LECTURE

27 NOVEMBER

9:00-10:00AM

MATH3/4/68181

- Quiz 5

$$ES_p(X) = \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

Find a substitution that will help you to get the answer

No need to derive $F^{-1}(t)$.

- syllabus for level 3 will be completed on Tues 4 Dec
- syllabus for levels 4,6 will be completed on Fri 7 Dec.
- Exam date will be out on 3 Dec.

Model 3

X_t = ~~random~~ Stock return at time t

X_0 = " " " " 0

Assume X_0 is known.

$$\frac{X_t}{X_0} = \underbrace{\frac{X_t}{X_{t-1}}}_{= Z_t} \cdot \underbrace{\frac{X_{t-1}}{X_{t-2}}}_{= Z_{t-1}} \cdots \underbrace{\frac{X_2}{X_1}}_{= Z_2} \cdot \underbrace{\frac{X_1}{X_0}}_{= Z_1}$$

$$= \prod_{i=1}^{t} Z_i$$

Assume Z_i are independent RVs.

$$E\left[\frac{X_t}{X_0}\right] = \prod_{i=1}^{t} E[Z_i]$$

$$E\left[\left(\frac{X_t}{X_0}\right)^2\right] = \prod_{i=1}^{t} E[Z_i^2]$$

$$Var\left[\frac{X_t}{X_0}\right] = \prod_{i=1}^{t} E[Z_i^2] - \prod_{i=1}^{t} \left(E[Z_i]\right)^2$$

Ex 1 Suppose $Z_i \sim LN(\mu_i, \sigma_i^2)$, $i = 1, 2, \dots, t$ independently.

What is the distribution of

$$\frac{X_t}{X_0} = \prod_{i=1}^{t} Z_i \text{ ?}$$

$$\Rightarrow \log \frac{X_t}{X_0} = \log \left[\prod_{i=1}^{t} Z_i \right]$$

$$= \sum_{i=1}^{t} \log Z_i$$

$$= \sum_{i=1}^{t} \log \left[LN(\mu_i, \sigma_i^2) \right]$$

$$= \sum_{i=1}^{t} N(\mu_i, \sigma_i^2)$$

$\boxed{\text{If } Y \sim LN(a, b) \Rightarrow \log Y \sim N(a, b)}$

$$= N\left(\sum_{i=1}^{t} \mu_i, \sum_{i=1}^{t} \sigma_i^2 \right)$$

$$\Rightarrow \frac{X_t}{X_0} \sim LN\left(\sum_{i=1}^{t} \mu_i, \sum_{i=1}^{t} \sigma_i^2 \right)$$

$$\boxed{\begin{array}{l}\text{If } Y \sim LN(a, b^2) \text{ then} \\ E[Y] = e^{a + \frac{b^2}{2}} \\ Var[Y] = \left[e^{b^2} - 1\right] e^{2a + b^2}\end{array}}$$

$$E\left[\frac{X_t}{X_0}\right] = \exp\left[\sum_{i=1}^{t} \mu_i + \frac{1}{2}\sum_{i=1}^{t} \sigma_i^2\right]$$

$$Var\left[\frac{X_t}{X_0}\right] = \left[\exp\left(\sum_{i=1}^{t} \sigma_i^2\right) - 1\right]$$

$$\cdot \exp\left[2\sum_{i=1}^{t} \mu_i + \sum_{i=1}^{t} \sigma_i^2\right].$$

In the IID case,

$$\frac{X_t}{X_0} \sim LN(t\mu, t\sigma^2)$$

$$E\left[\frac{X_t}{X_0}\right] = e^{t\mu + \frac{t\sigma^2}{2}}$$

$$Var\left[\frac{X_t}{X_0}\right] = \left[e^{t\sigma^2} - 1\right]\left[e^{2t\mu + t\sigma^2}\right]$$

Model 1 - More Examples

Ex 2

Suppose $X|\theta \sim Uni[-\theta, \theta]$

$\theta \sim Exp(a)$

What is the distribution of X?

$$f_X(x) = \int_0^\infty f_{X|\theta}(x|\theta)\, f(\theta)\, d\theta$$

$$= \int_0^\infty \frac{1}{2\theta} \cdot a e^{-a\theta}\, d\theta$$

$$= \frac{a}{2} \int_0^\infty \frac{1}{\theta} e^{-a\theta}\, d\theta$$

$$= \infty$$ [does not exist]

Ex 3

Suppose $X|\theta \sim \text{Uni}[-\theta, \theta]$

$\theta \sim \text{Uni}[a, b]$

$$f_X(x) = \int_a^b \frac{1}{2\theta} \cdot \frac{1}{b-a} \, d\theta$$

$$= \frac{1}{2(b-a)} \int_a^b \frac{1}{\theta} \, d\theta$$

$$= \frac{1}{2(b-a)} \Big[\log \theta\Big]_a^b$$

$$= \frac{1}{2(b-a)} \log \frac{b}{a}$$

Model 2 - More Examples

Ex 3

Suppose Y_i are IID Inverse Gaussian with PDF

$$\sqrt{\frac{\lambda}{2\pi y^3}}\; e^{-\frac{\lambda(y-\mu)^2}{2\mu^2 \cancel{y}}}, \quad y > 0$$

What is the distribution of

$$X_t - X_0 = \sum_{i=1}^{t} Y_i \;?$$

MGF Approach

$$M_{X_t - X_0}(s) = E\left[e^{s(X_t - X_0)}\right]$$

$$= E\left[e^{s\sum_{i=1}^{t} Y_i}\right]$$

$$= E\left[e^{sY_1}\right] E\left[e^{sY_2}\right] \cdots E\left[e^{sY_t}\right]$$

[Using independence]

$$= \left(E\left[e^{sY}\right]\right)^{t}$$

[Using identical assumption]

$$= \left[M_Y(s) \right]^t$$

$$= \left[\underbrace{e^{\frac{\lambda}{\mu}\left[1-\sqrt{1-\frac{2\mu^2 s}{\lambda}}\,\right]}}_{\text{MGF of an Inverse Gaussian RV}} \right]^t$$

$$= e^{\frac{\lambda t}{\mu}\left[1-\sqrt{1-\frac{2\mu^2 s}{\lambda}}\right]}$$

$$= e^{\frac{\lambda t^2}{\mu t}\left[1-\sqrt{1-\frac{2(\mu t)^2 s}{\lambda t^2}}\right]}$$

$$= e^{\frac{\lambda^*}{\mu^*}\left[1-\sqrt{1-\frac{2(\mu^*)^2 s}{\lambda^*}}\right]}$$

where $\lambda^* = \lambda t^2$

$\mu^* = \mu t$

Hence, $X_t - X_0$ has the Inverse Gaussian distribution with parameters λ^* and μ^*