LECTURE

7 DECEMBER

10:00-11:00AM

MATH4/68181

For a given $F(x, y)$ [the CDF of the data] can you find $G(x, y)$ if it exists?

Yes, you can.

(i) Find $F_X(x) = F(x, \infty)$

and $F_Y(y) = F(\infty, y)$

(ii) Find the max domain of attraction of F_X and F_Y.

(iii) If F_X and F_Y belong to the Gumbel domain set

$$a_n = \gamma\left(F_X^{-1}\left(1-\tfrac{1}{n}\right)\right), \quad b_n = F_X^{-1}\left(1-\tfrac{1}{n}\right)$$

$$c_n = \gamma\left(F_X^{-1}\left(1-\tfrac{1}{n}\right)\right), \quad d_n = F_Y^{-1}\left(1-\tfrac{1}{n}\right)$$

If F_X and F_Y belong to the Fréchet domain set

$$a_n = F_X^{-1}\left(1-\tfrac{1}{n}\right), \quad b_n = 0$$

$$c_n = F_Y^{-1}\left(1-\tfrac{1}{n}\right), \quad d_n = 0$$

If F_X and F_Y belong to the Weibull domain set

$$a_n = w(F_X) - F_X^{-1}\left(1-\tfrac{1}{n}\right), \quad b_n = w(F_X)$$

$$c_n = w(F_Y) - F_Y^{-1}\left(1-\tfrac{1}{n}\right), \quad d_n = w(F_Y).$$

(iv) Determine

$$G(x, y) = \lim_{n \to \infty} \left[F(a_n x + b_n, c_n y + d_n)\right]^n$$

Ex

$$F(x,y) = \left[1 + e^{-x} + e^{-y} + (1-a)e^{-x-y}\right]^{-1},$$

$$x > 0$$
$$y > 0$$

Find the corresponding $G(x,y)$.

(i) $F_X(x) = F(x, \infty) = \left[1 + e^{-x}\right]^{-1}$

$F_Y(y) = F(\infty, y) = \left[1 + e^{-y}\right]^{-1}$

(ii) $w(F_X) = \infty$

$w(F_Y) = \infty$

$$\lim_{t \to \infty} \frac{1 - F_X(t + x\gamma(t))}{1 - F_X(t)}$$

$$= \lim_{t \to \infty} \frac{1 - \left[1 - e^{-t - x\gamma(t)}\right]^{-1}}{1 - \left[1 - e^{-t}\right]^{-1}}$$

0

0

$$= \lim_{t \to \infty} \frac{1 - \left[1 + e^{-t - x\gamma(t)}\right]}{1 - \left[1 + e^{-t}\right]}$$

$$= \lim_{t \to \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}} = \lim_{t \to \infty} e^{-x\gamma(t)} = e^{-x}$$

if $\gamma(t) \equiv 1$.

Hence, F_X and F_Y belong to the Gumbel domain.

(iii) $$F_X(x) = \left[1 + e^{-x}\right]^{-1} = z$$

$$\Rightarrow \quad 1 + e^{-x} = \frac{1}{z}$$

$$\Rightarrow \quad e^{-x} = \frac{1-z}{z}$$

$$\Rightarrow \quad x = -\log\left(\frac{1-z}{z}\right)$$

$$\Rightarrow \quad F_X^{-1}(x) = -\log\left(\frac{1-x}{x}\right)$$

$$\Rightarrow \quad F_Y^{-1}(y) = -\log\left(\frac{1-y}{y}\right)$$

$$a_n = \gamma\left(F_X^{-1}\left(1-\tfrac{1}{n}\right)\right) = 1$$

$$b_n = F_X^{-1}\left(1-\tfrac{1}{n}\right) = -\log\left(\frac{1-\left(1-\frac{1}{n}\right)}{1-\frac{1}{n}}\right)$$

$$= \log(n-1)$$

$$c_n = 1$$

$$d_n = \log(n-1)$$

(iv) $$G(x,y) = \lim_{n\to\infty}\left[F(a_n x + b_n, c_n y + d_n)\right]^n$$

$$= \lim_{n\to\infty}\Big[1 + e^{-a_n x - b_n} + e^{-c_n y - d_n} + (1-a)\, e^{-a_n x - b_n - c_n y - d_n}\Big]^{-n}$$

$$= \lim_{n\to\infty}\Big[1 + e^{-x-\log(n-1)} + e^{-y-\log(n-1)} + (1-a)\, e^{-x-\log(n-1)-y-\log(n-1)}\Big]^{-n}$$

$$= \lim_{n \to \infty} \left[1 + \frac{e^{-x}}{n-1} + \frac{e^{-y}}{n-1} + (1-a)\frac{e^{-x-y}}{(n-1)^2} \right]^{-n}$$

$$= \lim_{n \to \infty} \left\{ 1 + \frac{1}{n} \left[\underbrace{\frac{ne^{-x}}{n-1}}_{\to 1} + \underbrace{\frac{ne^{-y}}{n-1}}_{\to 1} + \underbrace{\frac{(1-a)ne^{-x-y}}{(n-1)^2}}_{\to 0} \right] \right\}^{-n}$$

$$= \lim_{n \to \infty} \left\{ 1 + \frac{1}{n} \left[e^{-x} + e^{-y} \right] \right\}^{-n}$$

$$\boxed{\left(1 + \frac{z}{n}\right)^{-n} \longrightarrow e^{-z} \quad \text{as } n \to \infty}$$

$$= e^{-\left[e^{-x} + e^{-y}\right]}.$$

Representations for $G(x, y)$

1) If F_X and F_Y belong to the max domain of Gumbel then

$$G(x, y) = e^{-\int_0^1 \min\left[\frac{f_1(s)}{e^x}, \frac{f_2(s)}{e^y}\right] ds}$$

where f_1 and f_2 are integrable functions such that

$$\int_0^1 f_1(s) ds = 1$$

and

$$\int_0^1 f_2(s) ds = 1.$$

2) If F_X and F_Y belong to the max domain of Gumbel then

$$G(x,y) = e^{-(e^{-x}+e^{-y})\,k(y-x)}$$

where $k(\cdot)$ satisfies

$$\lim_{t \to \infty} k(t) = 1$$

$$\lim_{t \to -\infty} k(t) = 1$$

$$\frac{d}{dt}\left\{(1+e^{-t})\,k(t)\right\} \leq 0 \quad \forall t$$

$$\frac{d}{dt}\left\{(1+e^{t})\,k(t)\right\} \geq 0 \quad \forall t$$

$$(1+e^{-t})\frac{d^2}{dt^2}k(t) + (1-e^{-t})\frac{d}{dt}k(t) \geq 0 \quad \forall t$$

3) If F_X and F_Y belong to the max domain of Frechet domain

$$G(x, y) = e^{-\left(\frac{1}{x} + \frac{1}{y}\right) A\left(\frac{x}{x+y}\right)}$$

where $A(\cdot)$ satisfies

$A(0) = 1$

$A(1) = 1$

$\max(w, 1-w) \leq A(w) \leq 1 \quad \forall\, 0 \leq w \leq 1$

$A(\cdot)$ is a convex function

4) Suppose $F_X(x) = 1 - e^{-x}$ and

$$F_Y(y) = 1 - e^{-y}.$$

Then

$$\bar{G}(x, y) = e^{-(x+y)\, A\left(\frac{y}{x+y}\right)}$$

where $A(\cdot)$ satisfies

$A(0) = 1$

$A(1) = 1$

$\max(w, 1-w) \leq A(w) \leq 1 \ \forall\ 0 \leq w \leq 1$

$A(\cdot)$ is convex.

$$\boxed{G(x, y) = 1 - e^{-x} - e^{-y} + \bar{G}(x, y)}$$

Relation between G and $\bar{G}$.

What is the representation for $G(x, y)$ if F_X and F_Y belong to the Weibull margin?

Open problem.

MATH3/4/68181: Extreme values and financial risk
Semester 1
Problem sheet 12

1) Consider a bivariate distribution specified by the joint survival function

$$\overline{G}(x, y) = \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right]$$

for $x > 0$ and $y > 0$.

(a) show that the distribution is a bivariate extreme value distribution;

(b) derive the joint cdf;

(c) derive the conditional cdf if Y given $X = x$;

(d) derive the conditional cdf if X given $Y = y$;

(e) derive the joint pdf;

(f) derive the conditional pdf of Y given $X = x$;

(g) derive the conditional pdf of X given $Y = y$.

2) Consider a bivariate distribution specified by the joint survival function

$$\overline{F}(x, y) = \exp\left[\frac{\alpha xy}{x+y} - x - y\right]$$

for $x > 0$ and $y > 0$.

(a) show that the distribution is a bivariate extreme value distribution;

(b) derive the joint cdf;

(c) derive the conditional cdf if Y given $X = x$;

(d) derive the conditional cdf if X given $Y = y$;

(e) derive the joint pdf;

(f) derive the conditional pdf of Y given $X = x$;

(g) derive the conditional pdf of X given $Y = y$.

3) Consider a bivariate distribution specified by the joint survival function

$$\overline{F}(x, y) = \exp\left[-\left(x^a + y^a\right)^{1/a}\right]$$

for $x > 0$ and $y > 0$.

Q1

$$\bar{G}(x,y) = e^{-\frac{\theta y^2}{x+y} + \theta y - x - y}$$

$$\bar{G}(0,y) = e^{-\theta y + \theta y - y} = e^{-y}$$

$$\bar{G}(x,0) = e^{-x}$$

Marginal CDFs
~~Marginals~~ are $1-e^{-x}$ and $1-e^{-y}$.

This refers to representation 4.

$$\bar{G}(x,y) = e^{-(x+y)\left[\frac{\theta y^2}{(x+y)^2} - \frac{\theta y}{x+y} + 1\right]}$$

$$= e^{-(x+y)A\left(\frac{y}{x+y}\right)}$$

if $A(w) = \theta w^2 - \theta w + 1$

- $A(0) = \theta \cdot 0^2 - \theta \cdot 0 + 1 = 1$ ✓
- $A(1) = \theta \cdot 1^2 - \theta \cdot 1 + 1 = 1$ ✓
- $A(w) \leq 1$

$\Leftrightarrow \theta w^2 - \theta w + 1 \leq 1$

$\Leftrightarrow \theta w(w-1) \leq 0$

$\Leftrightarrow w - 1 \leq 0$

$\Leftrightarrow w \leq 1$ ✓

$A(w) \geq w$

$\Leftrightarrow \theta w^2 - \theta w + 1 \geq w$

$\Leftrightarrow \theta w(w-1) + 1 - w \geq 0$

$\Leftrightarrow (1-\theta w)(1-w) \geq 0$ ✓ if $\underline{\theta < 1}$

$A(w) \geq 1 - w$

$\Leftrightarrow \theta w^2 - \theta w + 1 \geq 1 - w$

$\Leftrightarrow \theta w^2 - \theta w + w \geq 0$

$\Leftrightarrow \theta w^2 + (1-\theta) w \geq 0$ ✓ if $\theta < 1$

- $A(w) = \theta w^2 - \theta w + 1$

  $A'(w) = 2\theta w - \theta$

  $A''(w) = 2\theta > 0$

  $\Rightarrow A(\cdot)$ is convex (because $0 < \theta < 1$)

Hence, $\bar{G}(x, y)$ is a bivariate extreme value distribution.