LECTURE

30 NOVEMBER

10:00-11:00AM

MATH4/68181

Bivariate Extreme Values

Often you are interested in extreme values of more than 1 variable.

Some examples are as follows.

Suppose $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ is a random sample from a joint CDF $F(x, y)$.

How to define an extreme value? There is no one way to define an extreme value. The most definition is

$$(M_{n,1}, M_{n,2})$$

$$= \left(\max(X_1, X_2, \ldots, X_n), \max(Y_1, Y_2, \ldots, Y_n) \right)$$

This definition may not be realistic because $(M_{n,1}, M_{n,2})$ may not be an actual observation.

Forest fires

Caused by extreme values of temperature and wind speed

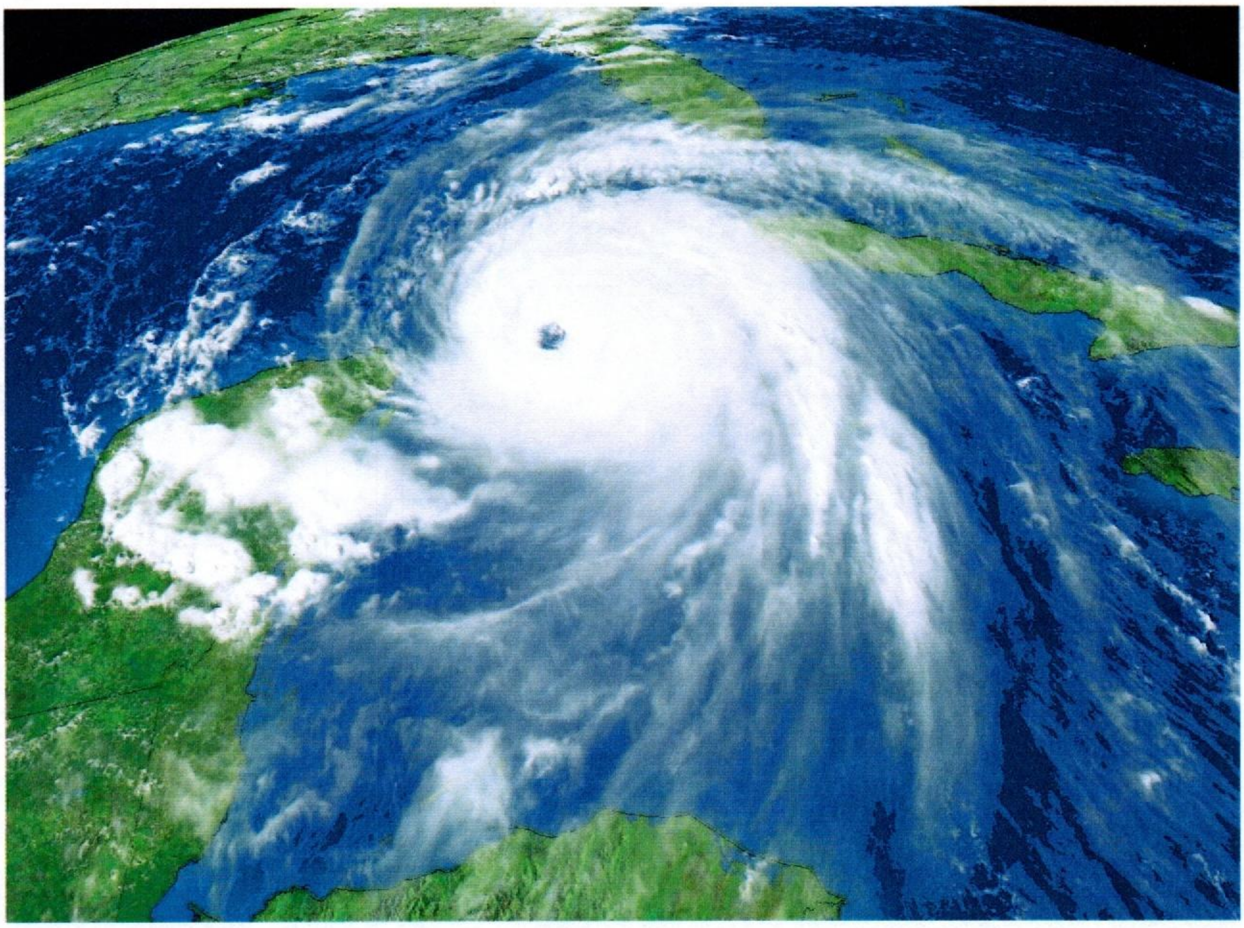
Tornado

Caused by extreme values of humidity and wind speed

Droughts

Caused by extreme values of rainfall and temperature

Hurricanes

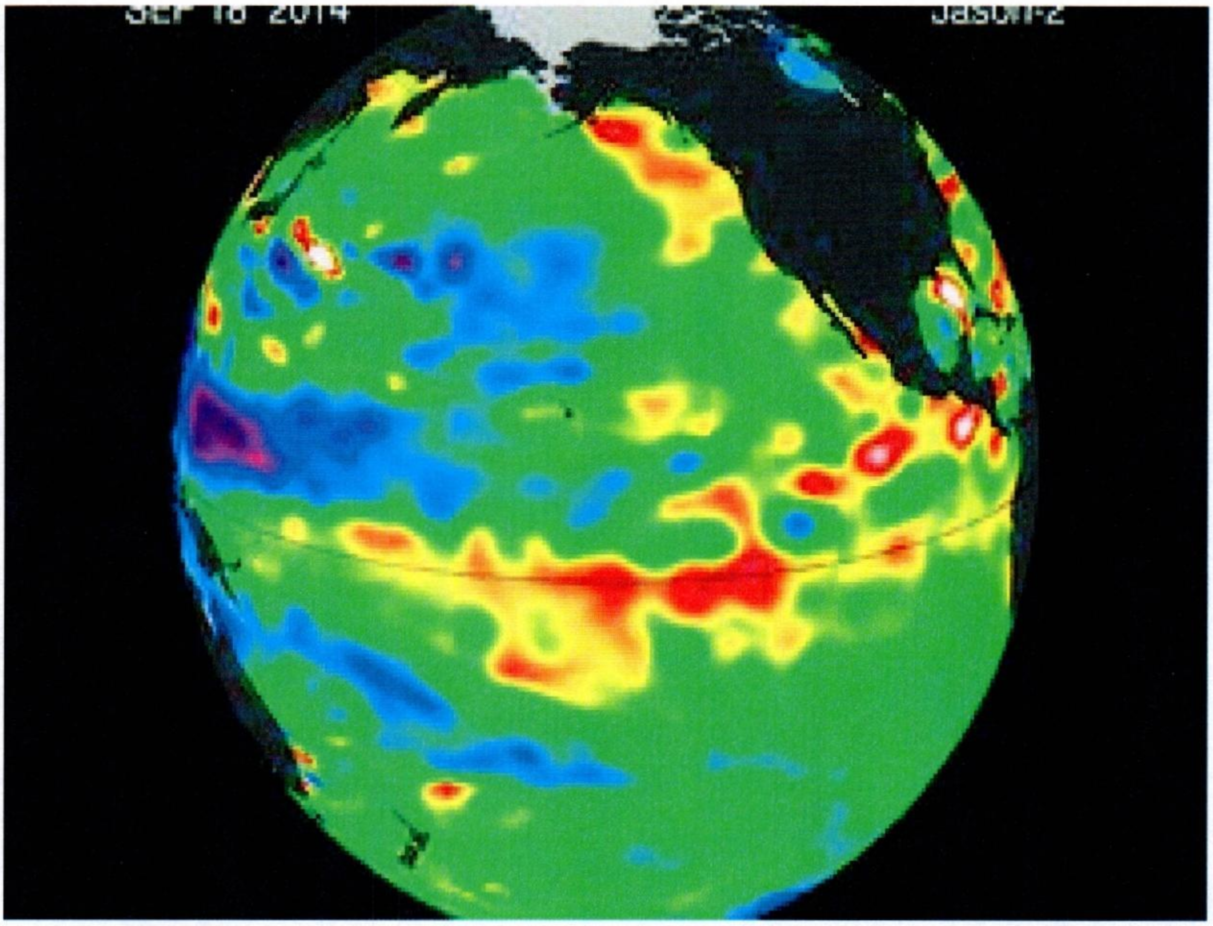


Caused by extreme values of sea temperature, rainfall and wind speed

Floods

Caused by extreme values of rainfall and wind speed

El-Nino



Caused by extreme values of sea temperature and air pressure

Ex

Data: $(1,3)$, $(2,0)$, $(-1,5)$, $(1,1)$, $(3,1)$

$(M_{n,1}, M_{n,2}) = (3,5)$

But $(3,5)$ is not an observation.

To model the data, we need to derive the distribution of

$$\left(\frac{M_{n,1} - b_n}{a_n}, \frac{M_{n,2} - d_n}{c_n} \right).$$

The joint CDF of this is

$$P\left(\frac{M_{n,1} - b_n}{a_n} \leq x, \frac{M_{n,2} - d_n}{c_n} \leq y \right)$$

$$= P\left(M_{n,1} \leq a_n x + b_n, M_{n,2} \leq c_n y + d_n \right)$$

$$= P\left(\max(X_1, \dots, X_n) \leq a_n x + b_n, \max(Y_1, \dots, Y_n) \leq c_n y + d_n \right)$$

$$= P\left(X_1 \leq a_n x + b_n, \dots, X_n \leq a_n x + b_n, Y_1 \leq c_n y + d_n, \dots, Y_n \leq c_n y + d_n \right)$$

$$= P\left(X_1 \leq a_n x + b_n, Y_1 \leq c_n y + d_n, \dots, X_n \leq a_n x + b_n, Y_n \leq c_n y + d_n \right)$$

$$\overset{\text{indep}}{=} P(X_1 \leq \cancel{a_n x + b_n)}\ a_n x + b_n, Y_1 \leq c_n y + d_n) \cdots P(X_n \leq a_n x + b_n, Y_n \leq c_n y + d_n)$$

$$= F(a_n x + b_n, c_n y + d_n)$$

$$\cdots F(a_n x + b_n, c_n y + d_n)$$

$$= [F(a_n x + b_n, c_n y + d_n)]^n.$$

We have shown that

$$P\left(\frac{M_{n,1} - b_n}{a_n} \leq x, \frac{M_{n,2} - d_n}{c_n} \leq y\right)$$

$$= [F(a_n x + b_n, c_n y + d_n)]^n.$$

We are interested in what happens as $n \longrightarrow \infty$. If there exist $a_n > 0$, $b_n \in \mathbb{R}$, $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$[F(a_n x + b_n, c_n y + d_n)]^n$$

$$\longrightarrow G(x, y)$$

as $n \longrightarrow \infty$ for a non-degenerate $G(x,y)$ then possible forms for $G(x,y)$ are <u>uncountably</u> <u>infinite</u>.

For a given $F(x, y)$ [the CDF of the data] can you find $G(x, y)$ if it exists?

Yes, you can.

(i) Find $F_X(x) = F(x, \infty)$

and $F_Y(y) = F(\infty, y)$

(ii) Find the max domain of attraction of F_X and F_Y.

(iii) If F_X and F_Y belong to the Gumbel domain set

$$a_n = \gamma\left(F_X^{-1}\left(1-\tfrac{1}{n}\right)\right), \quad b_n = F_X^{-1}\left(1-\tfrac{1}{n}\right)$$

$$c_n = \gamma\left(F_X^{-1}\left(1-\tfrac{1}{n}\right)\right), \quad d_n = F_Y^{-1}\left(1-\tfrac{1}{n}\right)$$

If F_X and F_Y belong to the Frechet domain set

$$a_n = F_X^{-1}\left(1-\tfrac{1}{n}\right), \quad b_n = 0$$

$$c_n = F_Y^{-1}\left(1-\tfrac{1}{n}\right), \quad d_n = 0$$

If F_X and F_Y belong to the Weibull domain set

$$a_n = w(F_X) - F_X^{-1}\left(1-\tfrac{1}{n}\right), \quad b_n = w(F_X)$$

$$c_n = w(F_Y) - F_Y^{-1}\left(1-\tfrac{1}{n}\right), \quad d_n = w(F_Y).$$

(iv) Determine

$$G(x, y) = \lim_{n\to\infty} \left[F(a_n x + b_n, c_n y + d_n)\right]^n$$

Ex

$$F(x, y) = \left[1 + e^{-x} + e^{-y} + (1-a)e^{-x-y}\right]^{-1},$$

$$x > 0$$
$$y > 0$$

Find the corresponding $G(x, y)$.

(i) $F_X(x) = F(x, \infty) = \left[1 + e^{-x}\right]^{-1}$

$F_Y(y) = F(\infty, y) = \left[1 + e^{-y}\right]^{-1}$

(ii) $w(F_X) = \infty$

$w(F_Y) = \infty$

$$\lim_{t \to \infty} \frac{1 - F_X(t + x\gamma(t))}{1 - F_X(t)}$$

$$= \lim_{t \to \infty} \frac{1 - \left[1 - \underbrace{e^{-t - x\gamma(t)}}_{\to 0}\right]^{-1}}{1 - \left[1 - \underbrace{e^{-t}}_{\to 0}\right]^{-1}}$$

$$= \lim_{t \to \infty} \frac{1 - \left[1 + e^{-t - x\gamma(t)}\right]}{1 - \left[1 + e^{-t}\right]}$$

$$= \lim_{t \to \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}} = \lim_{t \to \infty} e^{-x\gamma(t)} = e^{-x}$$

if $\gamma(t) \equiv 1$.

Hence, F_X and F_Y belong to the Gumbel domain.