EXAMPLE CLASS

4 DECEMBER

16:00-17:00PM

MATH3/4/68181

Q1

$$F(x) = e^{-(1+\xi x)^{-\frac{1}{\xi}}}$$ GEV cdf

(a) $$f(x) = \frac{d}{dx} F(x)$$

$$= (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}}$$

(b) $\boxed{\xi = 0}$: From lecture notes,

$$F(x) = e^{-e^{-x}}$$

$$f(x) = e^{-x} e^{-e^{-x}}$$

$$E(X^n) = \int_{-\infty}^{\infty} x^n e^{-x} e^{-e^{-x}} dx$$

$$\boxed{\begin{aligned} & y = e^{-x} \\ \Rightarrow \quad & x = -\log y \\ \Rightarrow \quad & \frac{dx}{dy} = -\frac{1}{y} \end{aligned}}$$

$$= \int_{\infty}^{0} (-\log y)^n \cancel{y} e^{-y} \left(-\frac{1}{\cancel{y}}\right) dy$$

$$= \int_{0}^{\infty} (-\log y)^n e^{-y} dy$$

$$\boxed{\left.\frac{d^n y^a}{da^n}\right|_{a=0} = (\log y)^n}$$

$$= \left[\int_0^\infty (-1)^n \frac{d^n y^a}{da^n} e^{-y}\, dy\right]\Bigg|_{a=0}$$

$$= (-1)^n \frac{d^n}{da^n}\left[\int_0^\infty y^a e^{-y}\, dy\right]\Bigg|_{a=0}$$

$$= (-1)^n \frac{d^n}{da^n} \Gamma(a+1)\Bigg|_{a=0}$$

$\boxed{\xi > 0}$ $\quad 1+\xi x > 0 \Rightarrow x > -\frac{1}{\xi}$

$$\Rightarrow x \in \left(-\frac{1}{\xi}, \infty\right)$$

$$E(X^n) = \int_{-\frac{1}{\xi}}^{\infty} x^n (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}}\, dx$$

Set $y = (1+\xi x)^{-\frac{1}{\xi}}$

$$\Rightarrow \frac{y^{-\xi}-1}{\xi} = x$$

$$\Rightarrow \frac{dx}{dy} = -y^{-\xi-1}$$

$$= \int_\infty^0 \left(\frac{y^{-\xi}-1}{\xi}\right)^n \cancel{y^{1+\xi}} e^{-y} (-1) \cancel{y^{-\xi-1}}\, dy$$

$$= \xi^{-n}\int_0^\infty (y^{-\xi}-1)^n e^{-y}\, dy$$

$$= \xi^{-n}\int_0^\infty \sum_{k=0}^{n}\binom{n}{k} y^{-k\xi}(-1)^{n-k} e^{-y}\, dy$$

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k} a^k b^{n-k}$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^\infty y^{-k\xi} e^{-y} \, dy$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Gamma(1-k\xi).$$

$\boxed{\xi < 0}$ $\quad 1 + \xi x > 0 \Rightarrow x < -\frac{1}{\xi}$

$$\Rightarrow x \in \left(-\infty, -\frac{1}{\xi}\right)$$

$$E(X^n) = \int_{-\infty}^{-\frac{1}{\xi}} x^n (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} \, dx$$

$$= \int_\infty^0 \left(\frac{y^{-\xi}-1}{\xi}\right)^n \cancel{y^{1+\xi}} e^{-y} (-1) \cancel{y^{-\xi-1}} \, dy$$

$$= \xi^{-n} \int_0^\infty (y^{-\xi}-1)^n e^{-y} \, dy$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^\infty y^{-k\xi} e^{-y} \, dy$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Gamma(1-k\xi).$$

Q2

$$F(x) = 1 - (1+\xi x)^{-\frac{1}{\xi}}$$

(a) $f(x) = \dfrac{dF(x)}{dx} = (1+\xi x)^{-\frac{1}{\xi}-1}$

(b) $\boxed{\xi = 0}$: From lecture notes,

$$F(x) = 1 - e^{-x}$$

$$f(x) = e^{-x}$$

$$E(X^n) = \int_0^\infty x^n e^{-x}\,dx$$

$$= \Gamma(n+1) = n!$$

$\boxed{\xi > 0}$: $E(X^n) = \displaystyle\int_0^\infty x^n (1+\xi x)^{-\frac{1}{\xi}-1} dx$

Set $y = 1 + \xi x$

$x = \dfrac{y-1}{\xi}$

$\dfrac{dx}{dy} = \dfrac{1}{\xi}$

$$= \int_1^\infty \left(\frac{y-1}{\xi}\right)^n y^{-\frac{1}{\xi}-1} \frac{1}{\xi}\,dy$$

$$= \frac{1}{\xi^{n+1}} \int_1^\infty (y-1)^n y^{-\frac{1}{\xi}-1}\,dy$$

$$= \frac{1}{\xi^{n+1}} \int_1^{\infty} \sum_{k=0}^{n} \binom{n}{k} y^k (-1)^{n-k} y^{-\frac{1}{\xi}-1} dy$$

$$= \frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_1^{\infty} y^{k-\frac{1}{\xi}-1} dy$$

$$= \frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{\frac{1}{\xi}-k} .$$

$\boxed{\xi < 0}$: $E(X^n) = \int_0^{-1/\xi} x^n (1+\xi x)^{-\frac{1}{\xi}-1} dx$

$$= \int_1^0 \left(\frac{y-1}{\xi}\right)^n y^{-\frac{1}{\xi}-1} \frac{1}{\xi} dy$$

$$= -\frac{1}{\xi^{n+1}} \int_0^1 (y-1)^n y^{-\frac{1}{\xi}-1} dy$$

$$= -\frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^1 y^{k-\frac{1}{\xi}-1} dy$$

$$= -\frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{k-\frac{1}{\xi}} .$$