EXAMPLE CLASS

2 OCTOBER

16:00-17:00PM

MATH3/4/68181

Q1

$$\Lambda(x) = e^{-e^{-x}}$$

$$\Lambda'(x) = e^{-e^{-x}} (-1) e^{-x} (-1)$$

$$= e^{-e^{-x}} e^{-x}$$

$$\Phi_\alpha(x) = \begin{cases} 0 & x < 0 \\ e^{-x^{-\alpha}} & x \geq 0 \end{cases}$$

$$\Phi_\alpha'(x) = \begin{cases} 0 & x < 0 \\ e^{-x^{-\alpha}} (-1)(-\alpha) x^{-\alpha-1} & x \geq 0 \end{cases}$$

$$= \begin{cases} 0 & x < 0 \\ \alpha x^{-\alpha-1} e^{-x^{-\alpha}} & x \geq 0 \end{cases}$$

$$\Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha} & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$$\Psi_\alpha'(x) = \begin{cases} e^{-(-x)^\alpha} (-1) \alpha (-x)^{\alpha-1} (-1) & x < 0 \\ 0 & x \geq 0 \end{cases}$$

$$= \begin{cases} \alpha (-x)^{\alpha-1} e^{-(-x)^\alpha} & x < 0 \\ 0 & x \geq 0 \end{cases}$$

Q2 $\Lambda(x) = e^{-e^{-x}}$

$$\Lambda'(x) = e^{-e^{-x}} e^{-x}$$

$$\text{Mean} = \int_{-\infty}^{\infty} x \cdot e^{-e^{-x}} e^{-x} \, dx$$

$$\boxed{\begin{array}{l} \text{Set } y = e^{-x} \Rightarrow x = -\log y \\ \frac{dx}{dy} = -\frac{1}{y} \end{array}}$$

$$= \int_{\infty}^{0} (-\log y) \, e^{-y} \, y \left(-\frac{1}{y}\right) dy$$

$$= -\int_{0}^{\infty} \log y \, e^{-y} \, dy$$

$$\boxed{\begin{array}{l} \frac{dy^a}{da} = y^a \log y \\ \Rightarrow \left.\frac{dy^a}{da}\right|_{a=0} = y^0 \log y = \log y \quad \ldots (*) \end{array}}$$

$$\overset{(*)}{=} -\int_{0}^{\infty} \left.\frac{dy^a}{da}\right|_{a=0} e^{-y} \, dy$$

$$= -\frac{d}{da}\left[\int_{0}^{\infty} y^a e^{-y} \, dy\right]\Bigg|_{a=0}$$

$$\boxed{\begin{array}{c} \Gamma(a+1) = \int_{0}^{\infty} y^a e^{-y} \, dy \\ \text{GAMMA FUNCTION} \end{array}}$$

$$= -\frac{d}{da}\left[\Gamma(a+1)\right]\Bigg|_{a=0}$$

$$= -\Gamma'(1)$$

Q3 $\quad \Lambda(x) = e^{-e^{-x}}$

$$\Lambda'(x) = e^{-e^{-x}} e^{-x}$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$
$$= E(X^2) - [\Gamma'(1)]^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 e^{-e^{-x}} e^{-x}\, dx$$

$$\boxed{y = e^{-x} \Rightarrow x = -\log y \Rightarrow \frac{dx}{dy} = -\frac{1}{y}}$$

$$= \int_{\infty}^{0} (\log y)^2 e^{-y}\, y \left(-\frac{1}{y}\right) dy$$

$$= \int_{0}^{\infty} (\log y)^2 e^{-y}\, dy$$

$$\frac{dy^a}{da} = y^a \log y$$
$$\Rightarrow \frac{d^2 y^a}{da^2} = \left(\frac{dy^a}{da}\right)\log y = (y^a \log y)\log y$$
$$= y^a (\log y)^2$$
$$\Rightarrow \left.\frac{d^2 y^a}{da^2}\right|_{a=0} = y^0 (\log y)^2 = (\log y)^2 \quad \ldots (*)$$

$$= \int_0^{\infty} \left.\frac{d^2 y^a}{da^2}\right|_{a=0} e^{-y}\, dy$$

$$= \left.\frac{d^2}{da^2}\left[\int_0^{\infty} y^a e^{-y}\, dy\right]\right|_{a=0}$$

$$= \left.\frac{d^2}{da^2} \Gamma(a+1)\right|_{a=0}$$

$$= \Gamma''(1)$$

$$\text{Variance} = \Gamma''(1) - [\Gamma'(1)]^2.$$

Q4 $\Lambda^n(x) = \Lambda(\alpha_n + \beta_n x)$

$\Leftrightarrow \left[e^{-e^{-x}}\right]^n = e^{-e^{-\alpha_n - \beta_n x}}$

$\Leftrightarrow e^{-ne^{-x}} = e^{-e^{-\alpha_n - \beta_n x}}$

$\xLeftrightarrow{\text{Take log}} -ne^{-x} = -e^{-\alpha_n - \beta_n x}$

$\Leftrightarrow ne^{-x} = e^{-\alpha_n - \beta_n x}$

$\xLeftrightarrow{\text{Take log}} \log n - x = -\alpha_n - \beta_n x$

$\Leftrightarrow$ Eq coeff of x: $-1 = -\beta_n \Rightarrow \beta_n = 1$

" constants : $\log n = -\alpha_n \Rightarrow \alpha_n = -\log n$.

Q5, Q6 are similar

Q7 done in lectures

Q8 $\quad F(x) = [1-e^{-x}]^{\alpha}, \quad x > 0$

$$F(x) = 1 \Rightarrow [1-e^{-x}]^{\alpha} = 1 \Rightarrow 1-e^{-x} = 1$$

$$\Rightarrow e^{-x} = 0 \Rightarrow x = \infty \Rightarrow \omega(F) = \infty$$

I: $$\lim_{t \to \infty} \frac{1-[1-e^{-t-x\gamma(t)}]^{\alpha}}{1-[1-e^{-t}]^{\alpha}}$$

$e^{-t-x\gamma(t)} \to 0$, $\quad e^{-t} \to 0$

$$\boxed{(1-y)^{\alpha} \approx 1-\alpha y \quad \text{as } y \to 0}$$

$$= \lim_{t \to \infty} \frac{\cancel{1}-[\cancel{1}-\cancel{\alpha} e^{-t-x\gamma(t)}]}{\cancel{1}-[\cancel{1}-\cancel{\alpha} e^{-t}]}$$

$$= \lim_{t \to \infty} \frac{e^{-t-x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t \to \infty} e^{-x\gamma(t)}$$

$$= e^{-x} \quad \text{if } \gamma(t) \equiv 1 \quad \forall t$$

Hence, cond (I) is satisfied.
By the ETT, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) \longrightarrow e^{-e^{-x}}$$

as $n \longrightarrow \infty$.