EXAMPLE CLASS

27 NOVEMBER

16:00-17:00PM

MATH3/4/68181

Q1

X = stock returns

$X \mid \lambda \sim \text{Exp}(\lambda)$

$\lambda \sim \text{Exp}(a)$

$$f_X(x) = \int_0^\infty f_{X|\lambda}(x|\lambda)\, f(\lambda)\, d\lambda$$

$$= \int_0^\infty \lambda e^{-\lambda x}\, a e^{-\lambda a}\, d\lambda$$

$$= a \int_0^\infty \lambda e^{-\lambda(x+a)}\, d\lambda$$

Set $y = \lambda(x+a)$

$\Rightarrow \lambda = \frac{y}{x+a}$

$\Rightarrow d\lambda = \frac{dy}{x+a}$

$$= a \int_0^\infty \frac{y}{x+a} e^{-y} \frac{dy}{x+a}$$

$$= \frac{a}{(x+a)^2} \int_0^\infty y e^{-y}\, dy$$

$$= \frac{a}{(x+a)^2} \Gamma(2) = \frac{a}{(x+a)^2}.$$

Q2

X = stock returns

$X|\lambda \sim Exp(\lambda)$

$\lambda \sim Uni[a,b]$

$$f_X(x) = \int_a^b \lambda e^{-\lambda x} \frac{1}{b-a} \, d\lambda$$

$$= \frac{1}{b-a} \int_a^b \lambda e^{-\lambda x} \, d\lambda$$

$$= \frac{1}{b-a} \left\{ \left[\lambda \frac{e^{-\lambda x}}{(-x)} \right]_a^b + \frac{1}{x} \int_a^b e^{-\lambda x} \, d\lambda \right\}$$

$$= \frac{1}{b-a} \left\{ \frac{b e^{-bx} - a e^{-ax}}{(-x)} + \frac{1}{x} \left[\frac{e^{-\lambda x}}{(-x)} \right]_a^b \right\}$$

$$= \frac{1}{b-a} \left\{ \frac{b e^{-bx} - a e^{-ax}}{(-x)} - \frac{e^{-bx} - e^{-ax}}{x^2} \right\}$$

Q3

$X|\lambda \sim Exp(\lambda)$

λ has PDF $a\lambda^{a-1}$, $0 < \lambda < 1$

$$f_X(x) = \int_0^1 \lambda e^{-\lambda x} \, a\lambda^{a-1} \, d\lambda$$

$$= a \int_0^1 \lambda^a e^{-\lambda x} \, d\lambda$$

Set $y = \lambda x \Rightarrow \lambda = \frac{y}{x} \Rightarrow d\lambda = \frac{dy}{x}$

$$= a \int_0^x \left(\frac{y}{x}\right)^a e^{-y} \frac{dy}{x}$$

$$= \frac{a}{x^{a+1}} \int_0^x y^a e^{-y} \, dy$$

$$= \frac{a}{x^{a+1}} \gamma(a+1, x)$$

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} \, dt$$

(Upper) Incomplete gamma function

Q4

$$X|\lambda \sim Exp(\lambda)$$

λ has PDF $\frac{ak^a}{\lambda^{a+1}}$, $\lambda > k$

$$f_X(x) = \int_k^\infty \lambda e^{-\lambda x} \frac{ak^a}{\lambda^{a+1}} d\lambda$$

$$= ak^a \int_k^\infty \frac{1}{\lambda^a} e^{-\lambda x} d\lambda$$

Set $y = \lambda x \Rightarrow \lambda = \frac{y}{x} \Rightarrow d\lambda = \frac{dy}{x}$

$$= ak^a \int_{kx}^\infty \frac{x^a}{y^a} e^{-y} \frac{dy}{x}$$

$$= ak^a x^{a-1} \int_{kx}^\infty y^{-a} e^{-y} dy$$

$$= ak^a x^{a-1} \Gamma(1-a, kx)$$

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$$

Lower incomplete gamma function