LECTURE

26 NOVEMBER

9:00-10:00AM

MATH3/4/68181

Quiz 5

No need to derive explicit expressions for VaR or ES

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{1}{p}\int_0^p \underbrace{F^{-1}(t)}\,dt$$

apply some transformation to get the answer.

Models for Stock Returns

1) Model based on varying volatility

(Model 1)

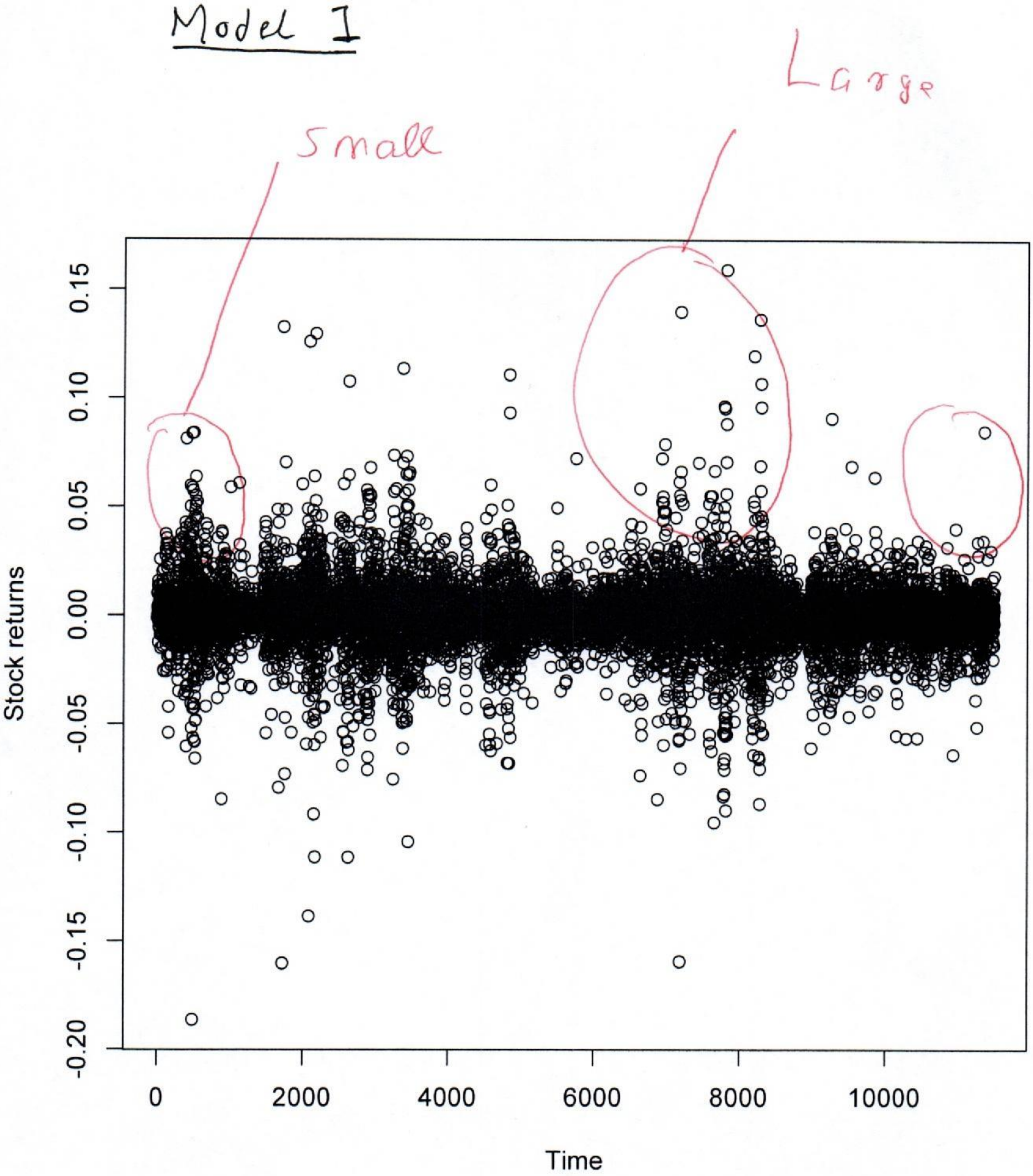
2) Model based on additive random walk

(Model 2)

3) Model based on multiplicative random walk

(Model 3)

Model I


Large
Small
Stock returns
0.15
0.10
0.05
0.00
-0.05
-0.10
-0.15
-0.20
0
2000
4000
6000
8000
10000
Time

Model I

X = stock returns

V = variability
(volatility)

V is a random variable itself

V = unobservable

$X|V$ = observable

Hence, the PDF of X is

$$f_X(x) = \int_0^\infty \underbrace{f_{X|V}(x|v)}_{\substack{\text{conditional} \\ \text{PDF of } X|V}} \underbrace{f_V(v)}_{\text{PDF of } V} dv$$

The CDF of X is

$$F_X(x) = \int_0^\infty F_{X|V}(x|v)\, f_V(v)\, dv$$

The nth moment of X is

$$E[X^n] = \int_0^\infty E[X^n | v] \, f_V(v) \, dv$$

$$= E[E[X^n | V]]$$

The mean of X is

$$E[X] = E[E[X | V]]$$

The variance of X is

$$Var[X] = E[X^2] - (E[X])^2$$

$$= E[E[X^2 | V]] - (E[E[X | V]])^2$$

Ex 1

Suppose

$$X \mid V \sim N(0, \sigma^2)$$

$$\boxed{V = \sigma} \quad \sigma \text{ is a RV}$$

σ has the PDF

$$\frac{2}{\sigma^3} e^{-\frac{1}{\sigma^2}}, \quad \sigma > 0.$$

What is the distribution of X?

$$f_X(x) = \int_0^\infty \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}}_{\substack{\text{cond PDF} \\ \text{of } X|V}} \underbrace{\frac{2}{\sigma^3} \cdot e^{-\frac{1}{\sigma^2}}}_{\substack{\text{PDF of} \\ V}} d\sigma$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sigma^4} e^{-\left(\frac{x^2}{2}+1\right)\frac{1}{\sigma^2}} d\sigma$$

Set $y = \left(\frac{x^2}{2}+1\right)\frac{1}{\sigma^2}$

$$\Rightarrow \sigma^2 = \frac{\frac{x^2}{2}+1}{y} \Rightarrow \sigma = \frac{\sqrt{\frac{x^2}{2}+1}}{\sqrt{y}}$$

$$\Rightarrow \frac{d\sigma}{dy} = -\frac{\sqrt{\frac{x^2}{2}+1}}{2y^{3/2}}$$

$$= -\sqrt{\frac{2}{\pi}} \int_\infty^0 \frac{y^2}{\left(\frac{x^2}{2}+1\right)^2} \cdot e^{-y} \frac{\sqrt{\frac{x^2}{2}+1}}{2y^{3/2}} dy$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{x^2}{2}+1\right)^{-\frac{3}{2}} \boxed{\int_0^\infty y^{\frac{1}{2}} e^{-y} dy} = \Gamma\left(\frac{3}{2}\right)$$

$$\boxed{\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}}$$

$$f_X(x) = \frac{1}{2\sqrt{2}}\left(\frac{x^2}{2}+1\right)^{-\frac{3}{2}}.$$

$$E[X] = E[E[X|V]]$$
$$= E[0]$$
$$= 0$$

$$E[X^2] = E[E[X^2|V]]$$
$$= E[\sigma^2]$$
$$= \int_0^\infty \sigma^2 \cdot \frac{2}{\sigma^3} \cdot e^{-\frac{1}{\sigma^2}} d\sigma$$
$$= 2\int_0^\infty \frac{1}{\sigma} e^{-\frac{1}{\sigma^2}} d\sigma$$

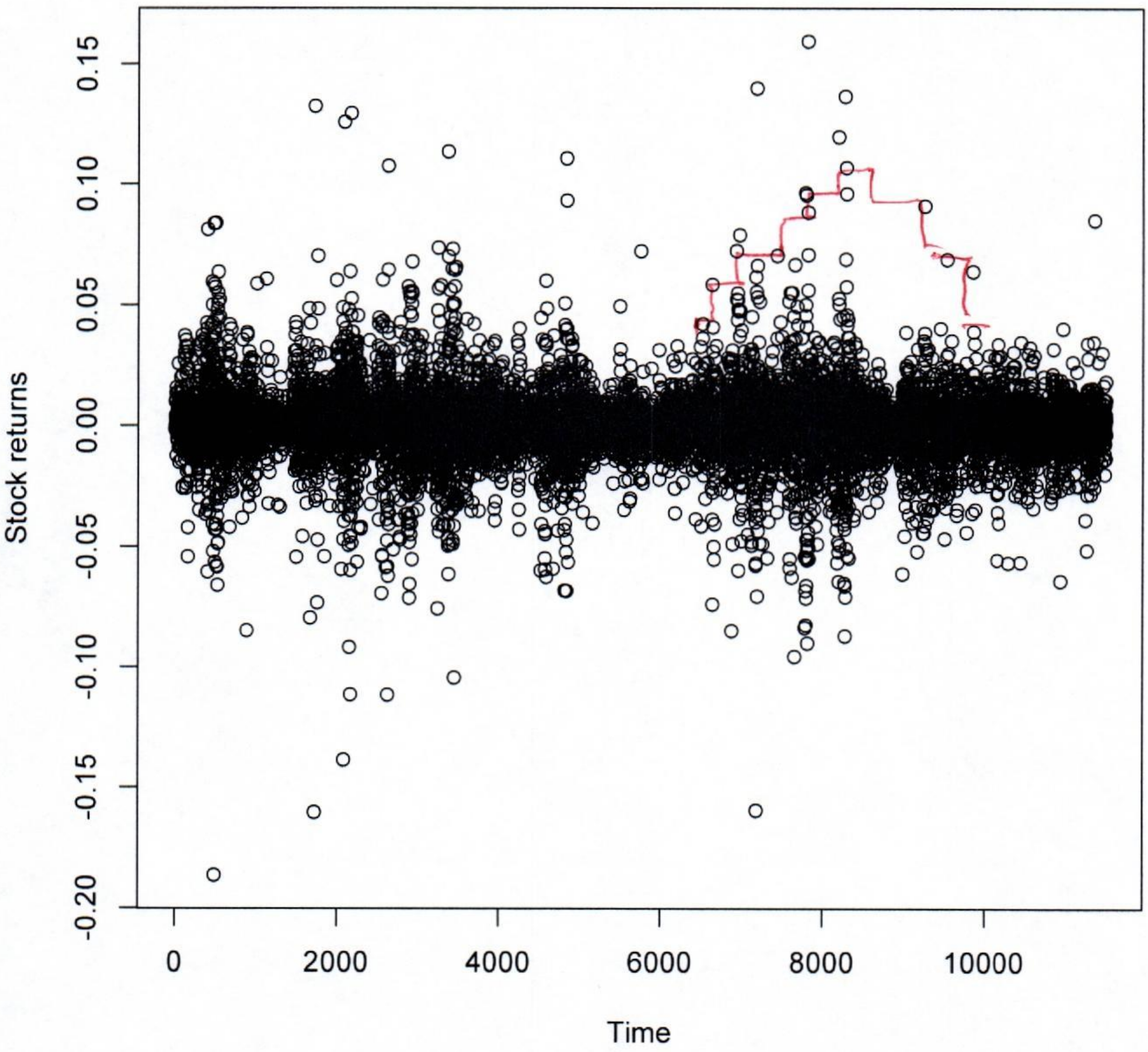
$$\boxed{\text{Set } y = \frac{1}{\sigma^2} \Rightarrow \sigma = \frac{1}{\sqrt{y}} \Rightarrow \frac{d\sigma}{dy} = -\frac{1}{2y^{3/2}}}$$

$$= -2\int_\infty^0 \sqrt{y}\, e^{-y}\left(\frac{1}{2y^{3/2}}\right) dy$$
$$= \frac{2}{2}\int_0^\infty \frac{1}{y} e^{-y} dy$$
$$= \Gamma(0) = \infty.$$

$$\Rightarrow \quad Var[X] = \infty.$$

Model 2

Additive random walk



Model 2

Let X_t = stock return at time t

Let X_0 = stock " " " 0.

Assume X_0 is known.

$$
\begin{aligned}
X_t - X_0 = & \ (X_t - X_{t-1}) = Y_t \\
& + (X_{t-1} - X_{t-2}) = Y_{t-1} \\
& + (X_{t-2} - X_{t-3}) = Y_{t-2} \\
& + \\
& \vdots \\
& + (X_2 - X_1) = Y_2 \\
& + (X_1 - X_0) = Y_1
\end{aligned}
$$

$$= \sum_{i=1}^{t} Y_i$$

Assume Y_i are independent RVs.

$$E[X_t - X_0] = \sum_{i=1}^{t} E[Y_i]$$

$$\Rightarrow \quad E[X_t] = X_0 + \sum_{i=1}^{t} E[Y_i]$$

$$Var[X_t - X_0] = \sum_{i=1}^{t} Var[Y_i]$$

$$\Rightarrow \quad Var[X_t] = \sum_{i=1}^{t} Var[Y_i]$$

If Y_i are IID then

$$E[X_t] = X_0 + t\,E[Y]$$

and

$$Var[X_t] = t\,Var[Y].$$

<u>Ex 1</u> Suppose $Y_i \sim N(\mu_i, \sigma_i^2)$ are indep RVs.

$$E[X_t] = X_0 + \sum_{i=1}^{t} \mu_i$$

$$Var[X_t] = \sum_{i=1}^{t} \sigma_i^2$$

If Y_i are IID $N(\mu, \sigma^2)$

$$E[X_t] = X_0 + t\mu$$

$$Var[X_t] = t \cdot \sigma^2$$

$$X_t - X_0 = \sum_{i=1}^{t} Y_i \sim N\left(\sum_{i=1}^{t} \mu_i, \sum_{i=1}^{t} \sigma_i^2\right)$$

If Y_i are IID $N(\mu, \sigma^2)$

$$X_t - X_0 \sim N(t\mu, t\sigma^2)$$

Ex 2

Suppose Y_i are IID $\Gamma(a, b)$

Gamma distribution with parameters a & b

$$E[X_t] = X_0 + t\,E[Y]$$

$$= X_0 + t\,\frac{a}{b}$$

$$Var[X_t] = t \cdot Var[Y]$$

$$= t \cdot \frac{a}{b^2}$$

$$X_t - X_0 = \sum_{i=1}^{t} Y_i \sim \Gamma(ta, b)$$

[by Math 20802]

Model 3

X_t = ~~random~~ Stock return at time t

X_0 = " " " " 0

Assume X_0 is known.

$$\frac{X_t}{X_0} = \underbrace{\frac{X_t}{X_{t-1}}}_{= Z_t} \cdot \underbrace{\frac{X_{t-1}}{X_{t-2}}}_{= Z_{t-1}} \cdots \underbrace{\frac{X_2}{X_1}}_{= Z_2} \cdot \underbrace{\frac{X_1}{X_0}}_{= Z_1}$$

$$= \prod_{i=1}^{t} Z_i$$

Assume Z_i are independent RVs.

$$E\left[\frac{X_t}{X_0}\right] = \prod_{i=1}^{t} E[Z_i]$$

$$E\left[\left(\frac{X_t}{X_0}\right)^2\right] = \prod_{i=1}^{t} E[Z_i^2]$$

$$Var\left[\frac{X_t}{X_0}\right] = \prod_{i=1}^{t} E[Z_i^2] - \prod_{i=1}^{t} \left(E[Z_i]\right)^2$$