REVISION CLASS

14 DECEMBER

10:00-11:00AM

MATH4/68181

REVISION

CLASS

FRI 14 DEC

10- 11 AM

A2 (a), Exam 2017/18

$$C(u_1, u_2) = \min\left(u_1^{1-\alpha} u_2,\ u_1 u_2^{1-\beta}\right), \quad \begin{matrix} 0<\alpha<1 \\ 0<\beta<1 \end{matrix}$$

(i) $C(0, u_2) = \min(0, 0) = 0$ ✓

(ii) $C(u_1, 0) = \min(0, 0) = 0$ ✓

(iii) $C(1, u_2) = \min(u_2, u_2^{1-\beta}) = u_2$ ✓

(iv) $C(u_1, 1) = \min(u_1^{1-\alpha}, u_1) = u_1$ ✓

(v) $$\frac{\partial C}{\partial u_1} = \frac{\partial}{\partial u_1} \begin{cases} u_1^{1-\alpha} u_2 & \text{if } u_1^{1-\alpha} u_2 < u_1 u_2^{1-\beta} \\ u_1 u_2^{1-\beta} & \text{if } u_1 u_2^{1-\beta} < u_1^{1-\alpha} u_2 \end{cases}$$

$$= \begin{cases} \underbrace{(1-\alpha) u_1^{-\alpha} u_2}_{\geq 0} & \text{if } u_1^{1-\alpha} u_2 < u_1 u_2^{1-\beta} \\ u_2^{1-\beta} \geq 0 & \text{if } u_1 u_2^{1-\beta} < u_1^{1-\alpha} u_2 \end{cases}$$

≥ 0 ✓

(vi) $$\frac{\partial C}{\partial u_2} = \frac{\partial}{\partial u_2} \begin{cases} u_1^{1-\alpha} u_2 & \text{if } u_1^{1-\alpha} u_2 < u_1 u_2^{1-\beta} \\ u_1 u_2^{1-\beta} & \text{if } u_1 u_2^{1-\beta} < u_1^{1-\alpha} u_2 \end{cases}$$

$$= \begin{cases} u_1^{1-\alpha} \geq 0 & \text{if } u_1^{1-\alpha} u_2 < u_1 u_2^{1-\beta} \\ (1-\beta) u_1 u_2^{-\beta} \geq 0 & \text{if } u_1 u_2^{1-\beta} < u_1^{1-\alpha} u_2 \end{cases}$$

≥ 0 ✓

Hence, C is a copula.

A2(d) Exam 2017/18

$$C(u_1, u_2) = \left\{\left[\left(u_1^{-\theta} - 1\right)^{\delta} + \left(u_2^{-\theta} - 1\right)^{\delta}\right]^{\frac{1}{\delta}} + 1\right\}^{-\frac{1}{\theta}}$$

$$\theta > 0$$
$$\delta \geq 1$$

(i) $$C(0, u_2) = \left\{\left[(\infty - 1)^{\delta} + \left(u_2^{-\theta} - 1\right)^{\delta}\right]^{\frac{1}{\delta}} + 1\right\}^{-\frac{1}{\theta}}$$
$$= 0 \quad ✓$$

(ii) $$C(u_1, 0) = \left\{\left[\left(u_1^{-\theta} - 1\right)^{\delta} + (\infty - 1)^{\delta}\right]^{\frac{1}{\delta}} + 1\right\}^{-\frac{1}{\theta}}$$
$$= 0 \quad ✓$$

(iii) $$C(1, u_2) = \left\{\left[(1-1)^{\delta} + \left(u_2^{-\theta} - 1\right)^{\delta}\right]^{\frac{1}{\delta}} + 1\right\}^{-\frac{1}{\theta}}$$
$$= \left\{u_2^{-\theta} - 1 + 1\right\}^{-\frac{1}{\theta}}$$
$$= u_2 \quad ✓$$

(iv) $$C(u_1, 1) = \left\{\left[\left(u_1^{-\theta} - 1\right)^{\delta} + (1-1)^{\delta}\right]^{\frac{1}{\delta}} + 1\right\}^{-\frac{1}{\theta}}$$
$$= \left\{u_1^{-\theta} - 1 + 1\right\}^{-\frac{1}{\theta}}$$
$$= u_1 \quad ✓$$

(v) $$\frac{\partial C}{\partial u_1} = \cancel{\left(-\frac{1}{\theta}\right)} \left\{\left[\left(u_1^{-\theta} - 1\right)^{\delta} + \left(u_2^{-\theta} - 1\right)^{\delta}\right]^{\frac{1}{\delta}} + 1\right\}^{-\frac{1}{\theta} - 1}$$
$$\cdot \cancel{\frac{1}{\delta}} \left[\left(u_1^{-\theta} - 1\right)^{\delta} + \left(u_2^{-\theta} - 1\right)^{\delta}\right]^{\frac{1}{\delta} - 1}$$
$$\cdot \cancel{\delta} \left(u_1^{-\theta} - 1\right)^{\delta - 1}$$
$$\cdot \cancel{(-\theta)} \, u_1^{-\theta - 1}$$

$$= \left\{ \left[(u_1^{-\theta}-1)^{\delta} + (u_2^{-\theta}-1)^{\delta} \right]^{\frac{1}{\delta}} + 1 \right\}^{-\frac{1}{\theta}-1} \geq 0$$

$$\cdot \left[(u_1^{-\theta}-1)^{\delta} + (u_2^{-\theta}-1)^{\delta} \right]^{\frac{1}{\delta}-1} \geq 0$$

$$\cdot \underbrace{(u_1^{-\theta}-1)^{\delta-1}}_{\geq 0} \underbrace{u_1^{-\theta-1}}_{\geq 0}$$

≥ 0 ✓

(vi) $$\frac{\partial C}{\partial u_2} = \left\{ \left[(u_1^{-\theta}-1)^{\delta} + (u_2^{-\theta}-1)^{\delta} \right]^{\frac{1}{\delta}} + 1 \right\}^{-\frac{1}{\theta}-1} \geq 0$$

$$\cdot \left[(u_1^{-\theta}-1)^{\delta} + (u_2^{-\theta}-1)^{\delta} \right]^{\frac{1}{\delta}-1} \geq 0$$

$$\cdot \underbrace{(u_2^{-\theta}-1)^{\delta-1}}_{\geq 0} \underbrace{u_2^{-\theta-1}}_{\geq 0}$$

≥ 0 ✓

Hence, C is a copula.

A1, Exam 2017/18

$$f_{X,Y}(x,y) = \frac{4xy + 2x + 2y + 1}{4}$$

$$0 < x < 1$$
$$0 < y < 1$$

(a) $F_{X,Y}(x,y) = \int_0^x \int_0^y f_{X,Y}(u,v)\, dv\, du$

$$= \frac{1}{4}\int_0^x \int_0^y [4uv + 2u + 2v + 1]\, dv\, du$$

$$= \frac{1}{4}\int_0^x \left[2uv^2 + 2uv + v^2 + v\right]_0^y du$$

$$= \frac{1}{4}\int_0^x [2uy^2 + 2uy + y^2 + y]\, du$$

$$= \frac{1}{4}\left[u^2y^2 + u^2y + uy^2 + uy\right]_0^x$$

$$= \frac{1}{4}\left[x^2y^2 + x^2y + xy^2 + xy\right]$$

(b) $F_X(x) = F_{X,Y}(x,1) = \frac{x^2 + x}{2}$

$F_Y(y) = F_{X,Y}(1,y) = \frac{y^2 + y}{2}$

(c) $w(F_X) = 1$,

$$\lim_{t \to 0} \frac{1 - F_X(1 - tx)}{1 - F_X(1 - t)}$$

$$= \lim_{t \to 0} \frac{1 - \frac{(1-tx)^2 + 1 - tx}{2}}{1 - \frac{(1-t)^2 + 1 - t}{2}}$$

$$= \lim_{t \to 0} \frac{\frac{3tx}{2} - \frac{(tx)^2}{2}}{\frac{3t}{2} - \frac{t^2}{2}}$$

$$= \lim_{t \to 0} \frac{\frac{3x}{2} - \frac{tx^2}{2} \to 0}{\frac{3}{2} - \frac{t}{2} \to 0}$$

$$= x$$

Hence F_X belongs to the Weibull max domain.

(d) $w(F_Y) = 1$

$$\lim_{t \to 0} \frac{1 - F_Y(1 - ty)}{1 - F_Y(1-t)}$$

$$= \lim_{t \to 0} \frac{\frac{3y}{2} - \frac{ty^2}{2} \to 0}{\frac{3}{2} - \frac{t}{2} \to 0}$$

$$= y$$

Hence F_Y belongs to the Weibull max domain.

(e) $a_n = w(F_X) - F_X^{-1}(1 - \frac{1}{n})$

$b_n = w(F_X)$

$$F_X(x) = \frac{x^2 + x}{2} = z$$

$$\Rightarrow x^2 + x - 2z = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1 + 8z}}{2}$$

$$\Rightarrow x = \frac{-1 + \sqrt{1 + 8z}}{2}$$

$$F_X^{-1}(x) = \frac{-1+\sqrt{1+8x}}{2}$$

$$F_X^{-1}\left(1-\frac{1}{n}\right) = \frac{-1+\sqrt{1+8\left(1-\frac{1}{n}\right)}}{2}$$

$$= \frac{-1+\sqrt{9-\frac{8}{n}}}{2}$$

$$\Rightarrow \quad a_n = \frac{3}{2} - \frac{1}{2}\sqrt{9-\frac{8}{n}}$$

$$b_n = 1$$

$$c_n = \frac{3}{2} - \frac{1}{2}\sqrt{9-\frac{8}{n}}$$

$$d_n = 1$$

(g) $$\lim_{n\to\infty} \left[F_{X,Y}(a_n x + b_n,\ c_n y + d_n)\right]^n$$

$$= \lim_{n\to\infty} \frac{(a_n x + b_n)^n (c_n y + d_n)^n}{4^n}$$

$$\cdot \left[(a_n x + b_n)(c_n y + d_n) + a_n x + b_n + c_n y + d_n + 1 \right]^n$$

FIND THE LIMIT OF THIS