MATH3/4/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTION TO QUIZ PROBLEM 4

Suppose a company has N departments functioning independently at a given time, where N is a truncated Poisson random variable with the probability mass function

$$\Pr(N = n) = \{\exp(\lambda) - 1\}^{-1} \frac{\lambda^n}{n!} \tag{1}$$

for $n = 1, 2, \ldots$. Suppose in addition to (1) that each department is made of α parallel units, so the department will fail to function if all of the units fail. Assume that the failure times of the units for the ith department, say $Z_{i,1}, Z_{i,2}, \ldots, Z_{i,\alpha}$, are independent and identical exponential random variables with the scale parameter β. Let Y_i denote the failure time of ith department. Let X denote the time to failure of the first out of the N functioning departments.

We can write $X = \min(Y_1, Y_2, \ldots, Y_N)$. Then the cumulative distribution function of X, say $G(x)$, can be derived as: the conditional cumulative distribution function of X given N is

$$\begin{aligned}
G(x|N) &= 1 - \Pr(X > x|N) \\
&= 1 - \Pr(Y_1 > x, Y_2 > x, \ldots, Y_N > x) \\
&= 1 - \Pr^N(Y_1 > x) \\
&= 1 - [1 - \Pr(Y_1 \leq x)]^N \\
&= 1 - [1 - \Pr(Z_{1,1} \leq x, Z_{1,2} \leq x, \ldots, Z_{1,\alpha} \leq x)]^N \\
&= 1 - [1 - \Pr^\alpha(Z_{1,1} \leq x)]^N \\
&= 1 - [1 - \{1 - \exp(-\beta x)\}^\alpha]^N,
\end{aligned}$$

and so the unconditional cumulative distribution function of X is

$$\begin{aligned}
G(x) &= \sum_{n=1}^{\infty} \{1 - [1 - \{1 - \exp(-\beta x)\}^\alpha]^n\} \{\exp(\lambda) - 1\}^{-1} \frac{\lambda^n}{n!} \\
&= \{\exp(\lambda) - 1\}^{-1} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} - \{\exp(\lambda) - 1\}^{-1} \sum_{n=1}^{\infty} [1 - \{1 - \exp(-\beta x)\}^\alpha]^n \frac{\lambda^n}{n!} \\
&= 1 - \{\exp(\lambda) - 1\}^{-1} \{\exp[\lambda - \lambda \{1 - \exp(-\beta x)\}^\alpha] - 1\} \\
&= \{1 - \exp(-\lambda)\}^{-1} \{1 - \exp[-\lambda \{1 - \exp(-\beta x)\}^\alpha]\}
\end{aligned}$$

for $x > 0$, $\lambda > 0$, $\beta > 0$ and $\alpha > 0$.

Setting $G(x) = u$, we have

$$\{1 - \exp(-\lambda)\}^{-1} \{1 - \exp[-\lambda \{1 - \exp(-\beta x)\}^\alpha]\} = u$$

$$
\begin{aligned}
&\Leftrightarrow \quad 1-\exp\left[-\lambda\left\{1-\exp\left(-\beta x\right)\right\}^{\alpha}\right]=\left\{1-\exp(-\lambda)\right\}u \\
&\Leftrightarrow \quad \exp\left[-\lambda\left\{1-\exp\left(-\beta x\right)\right\}^{\alpha}\right]=1-\left\{1-\exp(-\lambda)\right\}u \\
&\Leftrightarrow \quad -\lambda\left\{1-\exp\left(-\beta x\right)\right\}^{\alpha}=\log\left[1-\left\{1-\exp(-\lambda)\right\}u\right] \\
&\Leftrightarrow \quad \left\{1-\exp\left(-\beta x\right)\right\}^{\alpha}=-\frac{1}{\lambda}\log\left[1-\left\{1-\exp(-\lambda)\right\}u\right] \\
&\Leftrightarrow \quad 1-\exp\left(-\beta x\right)=\left\{-\frac{1}{\lambda}\log\left[1-\left\{1-\exp(-\lambda)\right\}u\right]\right\}^{1/\alpha} \\
&\Leftrightarrow \quad \exp\left(-\beta x\right)=1-\left\{-\frac{1}{\lambda}\log\left[1-\left\{1-\exp(-\lambda)\right\}u\right]\right\}^{1/\alpha} \\
&\Leftrightarrow \quad -\beta x=\log\left[1-\left\{-\frac{1}{\lambda}\log\left[1-\left\{1-\exp(-\lambda)\right\}u\right]\right\}^{1/\alpha}\right] \\
&\Leftrightarrow \quad x=-\frac{1}{\beta}\log\left[1-\left\{-\frac{1}{\lambda}\log\left[1-\left\{1-\exp(-\lambda)\right\}u\right]\right\}^{1/\alpha}\right].
\end{aligned}
$$

Hence,

$$
\mathrm{VaR}_u(X)=-\frac{1}{\beta}\log\left[1-\left\{-\frac{1}{\lambda}\log\left[1-\left\{1-\exp(-\lambda)\right\}u\right]\right\}^{1/\alpha}\right]
$$

for $0<u<1$.