EXAMPLE CLASS

26 NOVEMBER

12:00-13:00PM

MATH3/4/68181

Q1

X = stock returns

$X|\lambda \sim \text{Exp}(\lambda)$

$\lambda \sim \text{Exp}(a)$

$$f_X(x) = \int_0^\infty f_{X|\lambda}(x|\lambda)\, f(\lambda)\, d\lambda$$

$$= \int_0^\infty \lambda e^{-\lambda x}\, a e^{-a\lambda}\, d\lambda$$

$$= a \int_0^\infty \lambda e^{-(a+x)\lambda}\, d\lambda$$

Set $y = (a+x)\lambda$

$\Rightarrow \lambda = \frac{y}{a+x}$

$d\lambda = \frac{dy}{a+x}$

$$= a \int_0^\infty \frac{y}{a+x} e^{-y} \frac{dy}{a+x}$$

$$= \frac{a}{(a+x)^2} \boxed{\int_0^\infty y e^{-y}\, dy} = \Gamma(2) = 1$$

$$= \frac{a}{(a+x)^2}.$$

Q2

X = stock returns

$X \mid \lambda \sim Exp(\lambda)$

$\lambda \sim Uni[a, b]$

$$f_X(x) = \int_a^b \lambda e^{-\lambda x} \frac{1}{b-a} \, d\lambda$$

$$= \frac{1}{b-a} \int_a^b \lambda e^{-\lambda x} \, d\lambda$$

Set $y = \lambda x \Rightarrow \lambda = \frac{y}{x}$

$\Rightarrow d\lambda = \frac{dy}{x}$

$$= \frac{1}{b-a} \int_{ax}^{bx} \frac{y}{x} e^{-y} \frac{dy}{x}$$

$$= \frac{1}{(b-a)x^2} \int_{ax}^{bx} y e^{-y} \, dy$$

$$= \frac{1}{(b-a)x^2} \left\{ \left[y(-e^{-y}) \right]_{ax}^{bx} + \int_{ax}^{bx} e^{-y} \, dy \right\}$$

$$= \frac{1}{(b-a)x^2} \left\{ -bx e^{-bx} + ax e^{-ax} + \left[-e^{-y} \right]_{ax}^{bx} \right\}$$

$$= \frac{1}{(b-a)x^2} \left\{ -bx e^{-bx} + ax e^{-ax} - e^{-bx} + e^{-ax} \right\}.$$

Q3

$X \mid \lambda \sim Exp(\lambda)$

λ has PDF $a\lambda^{a-1}$, $0 < \lambda < 1$

$$f_X(x) = \int_0^1 \lambda e^{-\lambda x} \cdot a\lambda^{a-1} \, d\lambda$$

$$= a \int_0^1 \lambda^a e^{-\lambda x} \, d\lambda$$

Set $y = \lambda x \Rightarrow \lambda = \frac{y}{x}$

$\Rightarrow d\lambda = \frac{dy}{x}$

$$= a \int_0^x \left(\frac{y}{x}\right)^a e^{-y} \frac{dy}{x}$$

$$= \frac{a}{x^{a+1}} \int_0^x y^a e^{-y} \, dy$$

$$= \frac{a}{x^{a+1}} \gamma(a+1, x)$$

Upper Incomplete gamma function

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} \, dt$$

Q4

$X|\lambda \sim Exp(\lambda)$

λ has PDF $\frac{a k^a}{\lambda^{a+1}}, \quad \lambda > k$

$$f_X(x) = \int_k^\infty \lambda e^{-\lambda x} \frac{a k^a}{\lambda^{a+1}} d\lambda$$

$$= a k^a \int_k^\infty \frac{1}{\lambda^a} e^{-\lambda x} d\lambda$$

Set $y = \lambda x \Rightarrow \lambda = \frac{y}{x} \Rightarrow d\lambda = \frac{dy}{x}$

$$= a k^a \int_{kx}^\infty \frac{x^a}{y^a} e^{-y} \frac{dy}{x}$$

$$= a k^a x^{a-1} \int_{kx}^\infty y^{-a} e^{-y} dy$$

$$= a k^a x^{a-1} \Gamma(1-a, kx)$$

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$$

Lower incomplete gamma function