EXAMPLE CLASS

3 DECEMBER

12:00-13:00PM

MATH3/4/68181

Q1

$$F(x) = e^{-(1+\xi x)^{-1/\xi}}$$

(a) $$f(x) = \frac{dF(x)}{dx}$$

$$= (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}}$$

(b) $\boxed{\xi = 0}$ $$F(x) = e^{-e^{-x}}$$

$$f(x) = e^{-x} e^{-e^{-x}}$$

$$E(X^n) = \int_{-\infty}^{\infty} x^n e^{-x} e^{-e^{-x}} dx$$

$$\boxed{\begin{aligned} y &= e^{-x} \\ x &= -\log y \\ \frac{dx}{dy} &= -\frac{1}{y} \end{aligned}}$$

$$= \int_{\infty}^{0} (-\log y)^n \cancel{y} e^{-y} \left(-\frac{1}{\cancel{y}}\right) dy$$

$$= \int_{0}^{\infty} (-\log y)^n e^{-y} dy$$

$$\boxed{\left.\frac{d^n y^a}{da^n}\right|_{a=0} = (\log y)^n}$$

$$= (-1)^n \left[\int_0^{\infty} \frac{d^n y^a}{da^n} e^{-y} dy\right]\Bigg|_{a=0}$$

$$= (-1)^n \frac{d^n}{da^n} \Gamma(a+1) \Bigg|_{a=0}$$

$\boxed{\xi > 0}$: $1 + \xi x > 0$

$$x > -\frac{1}{\xi}$$

$$x \in \left(-\frac{1}{\xi}, \infty\right)$$

$$E(X^n) = \int_{-\frac{1}{\xi}}^{\infty} x^n (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

Set $y = (1+\xi x)^{-\frac{1}{\xi}}$

$$\Rightarrow \quad x = \frac{y^{-\xi}-1}{\xi}$$

$$\Rightarrow \quad \frac{dx}{dy} = -y^{-\xi-1}$$

$$= \int_{\infty}^{0} \left(\frac{y^{-\xi}-1}{\xi}\right)^n \cancel{y^{1+\xi}} e^{-y} (-1) \cancel{y^{-\xi-1}} dy$$

$$= \frac{1}{\xi^n} \int_0^{\infty} (y^{-\xi}-1)^n e^{-y} dy$$

Bin Thm

$$= \frac{1}{\xi^n} \int_0^{\infty} \sum_{k=0}^{n} \binom{n}{k} (y^{-\xi})^k (-1)^{n-k} e^{-y} dy$$

$$= \frac{1}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^{\infty} y^{-\xi k} e^{-y} dy$$

$$= \frac{1}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Gamma(1-\xi k)$$

$\boxed{\xi < 0}$

$$1 + \xi x > 0$$
$$x < -\frac{1}{\xi}$$
$$x \in \left(-\infty, -\frac{1}{\xi}\right)$$

$$E(X^n) = \int_{-\infty}^{-\frac{1}{\xi}} x^n (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

$$\boxed{\begin{aligned} &\text{Set} \quad y = (1+\xi x)^{-\frac{1}{\xi}} \\ &\Rightarrow x = \frac{y^{-\xi}-1}{\xi} \\ &\Rightarrow \frac{dx}{dy} = -\, y^{-\xi-1} \end{aligned}}$$

$$= \int_0^\infty \left(\frac{y^{-\xi}-1}{\xi}\right)^n \cancel{y^{1+\xi}} e^{-y} (-1) \cancel{y^{-\xi-1}} dy$$

$$= -\frac{1}{\xi^n} \int_0^\infty (y^{-\xi}-1)^n e^{-y} dy$$

$$= -\frac{1}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^\infty y^{-\xi k} e^{-y} dy$$

$$= -\frac{1}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Gamma(1-\xi k).$$

Q2

$$F(x) = 1-(1+\xi x)^{-\frac{1}{\xi}}$$

(a) $$f(x) = \frac{dF(x)}{dx} = (1+\xi x)^{-\frac{1}{\xi}-1}$$

(b) $\boxed{\xi = 0}$ $$F(x) = 1-\exp(-x)$$
$$f(x) = e^{-x}$$
$$E(X^n) = \int_0^\infty x^n e^{-x}\,dx$$
$$= \Gamma(n+1) = n!$$

$\boxed{\xi > 0}$ $$E(X^n) = \int_0^\infty x^n (1+\xi x)^{-\frac{1}{\xi}-1} dx$$

$$\boxed{\begin{array}{l} \text{Set } y = 1+\xi x \\ \Rightarrow x = \frac{y-1}{\xi} \\ \Rightarrow \frac{dx}{dy} = \frac{1}{\xi} \end{array}}$$

$$= \int_1^\infty \left(\frac{y-1}{\xi}\right)^n y^{-\frac{1}{\xi}-1} \frac{dy}{\xi}$$
$$= \frac{1}{\xi^{n+1}} \int_1^\infty (y-1)^n y^{-\frac{1}{\xi}-1} dy$$
$$= \frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_1^\infty y^{k-\frac{1}{\xi}-1} dy$$
$$= \frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{\frac{1}{\xi}-k}$$

$\boxed{\xi < 0}$

$$E(X^n) = \int_0^{-1/\xi} x^n (1+\xi x)^{-\frac{1}{\xi}-1} \, dx$$

$$= \int_1^0 \left(\frac{y-1}{\xi}\right)^n y^{-\frac{1}{\xi}-1} \frac{dy}{\xi}$$

$$= -\frac{1}{\xi^{n+1}} \int_0^1 (y-1)^n y^{-\frac{1}{\xi}-1} \, dy$$

$$= -\frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^1 y^{k-\frac{1}{\xi}-1} \, dy$$

$$= -\frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{k-\frac{1}{\xi}} .$$