MATH3/4/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 5

A random variable X is said to have the Student's t distribution if its probability density function is

$$f(x) = K \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}}$$

for $-\infty < x < \infty$ and $a > 0$, where

$$K = \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{a\pi}\,\Gamma\left(\frac{a}{2}\right)}.$$

Let $F(\cdot)$ denote the cumulative distribution function of X. Let $F^{-1}(\cdot)$ denote inverse of F. By definition, $\mathrm{ES}_p(X)$ is given by

$$\mathrm{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(u) du.$$

Set $y = F^{-1}(u)$. Then $u = F(y)$ and $du/dy = f(y)$. So, (1) can be written as

$$\begin{aligned}
\mathrm{ES}_p(X) &= \frac{1}{p} \int_{-\infty}^{F^{-1}(p)} y f(y) dy \\
&= \frac{K}{p} \int_{-\infty}^{F^{-1}(p)} y \left(1 + \frac{y^2}{a}\right)^{-\frac{a+1}{2}} dy \\
&= \frac{aK}{2p} \int_{-\infty}^{F^{-1}(p)} \frac{2y}{a} \left(1 + \frac{y^2}{a}\right)^{-\frac{a+1}{2}} dy \\
&= \frac{aK}{2p} \int_{-\infty}^{F^{-1}(p)} \left(1 + \frac{y^2}{a}\right)^{-\frac{a+1}{2}} d\left(1 + \frac{y^2}{a}\right) \\
&= \frac{aK}{2p} \left[\frac{2}{1-a} \left(1 + \frac{y^2}{a}\right)^{\frac{1-a}{2}}\right]_{-\infty}^{F^{-1}(p)} \\
&= \frac{aK}{p(1-a)} \left[\left(1 + \frac{y^2}{a}\right)^{\frac{1-a}{2}}\right]_{-\infty}^{F^{-1}(p)} \\
&= \frac{aK}{p(1-a)} \left[\left(1 + \frac{\left(F^{-1}(p)\right)^2}{a}\right)^{\frac{1-a}{2}} - 0\right]
\end{aligned}$$

$$= \frac{aK}{(1-a)p}\left\{1+\frac{1}{a}\left[\mathrm{VaR}_p(X)\right]^2\right\}^{\frac{1-a}{2}}$$

provided that $a > 1$.