(x) = e^{-e^{-x}}$

$\Rightarrow \Lambda(x)$ is a particular case of GEV when $\xi = 0$

ii) $\boxed{\xi > 0}$

$$e^{-\left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= e^{-\left(1 - \frac{\xi\mu}{\sigma} + \frac{\xi x}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$\boxed{\text{Set } \alpha = \frac{1}{\xi}}$$

$$= e^{-\left(1 - \frac{\mu}{\alpha\sigma} + \frac{x}{\sigma\alpha}\right)^{-\alpha}}$$

$$\boxed{a = \frac{1}{\sigma\alpha}, \; b = 1 - \frac{\mu}{\alpha\sigma}}$$

$a > 0$
$b \in \mathbb{R}$

$$= e^{-(b + a x)^{-\alpha}}$$

= the <u>same</u> <u>type</u> as $\Phi_\alpha(x)$

$\Rightarrow$ $\Phi_\alpha(x)$ is a particular case of GEV when $\xi > 0$.

iii) $\boxed{\xi < 0}$

$$e^{-\left(1+\xi\,\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= e^{-\left(1-\frac{\xi\mu}{\sigma}+\frac{\xi x}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$\boxed{\text{Set } \alpha = -\frac{1}{\xi}}$$

$$= e^{-\left(1+\frac{\mu}{\alpha\sigma}-\frac{x}{\alpha\sigma}\right)^{\alpha}}$$

$$= e^{-\left(-\left(-1-\frac{\mu}{\alpha\sigma}+\frac{x}{\alpha\sigma}\right)\right)^{\alpha}}$$

$$\boxed{a = \frac{1}{\alpha\sigma}, \quad b = -1-\frac{\mu}{\alpha\sigma}} \qquad \begin{array}{l} a > 0 \\ b \in \mathbb{R} \end{array}$$

$$= e^{-\left(-\left(b+ax\right)\right)^{\alpha}}$$

= same type as $\Psi_\alpha(x)$

$\Rightarrow$ $\Psi_\alpha(x)$ is a particular case of GEV when $\xi < 0$.

GEV Distribution

CDF

$$G(x) = e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

PDF

$$g(x) = \frac{d\,G(x)}{dx}$$

$$= \frac{1}{\sigma}\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \cdot e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

Quantile function

$$e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} = p, \quad 0 < p < 1$$

$$\Rightarrow \quad -\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} = \log p$$

$$\Rightarrow \quad \left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} = -\log p$$

$$\Rightarrow \quad 1+\xi\frac{x-\mu}{\sigma} = (-\log p)^{-\xi}$$

$$\Rightarrow \quad x = \mu + \frac{\sigma}{\xi}\left[(-\log p)^{-\xi} - 1\right]$$

p th quantile

For example, if $p = \frac{1}{2}$

$$\text{Median} = \mu + \frac{\sigma}{\xi}\left[(\log 2)^{-\xi} - 1 \right]$$

<u>Return</u> <u>level</u>

is the quantile when

$$p = 1 - \frac{1}{T}$$ — No of years

For example if $T = 100$ years then the quantile will be the extreme value that occurs on average once in 100 years.

Return level for T years

$$= \mu + \frac{\sigma}{\xi}\left[\left(-\log\left(1 - \frac{1}{T}\right)\right)^{-\xi} - 1 \right].$$

ML estimation

(Maximum Likelihood Estimation)

Suppose $x_1, x_2, \ldots, x_n$ is a random sample from the GEV distribution.

$$L(\mu, \sigma, \xi) = \prod_{i=1}^{n} g(x_i)$$

$$= \prod_{i=1}^{n} \left[\frac{1}{\sigma} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi} - 1} e^{-\left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi}}} \right]$$

$$= \frac{1}{\sigma^n} \left[\prod_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right) \right]^{-\frac{1}{\xi} - 1} e^{-\sum_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi}}} .$$

$$\log L(\mu, \sigma, \xi) = -n \log \sigma - \left(\frac{1}{\xi} + 1\right) \sum_{i=1}^{n} \log\left(1 + \xi \frac{x_i - \mu}{\sigma}\right) - \sum_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi}}$$

MLE equations for the GEV distribution

The MLEs of μ, σ and ξ are the simultaneous solutions of

$$\begin{aligned}\frac{\partial \log L}{\partial \mu} &= \frac{1+\xi}{\sigma} \sum_{i=1}^{n} \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-1} \\ &\quad -\frac{1}{\sigma} \sum_{i=1}^{n} \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0, \quad \ldots\ldots \qquad (1)\end{aligned}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{1+\xi}{\sigma^2} \sum_{i=1}^{n} (x_i-\mu) \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-1} \\ &\quad -\frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i-\mu) \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0, \quad \ldots\ldots\ldots \qquad (2)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \xi} &= \frac{1}{\xi^2} \sum_{i=1}^{n} \log\left(1+\xi \frac{x_i-\mu}{\sigma}\right) \\ &\quad -\frac{1+\xi}{\xi\sigma} \sum_{i=1}^{n} (x_i-\mu) \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-1} \\ &\quad -\frac{1}{\xi^2} \sum_{i=1}^{n} \log\left(1+\xi \frac{x_i-\mu}{\sigma}\right) \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}} \\ &\quad +\frac{1}{\xi\sigma} \sum_{i=1}^{n} (x_i-\mu) \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0. \quad \ldots\ldots\ldots\ldots \qquad (3)\end{aligned}$$

The MLEs of μ, σ and ξ are the simultaneous solutions of (1), (2) and (3).

$\boxed{\text{fgev}}$ in the R software will compute the MLEs $\hat{\mu}$, $\hat{\sigma}$ and $\hat{\xi}$.

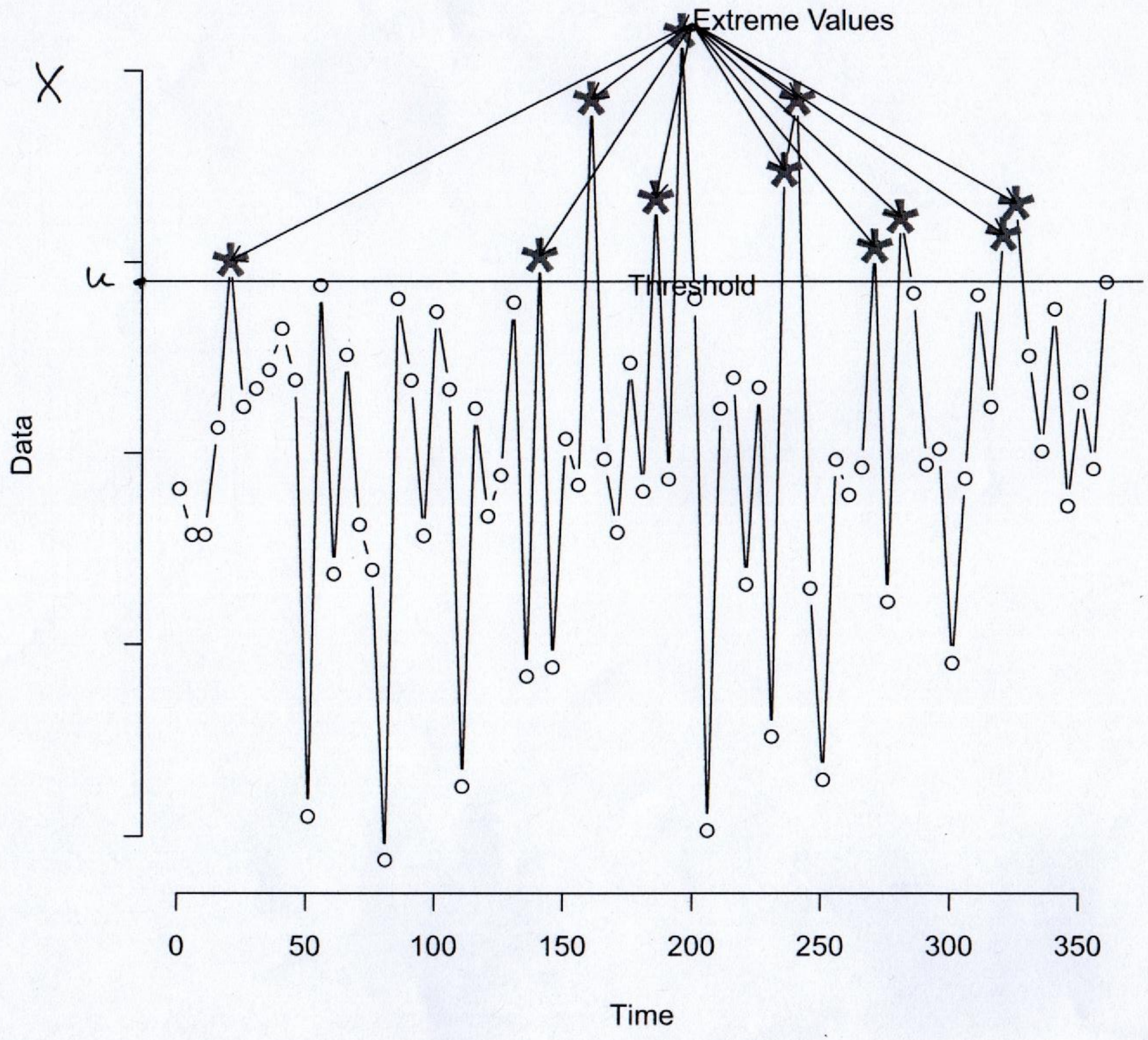
Definition 2

X = variable of interest (eg stock returns)

u = Threshold (considered high enough)

X is an extreme value if $X > u$

$X - u$ = Excess amount



Definition 2

What is the distribution of $X - u$ (the excess) if $X > u$ (that is X is an extreme value)?

$$P(X - u < x \mid X > u) = ?$$