LECTURE

28 SEPTEMBER

10:00-11:00AM

MATH4/68181

Sometime $f(x) = \frac{dF(x)}{dx}$ (the PDF)

is more convenient than $F(x)$.

Ex $X \sim N(0, 1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$F(x) = \Phi(x)$$

CONDITIONS IN TERMS OF PDF

Let
$$a(t) = F^{-1}\left(1 - \frac{1}{t}\right)$$
$$b(t) = t\, f(a(t))$$

I : $\dfrac{b(tx)}{b(t)} \longrightarrow 1$ as $t \to \infty$

II : $\omega(F) = \infty$ and $\lim_{t \to \infty} a(t)\, b(t) = \alpha$

III : $\omega(F) < \infty$ and $\lim_{t \to \infty} \{\omega(F) - a(t)\}\, b(t) = \alpha$

Only one of these conditions (if any) will be satisfied.

Ex 1

$$F(x) = 1 - e^{-x} = y \Rightarrow e^{-x} = 1 - y$$
$$\Rightarrow \quad x = -\log(1-y)$$

$$F^{-1}(x) = -\log(1-y)$$

$$a(t) = F^{-1}\left(1 - \frac{1}{t}\right) = -\log\left(1 - \left(1 - \frac{1}{t}\right)\right)$$
$$= -\log\left(\frac{1}{t}\right)$$
$$= \log t$$

$$f(x) = e^{-x}$$

$$b(t) = t \cdot f(a(t))$$
$$= t \cdot e^{-\log t}$$
$$= 1$$

I: $$\frac{b(tx)}{b(t)} = \frac{1}{1} = 1 \quad \text{for all } t$$

Hence (I) is satisfied

By the ETT, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) \longrightarrow e^{-e^{-x}}$$

as $n \longrightarrow \infty$.

Ex 2 $\quad F(x) = 1 - \frac{1}{x}, \quad x > 1$

$$w(F) = +\infty.$$

$$F(x) = y \Rightarrow 1 - \frac{1}{x} = y$$

$$\Rightarrow \frac{1}{x} = 1 - y$$

$$\Rightarrow x = \frac{1}{1-y}$$

$$a(t) = F^{-1}\left(1 - \frac{1}{t}\right) = \frac{1}{1 - \left(1 - \frac{1}{t}\right)} = t$$

$$f(x) = \frac{1}{x^2}$$

$$b(t) = t \cdot f(a(t)) = t \cdot \frac{1}{t^2} = \frac{1}{t}.$$

I : $\quad \dfrac{b(tx)}{b(t)} = \dfrac{\frac{1}{tx}}{\frac{1}{t}} = \dfrac{1}{x} \neq 1 \quad$ for all t

Cond (I) is not satisfied

II : $\quad w(F) = +\infty \quad \checkmark$

$$\lim_{t \to \infty} a(t)\, b(t) = \lim_{t \to \infty} t \cdot \frac{1}{t} = 1 = \kappa$$

Cond (II) is satisfied

By the ETT, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) \longrightarrow \begin{cases} 0, & x < 0 \\ e^{-x^{-1}}, & x \geq 0 \end{cases}$$

as $n \longrightarrow \infty$.

Ex 3

$X \sim \text{Uni}[0,1]$

$F(x) = x, \quad 0 < x < 1$

$\omega(F) = 1$

$f(x) = 1, \quad 0 < x < 1$

$$b(t) = t f(a(t)) = t$$

I : $$\frac{b(tx)}{b(t)} = \frac{tx}{t} = x \neq 1$$

Cond (I) is <u>not</u> satisfied.

II : $\omega(F) = 1 \neq \infty$

condition (II) is not satisfied

III : $F^{-1}(x) = x$

$$a(t) = F^{-1}\left(1 - \frac{1}{t}\right) = 1 - \frac{1}{t}$$

$$\lim_{t \to \infty} \{\omega(F) - a(t)\} b(t)$$

$$= \lim_{t \to \infty} \left(1 - \left(1 - \frac{1}{t}\right)\right) t$$

$$= 1 = \alpha$$

Condition (III) is satisfied.

By the ETT, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) \longrightarrow \begin{cases} e^x & x < 0 \\ 1 & x \geq 0 \end{cases}$$

as $n \longrightarrow \infty$.

The ETT uses a linear normalization.

$$P\left(\frac{M_n - b_n}{a_n} \leq x \right) = \left[F\left(a_n x + b_n \right) \right]^n$$

Try some other normalization. A <u>power</u> <u>normalization</u> is

$$P\left(\left| \frac{M_n}{a_n} \right|^{\frac{1}{b_n}} \operatorname{sign}(M_n) \leq x \right)$$

$$\boxed{\operatorname{sign}(y) = \begin{cases} +1 & y > 0 \\ -1 & y < 0 \\ 0 & y = 0 \end{cases}}$$

$$= F^n\left(a_n |x|^{b_n} \operatorname{sign}(x) \right) \quad (**)$$

ETT

for power normalization

If there exists $a_n > 0$ and $b_n > 0$ such that

$$(**) \longrightarrow G(x)$$

as $n \to \infty$ for a non-degenerate G then G must be of the same type as

$$\text{I} : \quad G(x) = \begin{cases} 0 & x \leq 1 \\ e^{-(\log x)^{-\alpha}} & x > 1 \end{cases}, \quad \alpha > 0$$

$$\text{II} : \quad G(x) = \begin{cases} 0 & x < 0 \\ e^{-|\log x|^{\alpha}} & 0 \leq x < 1 \\ 1 & x > 1 \end{cases} \quad \alpha > 0$$

$$\text{III} : \quad G(x) = \begin{cases} 0 & x \leq -1 \\ e^{-|\log |x||^{-\alpha}} & -1 < x < 0 \\ 1 & x \geq 0 \end{cases} \quad \alpha > 0$$

$$\text{IV} : \quad G(x) = \begin{cases} e^{-(\log |x|)^{\alpha}} & x < -1 \\ 1 & x \geq -1 \end{cases}$$

$$\text{V} : \quad G(x) = \Phi_1(x)$$

$$\text{VI} : \quad G(x) = \Psi_1(x).$$

G_1 & G_2 are of the same type if

$$G_1(x) = G_2(a|x|^b \operatorname{sign}(x))$$

for all x, where $a > 0$ and $b > 0$.