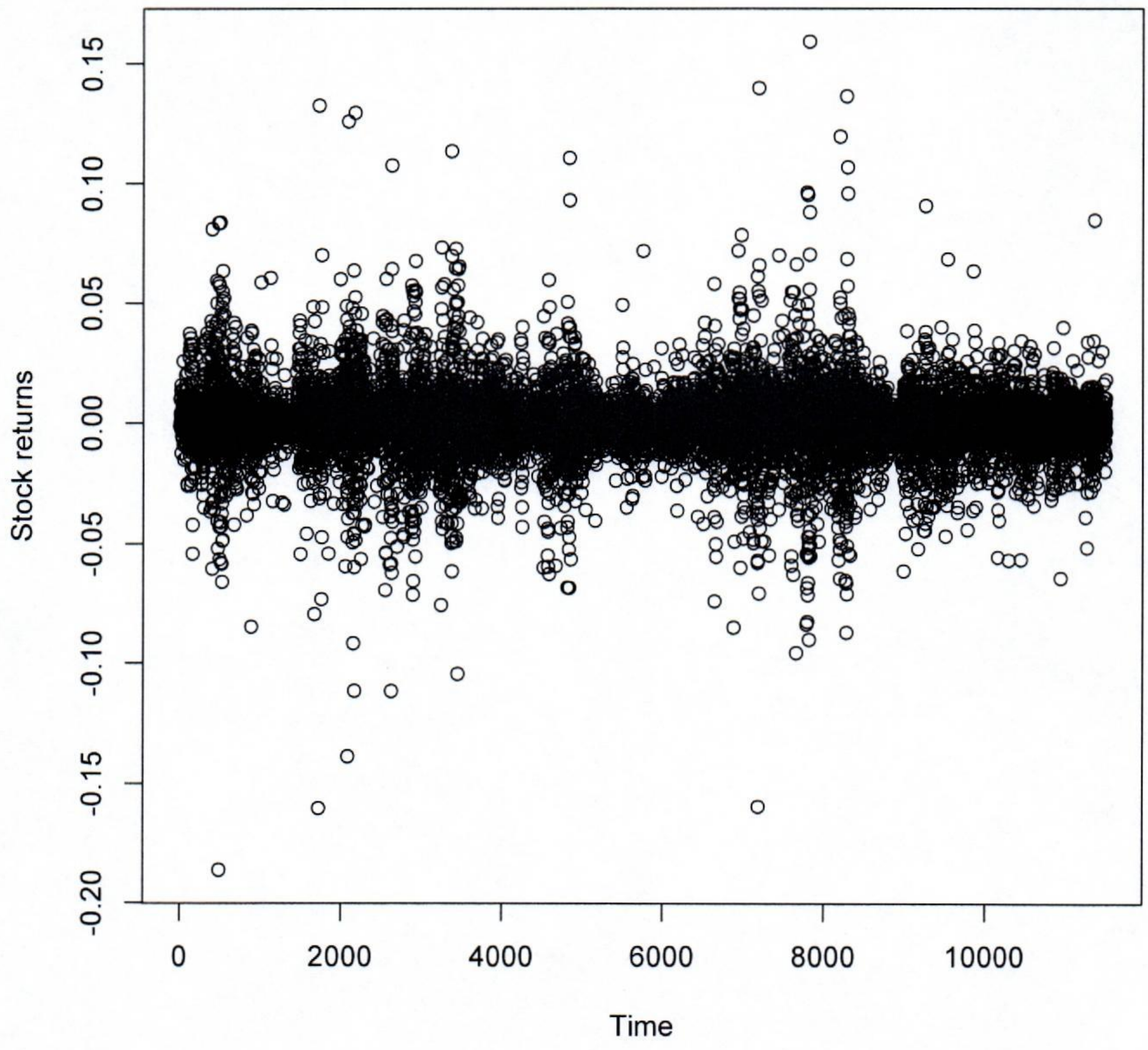
LECTURE

4 DECEMBER

9:00-10:00AM

MATH3/4/68181



0.15
0.10
0.05
0.00
-0.05
-0.10
-0.15
-0.20
Stock returns
0
2000
4000
6000
8000
10000
Time

VEQ

< 50% complete

Please ask your friends to complete asap.

Many thanks.

GARCH type models

$$X_t = \sigma_t Z_t$$

where

X_t = financial data at time t

σ_t = volatility at time t

Z_t = innovation at time t

Usually, $Z_t \sim N(0,1)$ IID.

$$E[X_t] = E[\sigma_t Z_t]$$

Total Law of Expectation

$$= E[E[\sigma_t Z_t \mid \sigma_t]]$$

$$= E[\sigma_t E[Z_t]]$$

$$= E[\sigma_t \cdot 0]$$

$$= 0$$

$$E[X_t^2] = E[\sigma_t^2 Z_t^2]$$

Total Law of Expectation $\rightarrow$ $= E\left[E[\sigma_t^2 Z_t^2 \mid \sigma_t]\right]$

$$= E\left[\sigma_t^2 E[Z_t^2]\right]$$

$$= E\left[\sigma_t^2 \{Var(Z_t) + [E(Z_t)]^2\}\right]$$

$$= E[\sigma_t^2].$$

$$\Rightarrow \quad Var[X_t] = E[X_t^2] - (E[X_t])^2$$

$$= E[\sigma_t^2] - 0$$

$$= E[\sigma_t^2].$$

In summary,

$$E[X_t] = 0,$$

$$Var[X_t] = E[\sigma_t^2].$$

a) ARCH(q) model

$$X_t = \sigma_t Z_t,$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2$$

[volatility at time t] is a function of the financial data at the previous q lags]

$$E[X_t] = 0$$

$$Var[X_t] = E[\sigma_t^2]$$

If the data are stationary

$$E[\sigma_t^2] = E[\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2] \; \forall t$$

$$\Rightarrow E[\sigma_t^2] = \alpha_0 + \alpha_1 E[X_{t-1}^2] + \cdots + \alpha_q E[X_{t-q}^2]$$

$$\Rightarrow E[\sigma_t^2] = \alpha_0 + \alpha_1 E[\sigma_{t-1}^2] + \cdots + \alpha_q E[\sigma_{t-q}^2]$$

$$\Rightarrow E[\sigma_t^2] = \alpha_0 + \alpha_1 E[\sigma_t^2] + \cdots + \alpha_q E[\sigma_t^2]$$

using stationarity

$$\Rightarrow E[\sigma_t^2] = \frac{\alpha_0}{1 - \alpha_1 - \cdots - \alpha_q}.$$

b) GARCH(p, q) model

$$X_t = \sigma_t Z_t$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2$$
$$+ \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2$$

[volatility at time t is a function of the data values at the previous q lags as well as the volatility values at the previous p values]

$$E[X_t] = 0$$

$$Var[X_t] = E[\sigma_t^2].$$

If the data are stationary

$$E[\sigma_t^2] = \alpha_0 + \alpha_1 E[X_{t-1}^2] + \cdots + \alpha_q E[X_{t-q}^2]$$
$$+ \beta_1 E[\sigma_{t-1}^2] + \cdots + \beta_p E[\sigma_{t-p}^2]$$

$$= \alpha_0 + \alpha_1 E[\sigma_{t-1}^2] + \cdots + \alpha_q E[\sigma_{t-q}^2]$$
$$+ \beta_1 E[\sigma_{t-1}^2] + \cdots + \beta_p E[\sigma_{t-p}^2]$$

$$\underset{\text{stationarity}}{\overset{\text{using}}{=}} \alpha_0 + (\alpha_1 + \cdots + \alpha_q + \beta_1 + \cdots + \beta_p) E[\sigma_t^2]$$

$$\Rightarrow E[\sigma_t^2] = \frac{\alpha_0}{1 - (\alpha_1 + \cdots + \alpha_q + \beta_1 + \cdots + \beta_p)}$$

c) NGARCH model

$$X_t = \sigma_t Z_t$$

$$\sigma_t^2 = \omega + \alpha(X_{t-1} - \theta\sigma_{t-1})^2 + \beta\sigma_{t-1}^2$$

[volatility at time t is a function of the previous data and volatility values]

$$E[X_t] = 0$$

$$Var[X_t] = \sigma_t^2$$

If the data are stationary

$$E[\sigma_t^2] = \omega + \alpha E[(X_{t-1} - \theta\sigma_{t-1})^2] + \beta E[\sigma_{t-1}^2]$$

$$= \omega + \alpha E[X_{t-1}^2] - 2\alpha\theta E[X_{t-1}\sigma_{t-1}] + \alpha\theta^2 E[\sigma_{t-1}^2] + \beta E[\sigma_{t-1}^2]$$

$$= \omega + \alpha E[X_{t-1}^2] - 2\alpha\theta E[E[X_{t-1}\sigma_{t-1} | \sigma_{t-1}]] + (\alpha\theta^2 + \beta) E[\sigma_{t-1}^2]$$

$$= \omega + \alpha E[X_{t-1}^2] - 2\alpha\theta E[\sigma_{t-1} E[X_{t-1}]] + (\alpha\theta^2 + \beta) E[\sigma_{t-1}^2]$$

$$= \omega + \alpha E[\sigma_{t-1}^2] - 0 + (\alpha\theta^2 + \beta) E[\sigma_{t-1}^2]$$

using

$$= \omega + (\alpha + \alpha\theta^2 + \beta) E[\sigma_t^2]$$

stationarity

$$\Rightarrow \quad E[\sigma_t^2] = \frac{\omega}{1 - (\alpha + \alpha\theta^2 + \beta)}.$$

d) QGARCH model

$$X_t = \sigma_t Z_t$$

$$\sigma_t^2 = K + \alpha X_{t-1}^2 + \beta \sigma_{t-1}^2 + \phi X_{t-1}$$

[This model takes account of the sign of X_{t-1}]

$$E[X_t] = 0$$

$$Var[X_t] = E[\sigma_t^2]$$

If the data are stationary,

$$E[\sigma_t^2] = K + \alpha E[X_{t-1}^2] + \beta E[\sigma_{t-1}^2] + \phi E[X_{t-1}]$$

$$\Rightarrow E[\sigma_t^2] = K + \alpha E[\sigma_{t-1}^2] + \beta E[\sigma_{t-1}^2] + 0$$

using stationarity $\Rightarrow$ $E[\sigma_t^2] = K + (\alpha + \beta) E[\sigma_t^2]$

$$\Rightarrow E[\sigma_t^2] = \frac{K}{1-(\alpha+\beta)}.$$

e) GJR-QGARCH model

(initials of the 3 authors)

$$X_t = \sigma_t Z_t$$

$$\sigma_t^2 = \kappa + \delta \sigma_{t-1}^2 + \alpha X_{t-1}^2 + \phi X_{t-1}^2 I_{t-1}$$

where

$$I_{t-1} = \begin{cases} 0 & \text{if } X_{t-1} \geq 0 \\ 1 & \text{if } X_{t-1} < 0 \end{cases}$$

[volatility has an additional term if $X_{t-1} < 0$]