# REVISION CLASS

# 10 DECEMBER

# 9:00-10:00AM

# MATH3/4/68181

# Revision Class

Monday

10 Dec

9 - 10 am

Questions please!

Q5, 2017/18 Exam

$$\bar{F}(x_1, \dots, x_k) = \left[\max\left(\frac{x_1}{a_1}, \dots, \frac{x_k}{a_k}\right)\right]^{-a}$$

(a) $F_T(t)$

$= P(T \leq t)$

$= 1 - P(T > t)$

$= 1 - P(\min(X_1, \dots, X_k) > t)$

$= 1 - P(X_1 > t, \dots, X_t > t)$

$= 1 - \bar{F}(t, \dots, t)$

$= 1 - \left[\max\left(\frac{t}{a_1}, \dots, \frac{t}{a_k}\right)\right]^{-a}$

$= 1 - \left[t \cdot \max\left(\frac{1}{a_1}, \dots, \frac{1}{a_k}\right)\right]^{-a}$

$= 1 - \left[t \cdot \frac{1}{\min(a_1, \dots, a_k)}\right]^{-a}$

$= 1 - [\min(a_1, \dots, a_k)]^a t^{-a}$, $t \geq \min(a_1, \dots, a_k)$

(b) $f_T(t) = \frac{d}{dt} F_T(t)$

$= a[\min(a_1, \dots, a_k)]^a t^{-a-1}$, $t \geq \min(a_1, \dots, a_k)$

(c) $F_T(t) = p$

$\Rightarrow 1 - [\min(a_1, \dots, a_k)]^a t^{-a} = p$

$\Rightarrow t^{-a} = \frac{1-p}{[\min(a_1, \dots, a_k)]^a}$

$\Rightarrow VaR_p(T) = \frac{(1-p)^{-1/a}}{[\min(a_1, \dots, a_k)]^{-1}}$

(d) $$ES_p(T) = \frac{1}{p}\int_0^p VaR_u(T)\,du$$

$$= \frac{\min(a_1,\ldots,a_k)}{p}\int_0^p (1-u)^{-\frac{1}{a}}\,du$$

$$= \frac{\min(a_1,\ldots,a_k)}{p}\left[\frac{(1-u)^{1-\frac{1}{a}}}{\frac{1}{a}-1}\right]_0^p$$

$$= \frac{\min(a_1,\ldots,a_k)}{p}\,\frac{(1-p)^{1-\frac{1}{a}}-1}{\frac{1}{a}-1}.$$

(e)

$$L(a,a_1,\ldots,a_k)$$

$$= \prod_{i=1}^{n}\left\{a[\min(a_1,\ldots,a_k)]^a\, t_i^{-a-1}\right\}$$

$$= a^n[\min(a_1,\ldots,a_k)]^{na}\left(\prod_{i=1}^{n} t_i\right)^{-a-1}$$

$$\prod_{i=1}^{n} I\{t_i \geqslant \min(a_1,\ldots,a_k)\}$$

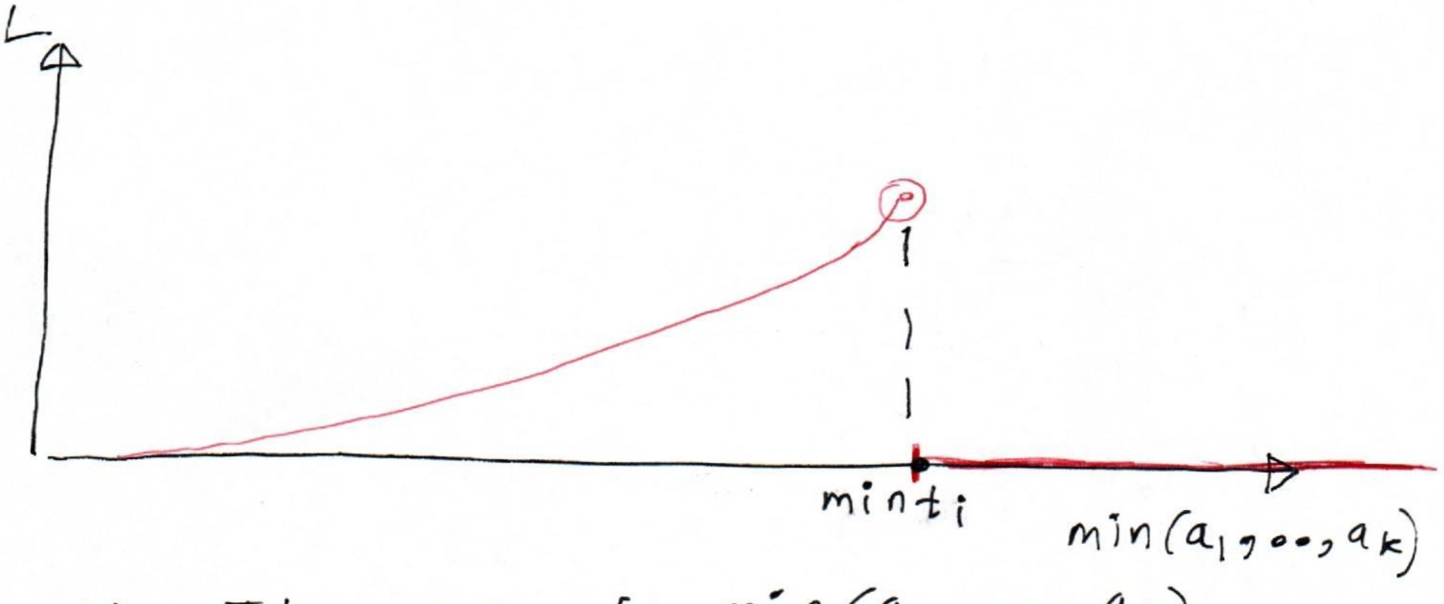
$$= a^n[\min(a_1,\ldots,a_k)]^{na}\left(\prod_{i=1}^{n} t_i\right)^{-a-1}$$

$$I\{\min t_i \geqslant \min(a_1,\ldots,a_k)\}$$

$$\log L = n \log a + na \log [\min(a_1, \ldots, a_k)] - (a+1) \sum_{i=1}^{n} \log t_i + \log I\{\min t_i \geq \min(a_1, \ldots, a_k)\}$$

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} + n \log [\min(a_1, \ldots, a_k)] - \sum_{i=1}^{n} \log t_i = 0$$

$$\Rightarrow \quad \hat{a} = n \left\{ \sum_{i=1}^{n} \log t_i - n \log [\min(a_1, \ldots, a_k)] \right\}^{-1}$$




$\Rightarrow$ The MLE of $\min(a_1, \ldots, a_k)$ is $\min(t_1, \ldots, t_n)$.

Quiz 5

$$ES_p(X) = \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

$$\boxed{\begin{aligned} &\text{Set } u = F^{-1}(t) \\ &t = F(u) \\ &\frac{dt}{du} = f(u) \end{aligned}}$$

$$= \frac{1}{p}\int_{F^{-1}(0)}^{F^{-1}(p)} u f(u)\,du$$

$$= \frac{1}{p}\int_{-\infty}^{F^{-1}(p)} u \cdot K\left(1+\frac{u^2}{a}\right)^{-\frac{a+1}{2}} du$$

$$= \frac{aK}{2p}\int_{-\infty}^{F^{-1}(p)} \frac{2u}{a}\left(1+\frac{u^2}{a}\right)^{-\frac{a+1}{2}} du$$

$$= \frac{aK}{2p}\int_{-\infty}^{F^{-1}(p)} \left(1+\frac{u^2}{a}\right)^{-\frac{a+1}{2}} d\left(1+\frac{u^2}{a}\right)$$

$$= \frac{aK}{2p}\left[\frac{\left(1+\frac{u^2}{a}\right)^{1-\frac{a+1}{2}}}{1-\frac{a+1}{2}}\right]_{-\infty}^{F^{-1}(p)}$$

$$= \frac{aK}{2p}\cdot\frac{2}{1-a}\left\{\left(1+\frac{[F^{-1}(p)]^2}{a}\right)^{\frac{1-a}{2}} - 0\right\}$$

$$= \frac{aK}{p(1-a)}\left(1+\frac{1}{a}[F^{-1}(p)]^2\right)^{\frac{1-a}{2}}.$$

## Alternative proof

Assume

$$ES_p(X) = \frac{aK}{(1-a)p}\left\{1+\frac{1}{a}\left[VaR_p(X)\right]^2\right\}^{\frac{1-a}{2}}$$

$$\Rightarrow \frac{1}{\cancel{p}}\int_0^p F^{-1}(u)\,du = \frac{aK}{(1-a)\cancel{p}}\left\{1+\frac{1}{a}\left[F^{-1}(p)\right]^2\right\}^{\frac{1-a}{2}}$$

$$\Rightarrow \int_0^p F^{-1}(u)\,du = \frac{aK}{1-a}\left\{1+\frac{1}{a}\left[F^{-1}(p)\right]^2\right\}^{\frac{1-a}{2}}$$

$$\Rightarrow \frac{d}{dp}\int_0^p F^{-1}(u)\,du = \frac{aK}{1-a}\frac{d}{dp}\left\{1+\frac{1}{a}\left[F^{-1}(p)\right]^2\right\}^{\frac{1-a}{2}}$$

$$\Rightarrow \cancel{F^{-1}(p)} = \frac{\cancel{a}K}{\cancel{1-a}}\,\frac{\cancel{1-a}}{\cancel{2}}\left\{1+\frac{1}{a}\left[F^{-1}(p)\right]^2\right\}^{\frac{1-a}{2}-1}$$

$$\times\frac{\cancel{2}}{\cancel{a}}\cancel{F^{-1}(p)}\frac{d}{dp}F^{-1}(p)$$

$$\Rightarrow 1 = K\left\{1+\frac{1}{a}\left[F^{-1}(p)\right]^2\right\}^{-\frac{a+1}{2}}\cdot\boxed{\frac{d}{dp}F^{-1}(p)}$$

$$\boxed{\frac{d}{dp}F^{-1}(p)} = \boxed{\frac{1}{f(F^{-1}(p))}}$$

$$\Rightarrow f(F^{-1}(p)) = K\left\{1+\frac{1}{a}\left[F^{-1}(p)\right]^2\right\}^{-\frac{a+1}{2}}$$

This is true.

Hence the assumption must be true.