MATH3/4/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 3

Consider a class of distributions defined by the cdf

$$F(x) = 1 - b^a \left[b + \frac{G(x)}{1 - G(x)} \right]^{-a}$$

where $a > 0$, $b > 0$ and $G(\cdot)$ is a valid cdf. Assume that F and G have the same upper end points.

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \to w(G)} \frac{1 - G(t + xh(t))}{1 - G(t)} = e^{-x}$$

for every $x > 0$. But

$$\begin{aligned}
& \lim_{t \to w(F)} \frac{1 - F(t + xh(t))}{1 - F(t)} \\
= & \lim_{t \to w(F)} \frac{1 - 1 + b^a \left[b + \frac{G(t+xh(t))}{1-G(t+xh(t))} \right]^{-a}}{1 - 1 + b^a \left[b + \frac{G(t)}{1-G(t)} \right]^{-a}} \\
= & \lim_{t \to w(F)} \frac{\left[b + \frac{G(t+xh(t))}{1-G(t+xh(t))} \right]^{-a}}{\left[b + \frac{G(t)}{1-G(t)} \right]^{-a}} \\
= & \lim_{t \to w(F)} \left[\frac{b + \frac{G(t+xh(t))}{1-G(t+xh(t))}}{b + \frac{G(t)}{1-G(t)}} \right]^{-a} \\
= & \lim_{t \to w(G)} \left\{ \frac{1 - G(t)}{1 - G(t + xh(t))} \frac{b\left[1 - G(t + xh(t))\right] + G(t + xh(t))}{b\left[1 - G(t)\right] + G(t)} \right\}^{-a} \\
= & \lim_{t \to w(G)} \left\{ \frac{1 - G(t)}{1 - G(t + xh(t))} \frac{b \cdot 0 + 1}{b \cdot 0 + 1} \right\}^{-a} \\
= & \lim_{t \to w(G)} \left\{ \frac{1 - G(t)}{1 - G(t + xh(t))} \right\}^{-a} \\
= & \lim_{t \to w(G)} \left\{ \exp(x) \right\}^{-a} \\
= & \exp(-ax)
\end{aligned}$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \le x \right) = \exp\left[-\exp(-ax) \right]$$

for some suitable norming constants $a_n > 0$ and b_n.

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t\to\infty} \frac{1 - G(tx)}{1 - G(t)} = x^{-\beta}$$

for every $x > 0$. But

$$\begin{aligned}
& \lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} \\
= & \lim_{t\to\infty} \frac{1 - 1 + b^a \left[b + \frac{G(tx)}{1-G(tx)}\right]^{-a}}{1 - 1 + b^a \left[b + \frac{G(t)}{1-G(t)}\right]^{-a}} \\
= & \lim_{t\to\infty} \frac{\left[b + \frac{G(tx)}{1-G(tx)}\right]^{-a}}{\left[b + \frac{G(t)}{1-G(t)}\right]^{-a}} \\
= & \lim_{t\to\infty} \left[\frac{b + \frac{G(tx)}{1-G(tx)}}{b + \frac{G(t)}{1-G(t)}}\right]^{-a} \\
= & \lim_{t\to\infty} \left\{\frac{1 - G(t)}{1 - G(tx)} \frac{b[1 - G(tx)] + G(tx)}{b[1 - G(t)] + G(t)}\right\}^{-a} \\
= & \lim_{t\to\infty} \left\{\frac{1 - G(t)}{1 - G(tx)} \frac{b \cdot 0 + 1}{b \cdot 0 + 1}\right\}^{-a} \\
= & \lim_{t\to\infty} \left\{\frac{1 - G(t)}{1 - G(tx)}\right\}^{-a} \\
= & \lim_{t\to\infty} \left\{x^{\beta}\right\}^{-a} \\
= & x^{-a\beta}
\end{aligned}$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \le x\right) = \exp\left(-x^{-a\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t\to 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^{\beta}$$

for every $x > 0$. But

$$\lim_{t\to 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)}$$

$$
\begin{aligned}
&= \lim_{t\to 0} \frac{1-1+b^a\left[b+\frac{G(w(F)-tx)}{1-G(w(F)-tx)}\right]^{-a}}{1-1+b^a\left[b+\frac{G(w(F)-t)}{1-G(w(F)-t)}\right]^{-a}} \\
&= \lim_{t\to 0} \frac{\left[b+\frac{G(w(F)-tx)}{1-G(w(F)-tx)}\right]^{-a}}{\left[b+\frac{G(w(F)-t)}{1-G(w(F)-t)}\right]^{-a}} \\
&= \lim_{t\to 0} \left[\frac{b+\frac{G(w(F)-tx)}{1-G(w(F)-tx)}}{b+\frac{G(w(F)-t)}{1-G(w(F)-t)}}\right]^{-a} \\
&= \lim_{t\to 0} \left\{\frac{1-G\left(w(G)-t\right))}{1-G\left(w(G)-tx\right)}\frac{b\left[1-G\left(w(G)-tx\right)\right]+G\left(w(G)-tx\right)}{b\left[1-G\left(w(G)-t\right)\right]+G\left(w(G)-t\right)}\right\}^{-a} \\
&= \lim_{t\to 0} \left\{\frac{1-G\left(w(G)-t\right)}{1-G\left(w(G)-tx\right)}\frac{b\cdot 0+1}{b\cdot 0+1}\right\}^{-a} \\
&= \lim_{t\to 0} \left\{\frac{1-G\left(w(G)-t\right)}{1-G\left(w(G)-tx\right)}\right\}^{-a} \\
&= \lim_{t\to 0} \left\{x^{-\beta}\right\}^{-a} \\
&= x^{a\beta}
\end{aligned}
$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$
\lim_{n\to\infty} P\left(\frac{M_n-b_n}{a_n} \leq x\right) = \exp\left(-(-x)^{a\beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.