LECTURE

1 OCTOBER

9:00-10:00AM

MATH3/4/68181

Ex 2 $\quad F(x) = \Phi(x), \quad -\infty < x < \infty$

$= \text{CDF of } N(0,1)$

$F(x) = 1 \Rightarrow \Phi(x) = 1 \Rightarrow x = \Phi^{-1}(1) = \infty$

$\Rightarrow \omega(F) = \infty$

$$\text{I}: \quad \lim_{t \to \infty} \frac{1 - \Phi(t + x\gamma(t))}{1 - \Phi(t)} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\boxed{\begin{array}{c} \text{L'Hopitals Rule} \\ \lim \dfrac{f(x)}{g(x)} = \lim \dfrac{f'(x)}{g'(x)} \\ \text{if } \lim f(x) = \lim g(x) = \pm\infty \\ \text{OR } \lim f(x) = \lim g(x) = 0 \end{array}}$$

$$\overset{LH}{=} \lim_{t \to \infty} \frac{-\phi(t + x\gamma(t)) \cdot (1 + x\gamma'(t))}{-\phi(t)}$$

$$= \lim_{t \to \infty} \frac{\cancel{\frac{1}{\sqrt{2\pi}}} e^{-\frac{(t + x\gamma(t))^2}{2}} (1 + x\gamma'(t))}{\cancel{\frac{1}{\sqrt{2\pi}}} e^{-\frac{t^2}{2}}}$$

$$= \lim_{t \to \infty} e^{\frac{t^2}{2} - \frac{(t + x\gamma(t))^2}{2}} \cdot (1 + x\gamma'(t))$$

$$= \lim_{t \to \infty} e^{-tx\gamma(t) - \frac{x^2(\gamma(t))^2}{2}} \cdot (1 + x\gamma'(t))$$

$$\boxed{\begin{array}{l} \text{Choose } \gamma(t) = \frac{1}{t} \\ \gamma'(t) = -\frac{1}{t^2} \\ (\gamma(t))^2 = \frac{1}{t^2} \end{array}}$$

$$= \lim_{t \to \infty} e^{-x - \frac{x^2}{2t^2}} \cdot \left(1 + \frac{x}{(-t^2)}\right)$$

$= e^{-x} \quad \Rightarrow$ Cond (I) is satisfied

By the ETT, there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\left[F(a_n x + b_n)\right]^n \longrightarrow e^{-e^{-x}}$$

as $n \to \infty$.

Ex 3 $\quad F(x) = e^{-\frac{1}{x}}, \quad x > 0$

$$F(x) = 1 \Rightarrow e^{-\frac{1}{x}} = 1 \Rightarrow -\frac{1}{x} = 0$$

$$\Rightarrow x = \infty \Rightarrow \omega(F) = \infty.$$

I: $$\lim_{t\to\infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t\to\infty} \frac{1 - e^{-\frac{1}{t + x\gamma(t)}}}{1 - e^{-\frac{1}{t}}}$$

$$\boxed{e^{-y} \approx 1 - y \quad \text{as } y \to 0}$$ ⊛

$$= \lim_{t\to\infty} \frac{\cancel{1} - \left(\cancel{1} - \frac{1}{t + x\gamma(t)}\right)}{\cancel{1} - \left(\cancel{1} - \frac{1}{t}\right)}$$

$$= \lim_{t\to\infty} \frac{t}{t + x\gamma(t)} \neq e^{-x}$$

$\Rightarrow$ Cond (I) is not satisfied

II: $$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t\to\infty} \frac{1 - e^{-\frac{1}{tx}}}{1 - e^{-\frac{1}{t}}}$$

$$\overset{⊛}{=} \lim_{t\to\infty} \frac{\cancel{1} - \left(\cancel{1} - \frac{1}{tx}\right)}{\cancel{1} - \left(\cancel{1} - \frac{1}{t}\right)} = \lim_{t\to\infty} \frac{\frac{1}{tx}}{\frac{1}{t}} = \frac{1}{x}$$

$\Rightarrow$ Cond (II) is satisfied with $\alpha = 1$

By the ETT, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) \longrightarrow \begin{cases} 0 & x < 0 \\ e^{-\frac{1}{x}} & x \geq 0 \end{cases}$$

AN EXAMPLE WHERE ETT FAILS

$$F(x) = 1 - \frac{1}{\log x}, \quad x > e$$

[log is log to the base e]

$$F(x) = 1 \Rightarrow 1 - \frac{1}{\log x} = 1 \Rightarrow \frac{1}{\log x} = 0$$

$$\Rightarrow \log x = \infty \Rightarrow x = \infty \Rightarrow \omega(F) = \infty.$$

I: $$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{\cancel{1} - \left[\cancel{1} - \frac{1}{\log(t + x\gamma(t))}\right]}{\cancel{1} - \left[\cancel{1} - \frac{1}{\log t}\right]}$$

$$= \lim_{t \to \infty} \frac{\overset{\to \infty}{\log t}}{\underset{\to \infty}{\log(t + x\gamma(t))}}$$

$$\overset{LH}{=} \lim_{t \to \infty} \frac{\frac{1}{t}}{\frac{1 + x\gamma'(t)}{t + x\gamma(t)}} = \lim_{t \to \infty} \frac{t + x\gamma(t)}{t[1 + x\gamma'(t)]}$$

$$\neq e^{-x}$$

$\Rightarrow$ Cond (I) is not satisfied

$\underline{\text{II}}$: $$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{\cancel{1} - \left[\cancel{1} - \frac{1}{\log(tx)}\right]}{\cancel{1} - \left[\cancel{1} - \frac{1}{\log t}\right]}$$

$$= \lim_{t \to \infty} \frac{\log t}{\log(tx)}$$

$$= \lim_{t \to \infty} \frac{\log t}{\log t + \log x}$$

$$= \lim_{t \to \infty} \frac{1}{1 + \frac{\log x}{\log t}} \qquad \frac{\log x}{\log t} \to 0 \qquad = 1 \neq x^{-\alpha}$$

$\Rightarrow$ Cond (II) is not satisfied

$\underline{\text{III}}$: $w(F) = \infty$ is not finite

$\Rightarrow$ Cond (III) is not satisfied

Hence, ETT fails.

How to Choose a_n and b_n

I : $a_n = \gamma\left(F^{-1}\left(1-\frac{1}{n}\right)\right)$ and $b_n = F^{-1}\left(1-\frac{1}{n}\right)$

II : $a_n = F^{-1}\left(1-\frac{1}{n}\right)$ and $b_n = 0$

III : $a_n = w(F) - F^{-1}\left(1-\frac{1}{n}\right)$ and $b_n = w(F)$.

Ex 1 $\quad F(x) = 1 - e^{-x}$

There exist $a_n > 0$ and $b_n \in R$ such that

$$\left[F(a_n x + b_n)\right]^n \rightarrow e^{-e^{-x}}$$

as $n \rightarrow \infty$.

$$F(x) = y \Rightarrow 1 - e^{-x} = y$$
$$\Rightarrow e^{-x} = 1 - y$$
$$\Rightarrow x = -\log(1-y)$$
$$\Rightarrow F^{-1}(x) = -\log(1-x)$$

$$a_n = \gamma\left(F^{-1}\left(1 - \frac{1}{n}\right)\right) \qquad [\gamma(t) \equiv 1]$$
$$= 1$$

$$b_n = F^{-1}\left(1 - \frac{1}{n}\right) = -\log\left(1 - \left(1 - \frac{1}{n}\right)\right)$$
$$= \log n$$

$$\Rightarrow \left[F(x + \log n)\right]^n \rightarrow e^{-e^{-x}}$$

as $n \rightarrow \infty$.

For a practitioner, dealing with 3 distributions is not convenient.

Is there a distribution that unifies the 3 distributions into a single form?

Yes.

GEV (Generalized Extreme Value) Distribution has the CDF

$$G(x) = e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

where $1+\xi\frac{x-\mu}{\sigma} > 0$

$-\infty < \xi < \infty$ "shape"

$-\infty < \mu < \infty$ "location"

$\sigma > 0$ "scale"