LECTURE

3 DECEMBER

9:00-10:00AM

MATH3/4/68181

Announcements

1. syllabus for level 3 will be completed tomorrow 4 Dec

2. syllabus for levels 4 & 6 will be completed on Friday 7 Dec

3. Next week will be revision class

4. QUIZ 5 due tomorrow

$$ES_p(X) = \frac{1}{p}\int_0^p \underbrace{F^{-1}(t)}\,dt$$

Find a substitution that will help you to get the answer

The proof should not be longer than 1 A4 page.

5. UEQ

Income modeling

X = True income

Z = Reported income

eg 1

X = £ 20,000 + 1 p

Z = £ 20,000

Under reporting

$Z = XY$ where $0 < Y < 1$

eg 2

X = £ 19,000 + 10 p

Z = £ 20,000

Over reporting

$Z = X/Y$ where $0 < Y < 1$.

Z is observable

X is not observable.

The most popular model for income is the Pareto distribution. Its CDF is (Italian economist)

given by

$$F_X(x) = 1 - \left(\frac{k}{x}\right)^a, \quad x > k$$

The corresponding PDF is

$$f_X(x) = \frac{a k^a}{x^{a+1}}, \quad x > k.$$

1) If X is Pareto distributed what is the distribution of Z ?

2) If Z is Pareto distributed what is the distribution of X ?

Assume Y is a power function RV with CDF

$$F_Y(y) = y^c, \quad 0 < y < 1$$

The corresponding PDF is

$$f_Y(y) = c\,y^{c-1}, \quad 0 < y < 1.$$

Theorem 1 (under reported income) Suppose $Z = XY$. Then X is Pareto distributed if and only if Z is also Pareto distributed.

Proof: Assume X is Pareto distributed with CDF

$$F_X(x) = 1 - \left(\frac{K}{x}\right)^a, \quad x > K.$$

The CDF of Z is

$$\begin{aligned}
F_Z(z) &= P(XY \leq z) \\
&= P\left(X \leq \frac{z}{Y}\right) \\
\text{Total Prob Rule} \quad &= \int_0^1 F_X\left(\frac{z}{y}\right) f_Y(y)\,dy \\
&= \int_0^1 \left[1 - \left(\frac{Ky}{z}\right)^a\right] c\,y^{c-1}\,dy \\
&= \int_0^1 c y^{c-1} dy - \frac{cK^a}{z^a} \int_0^1 y^{a+c-1}\,dy \\
&= 1 - \frac{cK^a}{(a+c)z^a} = 1 - \frac{\left[\frac{c^{1/a}}{(a+c)^{1/a}} K\right]^a}{z^a} = 1 - \left(\frac{K^*}{z}\right)^a
\end{aligned}$$

where $$K^{*} = \frac{c^{1/a}}{(a+c)^{1/a}} K .$$

Hence, Z is also Pareto distributed.

Assume Z is Pareto distributed with CDF

$$F_Z(z) = 1 - \left(\frac{K}{z}\right)^a , \quad z > K .$$

The CDF of X is

$$\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P\left(\frac{Z}{Y} \leq x\right) \\
&= P(Z \leq Yx)
\end{aligned}$$

Total Prob Rule

$$\begin{aligned}
&= \int_0^1 F_Z(yx)\, f_Y(y)\, dy \\
&= \int_0^1 \left[1 - \left(\frac{K}{yx}\right)^a\right] c y^{c-1}\, dy \\
&= \int_0^1 c y^{c-1}\, dy - \frac{cK^a}{x^a} \int_0^1 y^{c-a-1}\, dy \\
&= 1 - \frac{cK^a}{(c-a)x^a} \\
&= 1 - \left(\frac{K^{**}}{x}\right)^a
\end{aligned}$$

where $K^{*} = \frac{c^{\frac{1}{a}} K}{(c-a)^{\frac{1}{a}}}$.

Hence, X is also Pareto distributed.
Hence, Z is Pareto distributed if and only if X is also Pareto distributed.

Theorem 2 (Over reported income).
Suppose $Z = X/Y$. Then Z is Pareto distributed if and only if X is also Pareto distributed.

Proof: Assume Z is Pareto distributed with CDF

$$F_Z(z) = 1 - \left(\frac{K}{z}\right)^a, \quad z > K.$$

The CDF of X is

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(ZY \leq x) \\ &= P\left(Z \leq \frac{x}{Y}\right) \\ \text{Total Prob Rule} \searrow &= \int_0^1 F_Z\left(\frac{x}{y}\right) f_Y(y)\, dy \\ &= \int_0^1 \left[1 - \left(\frac{Ky}{x}\right)^a\right] c y^{c-1}\, dy \\ &= \int_0^1 c y^{c-1}\, dy - \frac{cK^a}{x^a} \int_0^1 y^{a+c-1}\, dy \\ &= 1 - \frac{cK^a}{(a+c)x^a} \\ &= 1 - \left(\frac{K^{***}}{x}\right)^a, \quad K^{***} = \frac{c^{\frac{1}{a}} K}{(a+c)^{\frac{1}{a}}} \end{aligned}$$

Hence, X is also Pareto distributed.

Assume X is Pareto distributed with CDF

$$F_X(x) = 1 - \left(\frac{K}{x}\right)^a, \quad x > K.$$

The CDF of Z is

$$\begin{aligned} F_Z(z) &= P(Z \le z) \\ &= P\left(\frac{X}{Y} \le z\right) \\ &= P(X \le Yz) \\ &= \int_0^1 F_X(yz)\, f_Y(y)\, dy \\ &= \int_0^1 \left[1 - \left(\frac{K}{yz}\right)^a\right] c y^{c-1}\, dy \\ &= 1 - \left(\frac{K^{****}}{z}\right)^a \end{aligned}$$

where $K^{****} = \dfrac{c^{\frac{1}{a}} K}{(c-a)^{\frac{1}{a}}}$.

Hence, Z is also Pareto distributed.