MATH3/4/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
A SOLUTION TO QUIZ PROBLEM 1

Suppose X is a random variable with cumulative distribution function

$$F(x) = 1 - \exp\left\{-a\left[\exp(bx) - 1\right]\right\}$$

for $x > 0$, $a > 0$ and $b > 0$. It is easy to see that $w(F) = \infty$.

Note that

$$\begin{aligned}
\lim_{t \to \infty} \frac{1 - F\left(t + xg(t)\right)}{1 - F(t)} &= \lim_{t \to \infty} \frac{1 - \left[1 - \exp\left\{-a\left[\exp\left(bt + bxg(t)\right) - 1\right]\right\}\right]}{1 - \left[1 - \exp\left\{-a\left[\exp(bt) - 1\right]\right\}\right]} \\
&= \lim_{t \to \infty} \frac{\exp\left\{-a\left[\exp\left(bt + bxg(t)\right) - 1\right]\right\}}{\exp\left\{-a\left[\exp(bt) - 1\right]\right\}} \\
&= \lim_{t \to \infty} \exp\left[a\exp(bt) - a\exp\left(bt + bxg(t)\right)\right] \\
&= \lim_{t \to \infty} \exp\left\{a\exp(bt)\left[1 - \exp\left(bxg(t)\right)\right]\right\} \\
&= \lim_{t \to \infty} \exp\left\{a\exp(bt)\left[1 - \left(1 + bxg(t) + \frac{\left(bxg(t)\right)^2}{2!} + \frac{\left(bxg(t)\right)^3}{3!} + \cdots\right)\right]\right\} \\
&= \lim_{t \to \infty} \exp\left\{-a\exp(bt)bxg(t) - a\exp(bt)\frac{\left(bxg(t)\right)^2}{2!} - a\exp(bt)\frac{\left(bxg(t)\right)^3}{3!} - \cdots\right\} \\
&= \exp(-x)
\end{aligned}$$

if $g(t) = 1/\left[ab\exp(bt)\right]$. Hence, F belongs to the Gumbel domain of attraction.

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AN ALTERNATIVE SOLUTION TO QUIZ PROBLEM 1

Suppose X is a random variable with cumulative distribution function

$$F(x) = 1 - \exp\left\{-a\left[\exp(bx) - 1\right]\right\}$$

for $x > 0$, $a > 0$ and $b > 0$. It is easy to see that $w(F) = \infty$.

Note that

$$\begin{aligned}
\lim_{t \to \infty} \frac{1 - F\left(t + xg(t)\right)}{1 - F(t)} &= \lim_{t \to \infty} \frac{1 - \left[1 - \exp\left\{-a\left[\exp\left(bt + bxg(t)\right) - 1\right]\right\}\right]}{1 - \left[1 - \exp\left\{-a\left[\exp(bt) - 1\right]\right\}\right]} \\
&= \lim_{t \to \infty} \frac{\exp\left\{-a\left[\exp\left(bt + bxg(t)\right) - 1\right]\right\}}{\exp\left\{-a\left[\exp(bt) - 1\right]\right\}} \\
&= \lim_{t \to \infty} \exp\left[a\exp(bt) - a\exp\left(bt + bxg(t)\right)\right] \\
&= \lim_{t \to \infty} \exp\left\{a\exp(bt)\left[1 - \exp\left(bxg(t)\right)\right]\right\}.
\end{aligned}$$

Now assume $g(t) \to 0$ as $t \to \infty$. Then using $\exp(y) \sim 1 + y$ as $y \to 0$, we have

$$\begin{aligned}
\lim_{t \to \infty} \frac{1 - F\left(t + xg(t)\right)}{1 - F(t)} &= \lim_{t \to \infty} \exp\left\{a\exp(bt)\left[1 - \left(1 + bxg(t)\right)\right]\right\} \\
&= \lim_{t \to \infty} \exp\left\{-ab\exp(bt)g(t)x\right\} \\
&= \exp\{-x\}
\end{aligned}$$

if $g(t) = 1/\left[ab\exp(bt)\right]$. Hence, F belongs to the Gumbel domain of attraction. Note that $g(t) \to 0$ as $t \to \infty$ as assumed.