LECTURE

25 SEPTEMBER

9:00-10:00AM

MATH3/4/68181

Definition I

Suppose $X_1, X_2, \ldots, X_n$ are IID with cumulative distribution function (CDF) $F(\cdot)$. The extreme value is

$$M_n = \max(X_1, \ldots, X_n).$$

For modeling purposes, we need to know the distribution of M_n

$$
\begin{aligned}
& P(M_n < x) \\
&= P(\max(X_1, \ldots, X_n) < x) \\
&= P(X_1 < x, \ldots, X_n < x) \\
&\overset{\text{indep}}{=} P(X_1 < x) \cdots P(X_n < x) \\
&\overset{\text{identical}}{=} F(x) \cdots F(x) \\
&= [F(x)]^n
\end{aligned}
$$

What is the limit of $P(M_n < x)$ as $n \to \infty$?

$$\lim_{n \to \infty} P(M_n < x) = \lim_{n \to \infty} [F(x)]^n = \begin{cases} 0 & \text{if } F(x) < 1 \\ 1 & \text{if } F(x) = 1 \end{cases}$$

DEGENERATE LIMIT

THIS IS NOT A USEFUL RESULT

SLLN

(Strong Law of Law Numbers)

Suppose $X_1, \dots, X_n$ are IID with population mean μ. Then

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow \mu$$

as $n \longrightarrow \infty$

HOW TO MAKE THIS RESULT INTO SOMETHING USEFUL?

CLT

(Central Limit Theorem)

Suppose $X_1, \dots, X_n$ are IID with population mean μ and population variance σ^2. Then

Scaled version of $\overline{X}$

$$\boxed{\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}} \xrightarrow{d} N(0,1)$$

as $n \longrightarrow \infty$

Let us look at the limit of a scaled version of M_n.

$$P\left(\frac{M_n - b_n}{a_n} \leqslant x \right)$$

scaled version of M_n

$$= P\left(M_n \leqslant a_n x + b_n \right)$$

$$= P\left(\max(X_1, \dots, X_n) \leqslant a_n x + b_n \right)$$

$$= P\left(X_1 \leqslant a_n x + b_n, \dots, X_n \leqslant a_n x + b_n \right)$$

$$\overset{\text{indep}}{=} P\left(X_1 \leqslant a_n x + b_n \right) \cdots P\left(X_n \leqslant a_n x + b_n \right)$$

$$\overset{\text{identical}}{=} F(a_n x + b_n) \cdots F(a_n x + b_n)$$

$$= \left[F(a_n x + b_n) \right]^n \quad \dots \quad (*)$$

WHAT IS THE LIMIT OF $(*)$ AS $n \longrightarrow \infty$?

ETT

(Extremal Types Theorem)

Suppose $X_1, \dots, X_n$ are IID with CDF $F(\cdot)$. Let $M_n = \max(X_1, \dots, X_n)$. If there exist $a_n > 0$ and $-\infty < b_n < \infty$ such that

$$(*) \longrightarrow G(x)$$

as $n \to \infty$ for a $\boxed{\text{non-degenerate}}$ G then it must be of the $\boxed{\text{same}}$ $\boxed{\text{type}}$ as

$$\text{I} : \Lambda(x) = e^{-e^{-x}}, \quad -\infty < x < \infty$$

GUMBEL

$$\text{II} : \Phi_\alpha(x) = \begin{cases} 0 & x < 0 \\ e^{-x^{-\alpha}} & x \geq 0 \end{cases}$$

FRÉCHET

$$\text{III} : \Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha} & x < 0 \\ 1 & x \geq 0 \end{cases}$$

WEIBULL

SAME TYPE

Two CDFs G_1 and G_2 are of the same type if

$$G_1(x) = G_2(ax+b)$$

for all x for some $a > 0$ and $b \in \mathbb{R}$

Ex 1

$$G_1(x) = e^{-e^{-x}}$$

$$G_2(x) = e^{-e^{-2x-3}}$$

G_1 & G_2 are of the same type

Ex 2

$$G_1(x) = e^{-\frac{1}{x}}$$

$$G_2(x) = e^{-\frac{1}{3-2x}}$$

G_1 & G_2 are not of the same type

HOW TO KNOW WHICH OF THE 3 LIMITS IS ATTAINED ?

$w(F) =$ "upper end point" of $F(\cdot)$

$$= \sup \{ x : F(x) < 1 \}$$

Set $F(x) = 1 \Rightarrow$ solve for x

Ex1 $F(x) = 1 - e^{-x}$

$F(x) = 1 \Rightarrow 1 - e^{-x} = 1 \Rightarrow e^{-x} = 0$

$\Rightarrow x = +\infty \Rightarrow w(F) = +\infty$

Ex2 $F(x) = 1 - \frac{1}{x}, \quad x > 1$

$F(x) = 1 \Rightarrow 1 - \frac{1}{x} = 1 \Rightarrow \frac{1}{x} = 0$

$\Rightarrow x = +\infty \Rightarrow w(F) = +\infty$

CONDITIONS TO CHECK FOR THE 3 LIMITS

I: there exist $\gamma(t) > 0$

such that

$$\lim_{t \to \omega(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = e^{-x}$$

II: $\omega(F) = +\infty$ and

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \quad \alpha > 0, \; x > 0$$

III: $\omega(F) < +\infty$ and

$$\lim_{t \to 0} \frac{1 - F(\omega(F) - tx)}{1 - F(\omega(F) - t)} = x^{\alpha}, \quad \alpha > 0, \; x > 0$$

Only one these conditions (if any) will be satisfied.

Ex 1

$$F(x) = 1 - e^{-x}$$

$$w(F) = +\infty$$

I : $$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{\cancel{1} - [\cancel{1} - e^{-t - x\gamma(t)}]}{\cancel{1} - [\cancel{1} - e^{-t}]}$$

$$= \lim_{t \to \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t \to \infty} e^{-x\gamma(t)}$$

$$= e^{-x} \quad \text{if} \quad \gamma(t) \equiv 1 \text{ for all } t.$$

$\Rightarrow$ Cond (I) is satisfied.

By the ETT, there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$[F(a_n x + b_n)]^n \longrightarrow e^{-e^{-x}}$$

as $n \longrightarrow \infty$.

(GUMBEL LIMIT)