LECTURE

24 SEPTEMBER

9:00-10:00AM

MATH3/4/68181

"EXTREME VALUES" & "FINANCIAL RISK"

Lecture times: Mon 9-10 am Cordingly Th
Tues 9-10am Roscoe 13
(MATH 38181)
Fri 10-11am ATB
(MATH 48181 / 68181)

Example class: Mon 12-13 ATB G10T
Tues 16-17 Chem G54
(Attend only 1 of these)

Website:
https://minerva.it.manchester.ac.uk/~saralees/
extremes3.html
(MATH 38181)
...../extremes4.html (MATH 48181)
...../extremes6.html (MATH 68181)

WELCOME

(BACK)

TO

MANCHESTER!

EMAIL: MBB55SN2@MANCHESTER.AC.UK

PHONE: 0161 275 5912

OFFICE: ATB 2.223

MON – FRI 11AM – 5PM (NO APPOINTMENT NEEDED)

WEEKENDS BY APPOINTMENT

EMAIL 24/7

- podcasted
- scanned handwritten notes

I WILL NOT USE BLACKBOARD.

Credits - 10 Math 38181
15 Math 48181
15 Math 68181

Pre-requisites - Math20802 (Stat Methods)
- Statistical Inference
(not essential you take this course)

Assessment - 5 quizzes each worth 1 marks
- in-class test 10 marks
- final exam 80 "

WHAT IS THE PURPOSE OF THIS COURSE?

1) minimize the chance of financial disasters happening

2) try to identify anomalies in financial markets (eg. using machine learning)

Extreme Values

What is an extreme value?

I will give you

3 different definitions

of an "extreme value"

Definition I

Suppose $X_1, X_2, \ldots, X_n$ are IID with cumulative distribution function (CDF) $F(\cdot)$. The extreme value is

$$M_n = \max(X_1, \ldots, X_n).$$

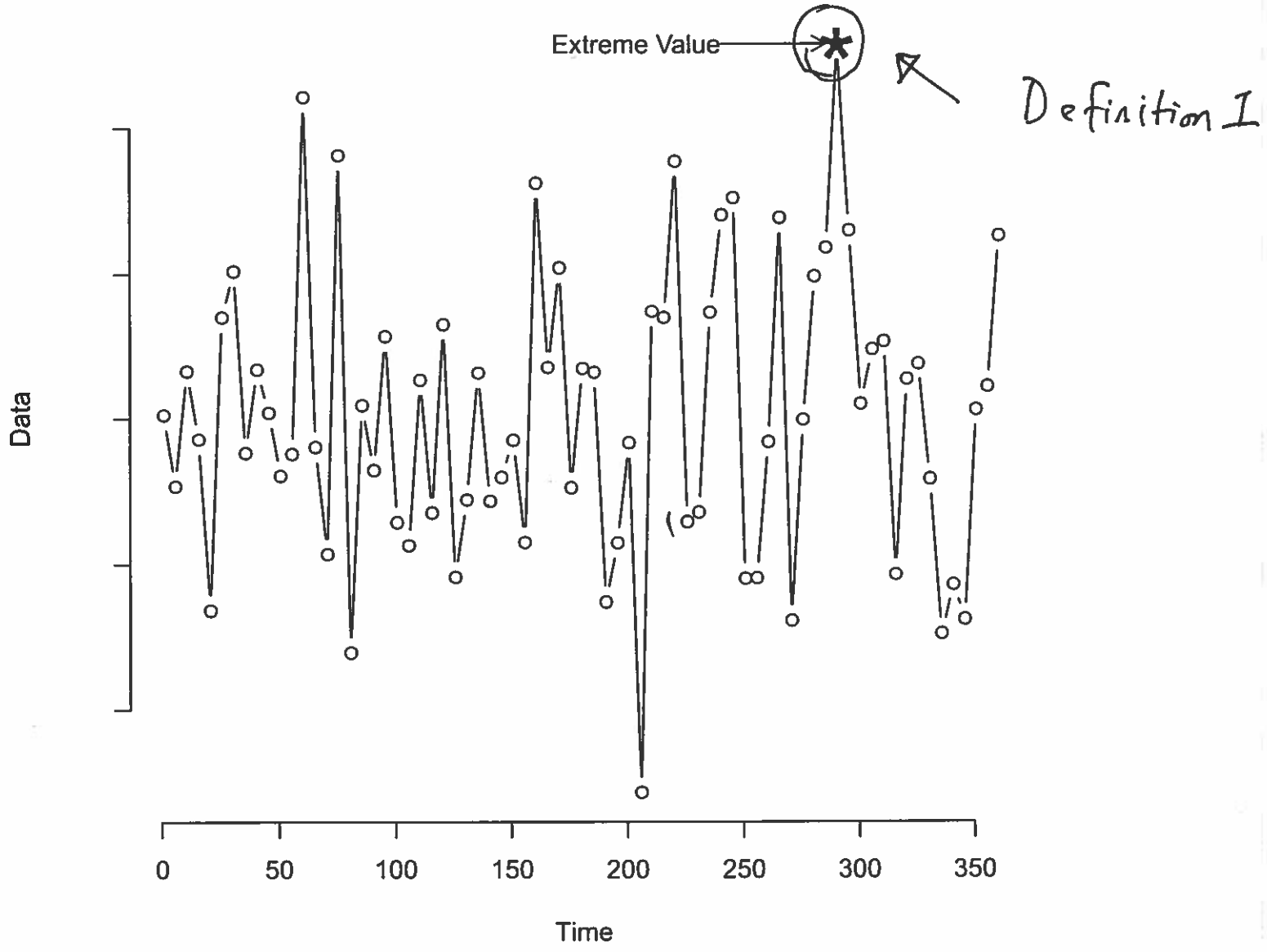
For modeling purposes, we need to know the distribution of M_n

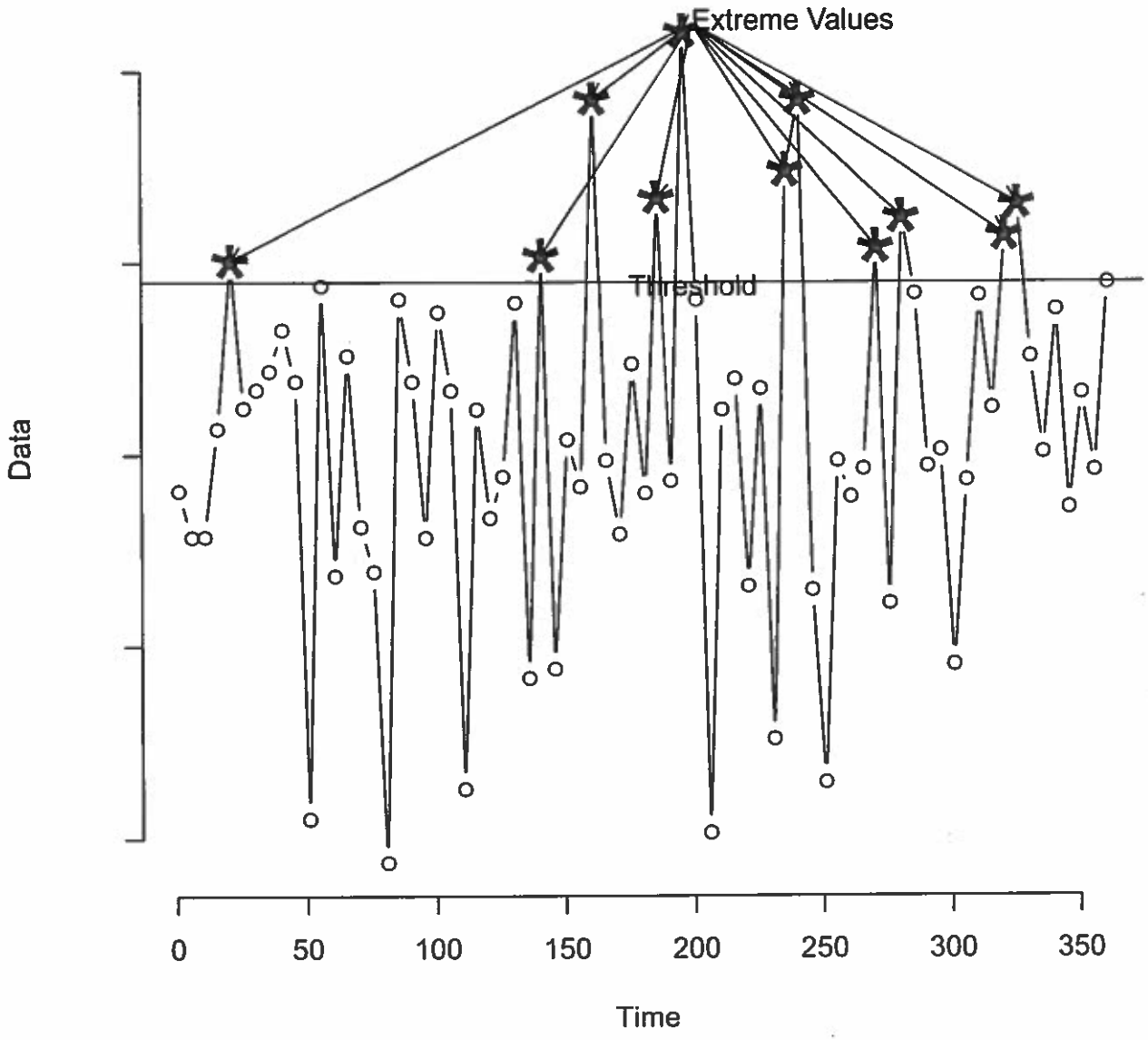
$$\begin{aligned} & P(M_n \leq x) \\ = \; & P(\max(X_1, \ldots, X_n) < x) \\ = \; & P(X_1 < x, \ldots, X_n < x) \\ \overset{\text{indep}}{=} \; & P(X_1 < x) \cdots P(X_n < x) \\ \overset{\text{identical}}{=} \; & F(x) \cdots F(x) \\ = \; & [F(x)]^n \end{aligned}$$

What is the limit of $P(M_n < x)$ as $n \longrightarrow \infty$?

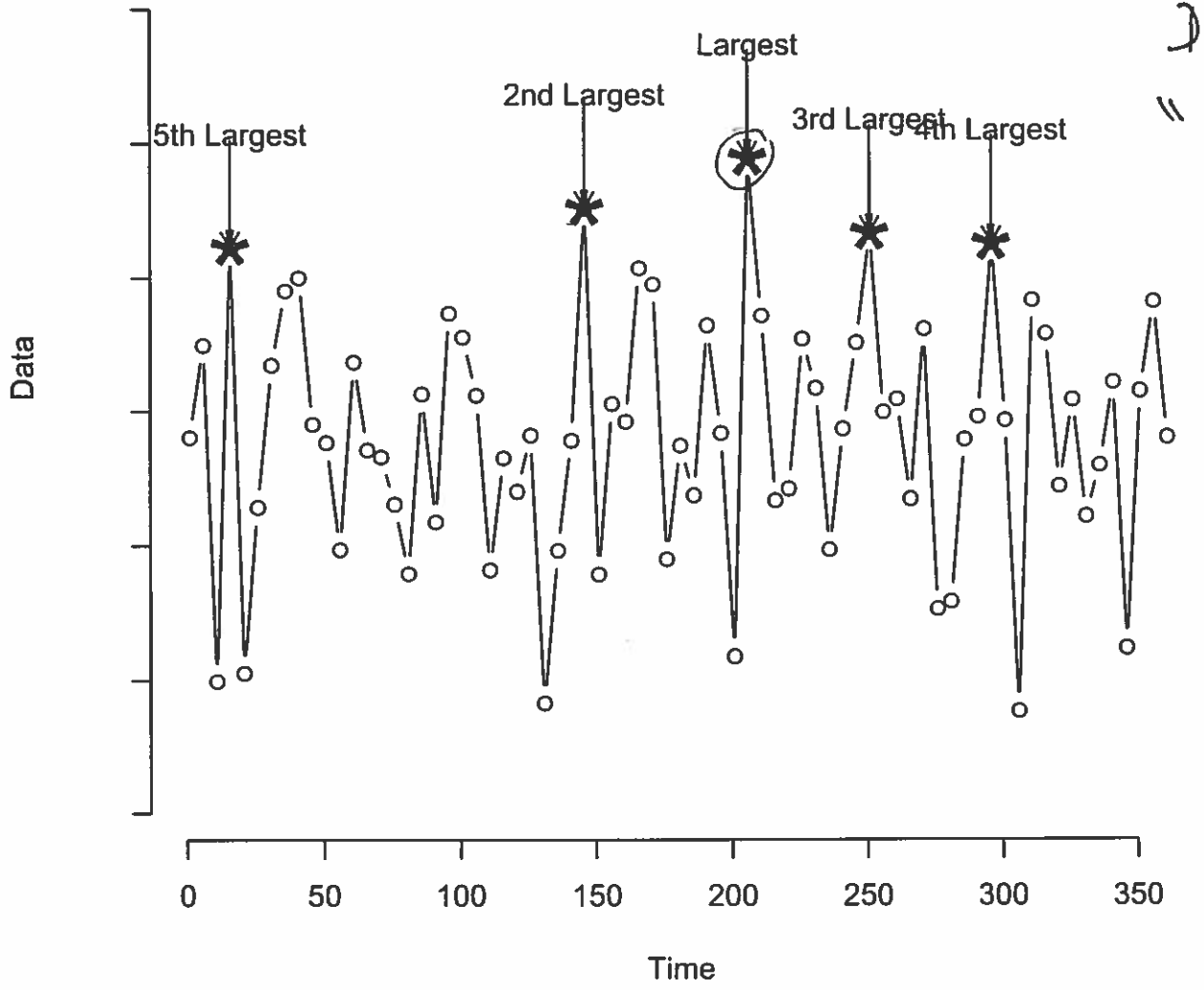
$$\lim_{n \to \infty} P(M_n < x) = \lim_{n \to \infty} [F(x)]^n = \begin{cases} 0 & \text{if } F(x) < 1 \\ 1 & \text{if } F(x) = 1 \end{cases}$$

THIS IS NOT A USEFUL RESULT


Extreme Value
Definition I
Data
0
50
100
150
200
250
300
350
Time



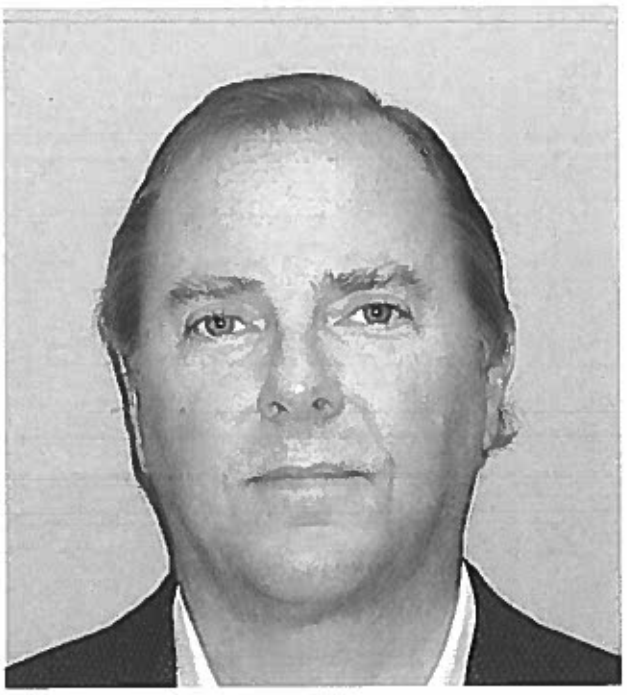
Definition 2



Definition 3

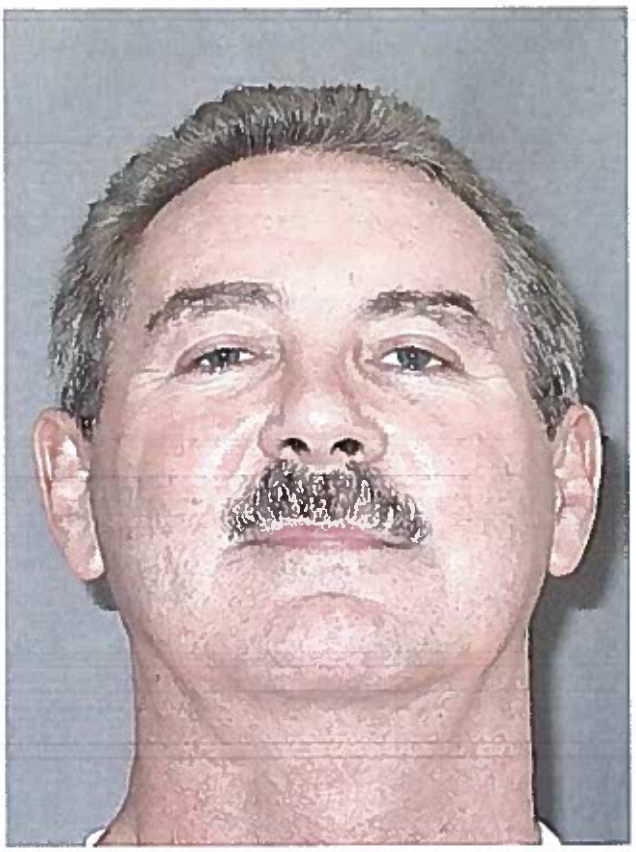
"r largest"

(r = 5)



Jeff Skilling

Jeff was the CEO of Enron Corp. He was convicted of federal felony charges relating to Enron's financial collapse and is currently serving 14 years prison sentence.



Allen Stanford

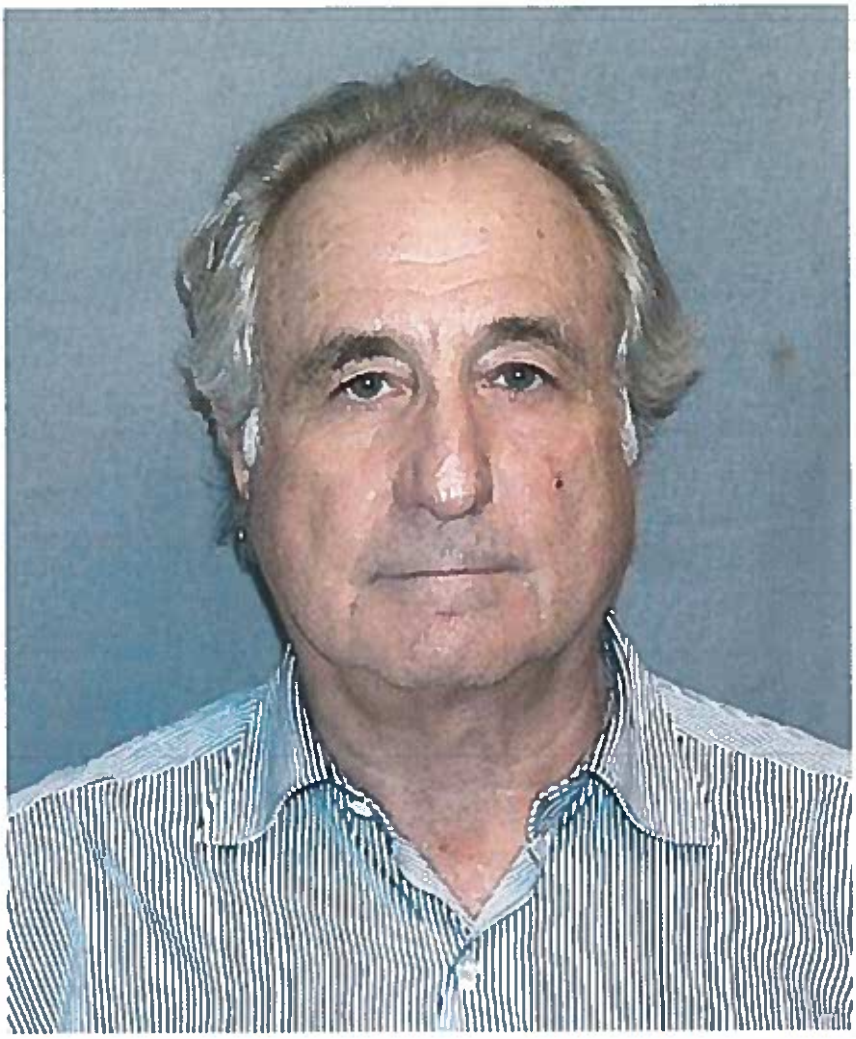
Stanford is serving a 110 year prison sentence, having been convicted of charges of fraud. Stanford was the chairman of the now defunct Standord Financial Group of companies.

Jordan Belfort

Belfort founded the brokeage firm Stratton Oakmont, and went on to defraud investors through sales of stock while employing more than 1,000 people. The firm was shut in 1998. Belfort was charged with money laundering and fraud.

Bruno Iksil

Former trader for JPMorgan Chase & Co and also known as "the London Whale". His position in the credit default swap market caused more than $6.2 billion in losses for the firm in 2012.



Bernard Madoff

Former stockbroker and investment adviser. His name is connected to the most well known Ponzi scheme fraud. He earned $62 billion through such fraud and is currently sentenced to a maximum of 150 years in prison.

Ivan Boesky

Boesky is a former American stock trader who is notable for his prominent role in a Wall Street insider trading scandal that occurred in the US in the mid-1980s. In 1987, Boesky was sentenced to 3 years in prison.

Barry Minkow

Founded ZZZZ Best which appeared to be an immensely successful carpet cleaning and restoration company. However, it was actually a front to attract investment for a massive Ponzi scheme. It collapsed in 1987 costing investors and lenders $100 million.

Martha Stewart

Martha is an American businesswoman, writer and television personality. In 2004, Stewart was convicted relating to the ImClone insider trading affair and sentenced to prison.

Bernard Ebbers

A Canadian born buisnessman and cofounded the telecommunications company WorlCom. In 2005, he was convicted of fraud and conspiracy as a result of WorldCom's false financial reporting. He was sentenced to 25 years for his role in an $11 billion accounting fraud. It is the largest accounting scandal in US history.

Donald Johnson

Executive of stock exchange for Nasdaq's. Convicted in 2011 of using insider information to trade shares of United Therapeutics, Honda and other companies from 2006 to 2009. He was sentenced to 3 and half years in prison for a total of $755,000 of fraud.

Doug Whitman

Managing more than $100 million through his Whitman Capital hedge fund. He was convicted in August 2012 of insider trading charges involving stocks of Google, Polycom and Marvell Technology Group. The total value of his fraud accumulated to $1 million and lead to 2 years in prison.

Raj Rajaratnam

Former head of Galleon Capital. He received the longest insider trading sentence in history of 11 years for securities fraud and five counts of conspiracy. He was ordered to pay back the $53.8 million of fraud.

Albert H. Wiggin

Career spaning from head of Chase National Bank and now JP Morgan chase. He shorted his own company with 40000 shares in 1929 and profited immensely to the tune of $4 million and wasn't even found guilty of breaking any laws.

LECTURE

25 SEPTEMBER

9:00-10:00AM

MATH3/4/68181

Definition I

Suppose $X_1, X_2, \ldots, X_n$ are IID with cumulative distribution function (CDF) $F(\cdot)$. The extreme value is

$$M_n = \max(X_1, \ldots, X_n).$$

For modeling purposes, we need to know the distribution of M_n

$$P(M_n \leq x)$$

$$= P(\max(X_1, \ldots, X_n) < x)$$

$$= P(X_1 < x, \ldots, X_n < x)$$

$$\overset{\text{indep}}{=} P(X_1 < x) \cdots P(X_n < x)$$

$$\overset{\text{identical}}{=} F(x) \cdots F(x)$$

$$= [F(x)]^n$$

What is the limit of $P(M_n < x)$ as $n \longrightarrow \infty$?

$$\lim_{n \to \infty} P(M_n < x) = \lim_{n \to \infty} [F(x)]^n = \begin{cases} 0 & \text{if } F(x) < 1 \\ 1 & \text{if } F(x) = 1 \end{cases}$$

DEGENERATE LIMIT

THIS IS NOT A USEFUL RESULT

SLLN

(Strong Law of Law Numbers)

Suppose $X_1, \ldots, X_n$ are IID with population mean μ. Then

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \longrightarrow \mu$$

as $n \longrightarrow \infty$

HOW TO MAKE THIS RESULT INTO SOMETHING USEFUL?

CLT

(Central Limit Theorem)

Suppose $X_1, \ldots, X_n$ are IID with population mean μ and population variance σ^2. Then

Scaled version of $\overline{X}$

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0, 1)$$

as $n \longrightarrow \infty$

Let us look at the limit of a scaled version of M_n.

$$P\left(\frac{M_n - b_n}{a_n} \leq x \right)$$

scaled version of M_n

$$= P(M_n \leq a_n x + b_n)$$

$$= P(\max(X_1, \ldots, X_n) \leq a_n x + b_n)$$

$$= P(X_1 \leq a_n x + b_n, \ldots, X_n \leq a_n x + b_n)$$

$$\overset{\text{indep}}{=} P(X_1 \leq a_n x + b_n) \ldots P(X_n \leq a_n x + b_n)$$

$$\overset{\text{identical}}{=} F(a_n x + b_n) \ldots F(a_n x + b_n)$$

$$= [F(a_n x + b_n)]^n \quad \ldots \quad (*)$$

WHAT IS THE LIMIT OF $(*)$ AS $n \longrightarrow \infty$?

ETT

(Extremal Types Theorem)

Suppose $X_1, \ldots, X_n$ are IID with CDF $F(\cdot)$. Let $M_n = \max(X_1, \ldots, X_n)$. If there exist $a_n > 0$ and $-\infty < b_n < \infty$ such that

$$(*) \longrightarrow G(x)$$

as $n \to \infty$ for a non-degenerate G then it must be of the same type as

$$\text{I}: \quad \Lambda(x) = e^{-e^{-x}}, \quad -\infty < x < \infty$$

GUMBEL

$$\text{II}: \quad \Phi_\alpha(x) = \begin{cases} 0 & x < 0 \\ e^{-x^{-\alpha}} & x \geq 0 \end{cases}$$

FRÉCHET

$$\text{III}: \quad \Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha} & x < 0 \\ 1 & x \geq 0 \end{cases}$$

WEIBULL

SAME TYPE

Two CDFs G_1 and G_2 are of the same type if

$$G_1(x) = G_2(ax+b)$$

for all x for some $a > 0$ and $b \in \mathbb{R}$

Ex 1

$$G_1(x) = e^{-e^{-x}}$$

$$G_2(x) = e^{-e^{-2x-3}}$$

G_1 & G_2 are of the same type

Ex 2

$$G_1(x) = e^{-\frac{1}{x}}$$

$$G_2(x) = e^{-\frac{1}{3-2x}}$$

G_1 & G_2 are not of the same type

HOW TO KNOW WHICH OF THE 3 LIMITS IS ATTAINED?

$w(F)$ = "upper end point" of $F(\cdot)$

$= \sup \{ x : F(x) < 1 \}$

Set $F(x) = 1 \Rightarrow$ solve for x

Ex1 $F(x) = 1 - e^{-x}$

$F(x) = 1 \Rightarrow 1 - e^{-x} = 1 \Rightarrow e^{-x} = 0$

$\Rightarrow x = +\infty \Rightarrow w(F) = +\infty$

Ex2 $F(x) = 1 - \frac{1}{x}, \quad x > 1$

$F(x) = 1 \Rightarrow 1 - \frac{1}{x} = 1 \Rightarrow \frac{1}{x} = 0$

$\Rightarrow x = +\infty \Rightarrow w(F) = +\infty$

CONDITIONS TO CHECK FOR THE 3 LIMITS

I: there exist $\gamma(t) > 0$

such that

$$\lim_{t \to w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = e^{-x}$$

II: $w(F) = +\infty$ and

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \quad \alpha > 0, \; x > 0$$

III: $w(F) < +\infty$ and

$$\lim_{t \to 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^{\alpha}, \quad \alpha > 0, \; x > 0$$

Only one these conditions (if any) will be satisfied.

Ex 1

$$F(x) = 1 - e^{-x}$$

$$\omega(F) = +\infty$$

I : $$\lim_{t \to \omega} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{1 - [1 - e^{-t - x\gamma(t)}]}{1 - [1 - e^{-t}]}$$

$$= \lim_{t \to \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t \to \infty} e^{-x\gamma(t)}$$

$$= e^{-x} \quad \text{if} \quad \gamma(t) \equiv 1 \text{ for all } t.$$

$\Rightarrow$ Cond (I) is satisfied.

By the ETT, there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$[F(a_n x + b_n)]^n \longrightarrow e^{-e^{-x}}$$

as $n \longrightarrow \infty$.

(GUMBEL LIMIT)

LECTURE

28 SEPTEMBER

10:00-11:00AM

MATH4/68181

Sometime $f(x) = \frac{dF(x)}{dx}$ (the PDF)

is more convenient than $F(x)$.

Ex $X \sim N(0, 1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$F(x) = \Phi(x)$$

CONDITIONS IN TERMS OF PDF

Let $a(t) = F^{-1}\left(1 - \frac{1}{t}\right)$

$b(t) = t\, f(a(t))$

I : $\frac{b(tx)}{b(t)} \longrightarrow 1$ as $t \to \infty$

II : $\omega(F) = \infty$ and $\lim_{t \to \infty} a(t)\, b(t) = \alpha$

III : $\omega(F) < \infty$ and $\lim_{t \to \infty} \{\omega(F) - a(t)\}\, b(t) = \alpha$

Only one of these conditions (if any) will be satisfied.

Ex 1

$$F(x) = 1 - e^{-x} = y \Rightarrow e^{-x} = 1-y$$
$$\Rightarrow x = -\log(1-y)$$

$$F^{-1}(x) = -\log(1-y)$$

$$a(t) = F^{-1}\left(1-\frac{1}{t}\right) = -\log\left(1-\left(1-\frac{1}{t}\right)\right)$$
$$= -\log\left(\frac{1}{t}\right)$$
$$= \log t$$

$$f(x) = e^{-x}$$

$$b(t) = t \cdot f(a(t))$$
$$= t \cdot e^{-\log t}$$
$$= 1$$

I: $\dfrac{b(tx)}{b(t)} = \dfrac{1}{1} = 1$ for all t

Hence (I) is satisfied

By the ETT, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) \longrightarrow e^{-e^{-x}}$$

as $n \longrightarrow \infty$.

Ex 2 $F(x) = 1 - \frac{1}{x}, \quad x > 1$

$w(F) = +\infty.$

$F(x) = y \Rightarrow 1 - \frac{1}{x} = y$

$\Rightarrow \frac{1}{x} = 1 - y$

$\Rightarrow x = \frac{1}{1-y}$

$a(t) = F^{-1}\left(1 - \frac{1}{t}\right) = \frac{1}{1-\left(1-\frac{1}{t}\right)} = t$

$f(x) = \frac{1}{x^2}$

$b(t) = t \cdot f(a(t)) = t \cdot \frac{1}{t^2} = \frac{1}{t}.$

I: $\frac{b(tx)}{b(t)} = \frac{\frac{1}{tx}}{\frac{1}{t}} = \frac{1}{x}$ for all t

$\neq 1$

Cond (I) is not satisfied

II: $w(F) = +\infty$ ✓

$\lim_{t \to \infty} a(t)\, b(t) = \lim_{t \to \infty} t \cdot \frac{1}{t} = 1 = \kappa$

Cond (II) is satisfied

By the ETT, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) \longrightarrow \begin{cases} 0, & x < 0 \\ e^{-x^{-1}}, & x \geq 0 \end{cases}$$

as $n \longrightarrow \infty$.

Ex 3

$X \sim \text{Uni}[0, 1]$

$F(x) = x, \quad 0 < x < 1$

$\omega(F) = 1$

$f(x) = 1, \quad 0 < x < 1$

$b(t) = t\, f(a(t)) = t$

I : $\dfrac{b(tx)}{b(t)} = \dfrac{tx}{t} = x \neq 1$

Cond (I) is <u>not</u> satisfied.

II : $\omega(F) = 1 \neq \infty$

condition (II) is not satisfied

III : $F^{-1}(x) = x$

$a(t) = F^{-1}\left(1 - \frac{1}{t}\right) = 1 - \frac{1}{t}$

$\lim_{t \to \infty} \{\omega(F) - a(t)\}\, b(t)$

$= \lim_{t \to \infty} \left(1 - \left(1 - \frac{1}{t}\right)\right) t$

$= 1 = \alpha$

condition (III) is satisfied.

By the ETT, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) \longrightarrow \begin{cases} e^x & x < 0 \\ 1 & x \geq 0 \end{cases}$$

as $n \longrightarrow \infty$.

The ETT uses a linear normalization.

$$P\left(\frac{M_n - b_n}{a_n} \leq x \right) = \left[F(a_n x + b_n) \right]^n$$

Try some other normalization. A power normalization is

$$P\left(\left| \frac{M_n}{a_n} \right|^{\frac{1}{b_n}} \operatorname{sign}(M_n) \leq x \right)$$

$$\operatorname{sign}(y) = \begin{cases} +1 & y > 0 \\ -1 & y < 0 \\ 0 & y = 0 \end{cases}$$

$$= F^n\left(a_n |x|^{b_n} \operatorname{sign}(x) \right) \qquad (**)$$

E T T

for power normalization

If there exists $a_n > 0$ and $b_n > 0$ such that

$$(**) \longrightarrow G(x)$$

as $n \to \infty$ for a non-degenerate G then G must be of the same type as

$$\text{I}: \quad G(x) = \begin{cases} 0 & x \le 1 \\ e^{-(\log x)^{-\alpha}} & x > 1 \end{cases}, \quad \alpha > 0$$

$$\text{II}: \quad G(x) = \begin{cases} 0 & x < 0 \\ e^{-|\log x|^{\alpha}} & 0 \le x < 1 \\ 1 & x > 1 \end{cases} \quad \alpha > 0$$

$$\text{III}: \quad G(x) = \begin{cases} 0 & x \le -1 \\ e^{-|\log |x||^{-\alpha}} & -1 < x < 0 \\ 1 & x \ge 0 \end{cases} \quad \alpha > 0$$

$$\text{IV}: \quad G(x) = \begin{cases} e^{-(\log |x|)^{\alpha}} & x < -1 \\ 1 & x \ge -1 \end{cases}$$

$$\text{V}: \quad G(x) = \Phi_1(x)$$

$$\text{VI}: \quad G(x) = \Psi_1(x).$$

G_1 & G_2 are of the same type if

$$G_1(x) = G_2(a|x|^b \operatorname{sign}(x))$$

for all x, where $a > 0$ and $b > 0$.

LECTURE

1 OCTOBER

9:00-10:00AM

MATH3/4/68181

Ex 2 $F(x) = \Phi(x), \quad -\infty < x < \infty$

$= \text{CDF of } N(0,1)$

$F(x) = 1 \Rightarrow \Phi(x) = 1 \Rightarrow x = \Phi^{-1}(1) = \infty$

$\Rightarrow \omega(F) = \infty$

I: $\lim_{t \to \infty} \dfrac{1 - \Phi(t + x\gamma(t))}{1 - \Phi(t)}$ → 1 / → 1

$= \dfrac{1-1}{1-1} = \dfrac{0}{0}$

L'Hopitals Rule

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

if $\lim f(x) = \lim g(x) = \pm\infty$

or $\lim f(x) = \lim g(x) = 0$

$\overset{LH}{=} \lim_{t \to \infty} \dfrac{-\phi(t + x\gamma(t)) \cdot (1 + x\gamma'(t))}{-\phi(t)}$

$= \lim_{t \to \infty} \dfrac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(t+x\gamma(t))^2}{2}} (1 + x\gamma'(t))}{\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}}$

$= \lim_{t \to \infty} e^{\frac{t^2}{2} - \frac{(t + x\gamma(t))^2}{2}} \cdot (1 + x\gamma'(t))$

$= \lim_{t \to \infty} e^{-t x\gamma(t) - \frac{x^2(\gamma(t))^2}{2}} \cdot (1 + x\gamma'(t))$

Choose $\gamma(t) = \frac{1}{t}$

$\gamma'(t) = -\frac{1}{t^2}$

$(\gamma(t))^2 = \frac{1}{t^2}$

$= \lim_{t \to \infty} e^{-x - \frac{x^2}{2t^2}} \cdot \left(1 + \frac{x}{(-t^2)}\right)$

$= e^{-x} \Rightarrow$ cond (I) is satisfied

By the ETT, there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\left[F(a_n x + b_n)\right]^n \longrightarrow e^{-e^{-x}}$$

as $n \longrightarrow \infty$.

Ex 3 $F(x) = e^{-\frac{1}{x}}$, $x > 0$

$F(x) = 1 \Rightarrow e^{-\frac{1}{x}} = 1 \Rightarrow -\frac{1}{x} = 0$

$\Rightarrow x = \infty \Rightarrow \omega(F) = \infty$.

I: $$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{1 - e^{-\frac{1}{t + x\gamma(t)}}}{1 - e^{-\frac{1}{t}}}$$

$e^{-y} \approx 1 - y$ as $y \to 0$ (*)

$$= \lim_{t \to \infty} \frac{1 - \left(1 - \frac{1}{t + x\gamma(t)}\right)}{1 - \left(1 - \frac{1}{t}\right)}$$

$$= \lim_{t \to \infty} \frac{t}{t + x\gamma(t)} \neq e^{-x}$$

$\Rightarrow$ cond (I) is not satisfied

II: $$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{1 - e^{-\frac{1}{tx}}}{1 - e^{-\frac{1}{t}}}$$

(*) $$= \lim_{t \to \infty} \frac{1 - \left(1 - \frac{1}{tx}\right)}{1 - \left(1 - \frac{1}{t}\right)} = \lim_{t \to \infty} \frac{\frac{1}{tx}}{\frac{1}{t}} = \frac{1}{x}$$

$\Rightarrow$ Cond (II) is satisfied with $\alpha = 1$

By the ETT, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) \longrightarrow \begin{cases} 0 & x < 0 \\ e^{-\frac{1}{x}} & x \geq 0 \end{cases}$$

AN EXAMPLE WHERE ETT FAILS

$$F(x) = 1 - \frac{1}{\log x}, \quad x > e$$

[log is log to the base e]

$$F(x) = 1 \Rightarrow 1 - \frac{1}{\log x} = 1 \Rightarrow \frac{1}{\log x} = 0$$

$$\Rightarrow \log x = \infty \Rightarrow x = \infty \Rightarrow w(F) = \infty.$$

I: $$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{1 - \left[1 - \frac{1}{\log(t + x\gamma(t))}\right]}{1 - \left[1 - \frac{1}{\log t}\right]}$$

$$= \lim_{t \to \infty} \frac{\log t}{\log(t + x\gamma(t))} \quad \left(\log t \to \infty,\ \log(t + x\gamma(t)) \to \infty\right)$$

$$\overset{LH}{=} \lim_{t \to \infty} \frac{\frac{1}{t}}{\frac{1 + x\gamma'(t)}{t + x\gamma(t)}} = \lim_{t \to \infty} \frac{t + x\gamma(t)}{t[1 + x\gamma'(t)]}$$

$$\neq e^{-x}$$

$\Rightarrow$ Cond (I) is not satisfied

II : $$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{1 - \left[1 - \frac{1}{\log(tx)}\right]}{1 - \left[1 - \frac{1}{\log t}\right]}$$

$$= \lim_{t \to \infty} \frac{\log t}{\log(tx)}$$

$$= \lim_{t \to \infty} \frac{\log t}{\log t + \log x}$$

$$= \lim_{t \to \infty} \frac{1}{1 + \frac{\log x}{\log t}} \to 0 \neq 1 - x^{-\alpha}$$

$\Rightarrow$ Cond (II) is not satisfied

III : $w(F) = \infty$ is not finite

$\Rightarrow$ Cond (III) is not satisfied

Hence, ETT fails.

How to Choose a_n and b_n

I : $a_n = \gamma\left(F^{-1}\left(1-\frac{1}{n}\right)\right)$ and $b_n = F^{-1}\left(1-\frac{1}{n}\right)$

II : $a_n = F^{-1}\left(1-\frac{1}{n}\right)$ and $b_n = 0$

III : $a_n = w(F) - F^{-1}\left(1-\frac{1}{n}\right)$ and $b_n = w(F)$.

Ex 1 $F(x) = 1 - e^{-x}$

There exist $a_n > 0$ and $b_n \in R$ such that

$$[F(a_n x + b_n)]^n \longrightarrow e^{-e^{-x}}$$

as $n \longrightarrow \infty$.

$$F(x) = y \Rightarrow 1 - e^{-x} = y$$

$$\Rightarrow e^{-x} = 1 - y$$

$$\Rightarrow x = -\log(1-y)$$

$$\Rightarrow F^{-1}(x) = -\log(1-x)$$

$$a_n = \gamma\left(F^{-1}\left(1 - \frac{1}{n}\right)\right) \qquad [\gamma(t) \equiv 1]$$

$$= 1$$

$$b_n = F^{-1}\left(1 - \frac{1}{n}\right) = -\log\left(1 - \left(1 - \frac{1}{n}\right)\right)$$

$$= \log n$$

$$\Rightarrow \left[F(x + \log n)\right]^n \longrightarrow e^{-e^{-x}}$$

as $n \longrightarrow \infty$.

For a practitioner, dealing with 3 distributions is not convenient.

Is there a distribution that unifies the 3 distributions into a single form?

Yes.

GEV (Generalized Extreme Value) Distribution has the CDF

$$G(x) = e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

where $1+\xi\frac{x-\mu}{\sigma} > 0$

$-\infty < \xi < \infty$ "shape"

$-\infty \quad \mu < \infty$ "location"

$\sigma > 0$ "scale"

LECTURE

2 OCTOBER

9:00-10:00AM

MATH3/4/68181

For a practitioner, dealing with 3 distributions is not convenient.

Is there a distribution that unifies the 3 distributions into a single form?

Yes.

GEV (Generalized Extreme Value) Distribution has the CDF

$$G(x) = e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

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$-\infty < \xi < \infty$ "shape"

$-\infty < \mu < \infty$ "location"

$\sigma > 0$ "scale"

Particular Cases

i) $\boxed{\xi = 0}$

$$\lim_{\xi \to 0} e^{-\left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= \lim_{\xi \to 0} e^{-\left(1+\frac{\frac{x-\mu}{\sigma}}{\frac{1}{\xi}}\right)^{-\frac{1}{\xi}}}$$

Set $m = \frac{1}{\xi}$

$$= \lim_{m \to \infty} e^{-\left(1+\frac{\frac{x-\mu}{\sigma}}{m}\right)^{-m}}$$

$$= \lim_{m \to \infty} e^{-\left[\left(1+\frac{x-\mu}{m}\right)^{m}\right]^{-1}}$$

$\left(1+\frac{y}{m}\right)^{m} \to e^{y}$ as $m \to \infty$

$$= e^{-\left[e^{\frac{x-\mu}{\sigma}}\right]^{-1}}$$

$$= e^{-e^{-\frac{x-\mu}{\sigma}}}$$

the same type as $\Lambda(x) = e^{-e^{-x}}$

$\Rightarrow \Lambda(x)$ is a particular case of GEV when $\xi = 0$

ii) $\boxed{\xi > 0}$

$$e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= e^{-\left(1-\frac{\xi\mu}{\sigma}+\frac{\xi x}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$\boxed{\text{Set} \quad \alpha = \frac{1}{\xi}}$$

$$= e^{-\left(1-\frac{\mu}{\alpha\sigma}+\frac{x}{\sigma\alpha}\right)^{-\alpha}}$$

$$\boxed{a = \frac{1}{\sigma\alpha}, \; b = 1-\frac{\mu}{\alpha\sigma}} \qquad a > 0, \; b \in \mathbb{R}$$

$$= e^{-(b+ax)^{-\alpha}}$$

= the same type as $\Phi_\alpha(x)$.

$\Rightarrow$ $\Phi_\alpha(x)$ is a particular case of GEV when $\xi > 0$.

iii) $\boxed{\xi < 0}$

$$e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= e^{-\left(1-\frac{\xi\mu}{\sigma}+\frac{\xi x}{\sigma}\right)^{-\frac{1}{\xi}}}$$

Set $\alpha = -\frac{1}{\xi}$

$$= e^{-\left(1+\frac{\mu}{\alpha\sigma}-\frac{x}{\alpha\sigma}\right)^{\alpha}}$$

$$= e^{-\left(-\left(-1-\frac{\mu}{\alpha\sigma}+\frac{x}{\alpha\sigma}\right)\right)^{\alpha}}$$

$a = \frac{1}{\alpha\sigma}$, $b = -1-\frac{\mu}{\alpha\sigma}$ $\quad a > 0$, $b \in \mathbb{R}$

$$= e^{-\left(-\left(b+ax\right)\right)^{\alpha}}$$

$$= \text{same type as } \Psi_\alpha(x)$$

$\Rightarrow$ $\Psi_\alpha(x)$ is a particular case of GEV when $\xi < 0$.

GEV Distribution

CDF

$$G(x) = e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

PDF

$$g(x) = \frac{d\,G(x)}{dx}$$

$$= \frac{1}{\sigma}\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \cdot e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

Quantile function

$$e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} = p, \quad 0 < p < 1$$

$$\Rightarrow -\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} = \log p$$

$$\Rightarrow \left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} = -\log p$$

$$\Rightarrow 1+\xi\frac{x-\mu}{\sigma} = (-\log p)^{-\xi}$$

$$\Rightarrow x = \mu + \frac{\sigma}{\xi}\left[(-\log p)^{-\xi} - 1\right]$$

pth quantile

For example, if $p = \frac{1}{2}$

$$\text{Median} = \mu + \frac{\sigma}{\xi}\left[(\log 2)^{-\xi} - 1\right]$$

Return Level

is the quantile when

$$p = 1 - \frac{1}{T}$$

T — No. of years

For example if $T = 100$ years then the quantile will be the extreme valve that occurs on average once in 100 years.

Return level for T years

$$= \mu + \frac{\sigma}{\xi}\left[\left(-\log\left(1 - \frac{1}{T}\right)\right)^{-\xi} - 1\right].$$

ML estimation

(Maximum Likelihood Estimation)

Suppose $x_1, x_2, \ldots, x_n$ is a random sample from the GEV distribution.

$$L(\mu, \sigma, \xi) = \prod_{i=1}^{n} g(x_i)$$

$$= \prod_{i=1}^{n} \left[\frac{1}{\sigma} \left(1 + \xi \frac{x_i - \mu}{\sigma} \right)^{-\frac{1}{\xi} - 1} e^{-\left(1 + \xi \frac{x_i - \mu}{\sigma} \right)^{-\frac{1}{\xi}}} \right]$$

$$= \frac{1}{\sigma^n} \left[\prod_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma} \right) \right]^{-\frac{1}{\xi} - 1} e^{-\sum_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma} \right)^{-\frac{1}{\xi}}}.$$

$$\log L(\mu, \sigma, \xi) = -n \log \sigma - \left(\frac{1}{\xi} + 1 \right) \sum_{i=1}^{n} \log \left(1 + \xi \frac{x_i - \mu}{\sigma} \right) - \sum_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma} \right)^{-\frac{1}{\xi}}$$

MLE equations for the GEV distribution

The MLEs of μ, σ and ξ are the simultaneous solutions of

$$\begin{aligned}\frac{\partial \log L}{\partial \mu} &= \frac{1+\xi}{\sigma}\sum_{i=1}^{n}\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-1} \\ &\quad -\frac{1}{\sigma}\sum_{i=1}^{n}\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0, \qquad \cdots \qquad (1)\end{aligned}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \sigma} &= -\frac{n}{\sigma}+\frac{1+\xi}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-1} \\ &\quad -\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0, \qquad \cdots \qquad (2)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \xi} &= \frac{1}{\xi^2}\sum_{i=1}^{n}\log\left(1+\xi\frac{x_i-\mu}{\sigma}\right) \\ &\quad -\frac{1+\xi}{\xi\sigma}\sum_{i=1}^{n}(x_i-\mu)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-1} \\ &\quad -\frac{1}{\xi^2}\sum_{i=1}^{n}\log\left(1+\xi\frac{x_i-\mu}{\sigma}\right)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}} \\ &\quad +\frac{1}{\xi\sigma}\sum_{i=1}^{n}(x_i-\mu)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0. \qquad \cdots \qquad (3)\end{aligned}$$

The MLEs of μ, σ and ξ are the simultaneous solutions of (1), (2) and (3).

`fgev` in the R software will compute the MLEs $\hat{\mu}$, $\hat{\sigma}$ and $\hat{\xi}$.

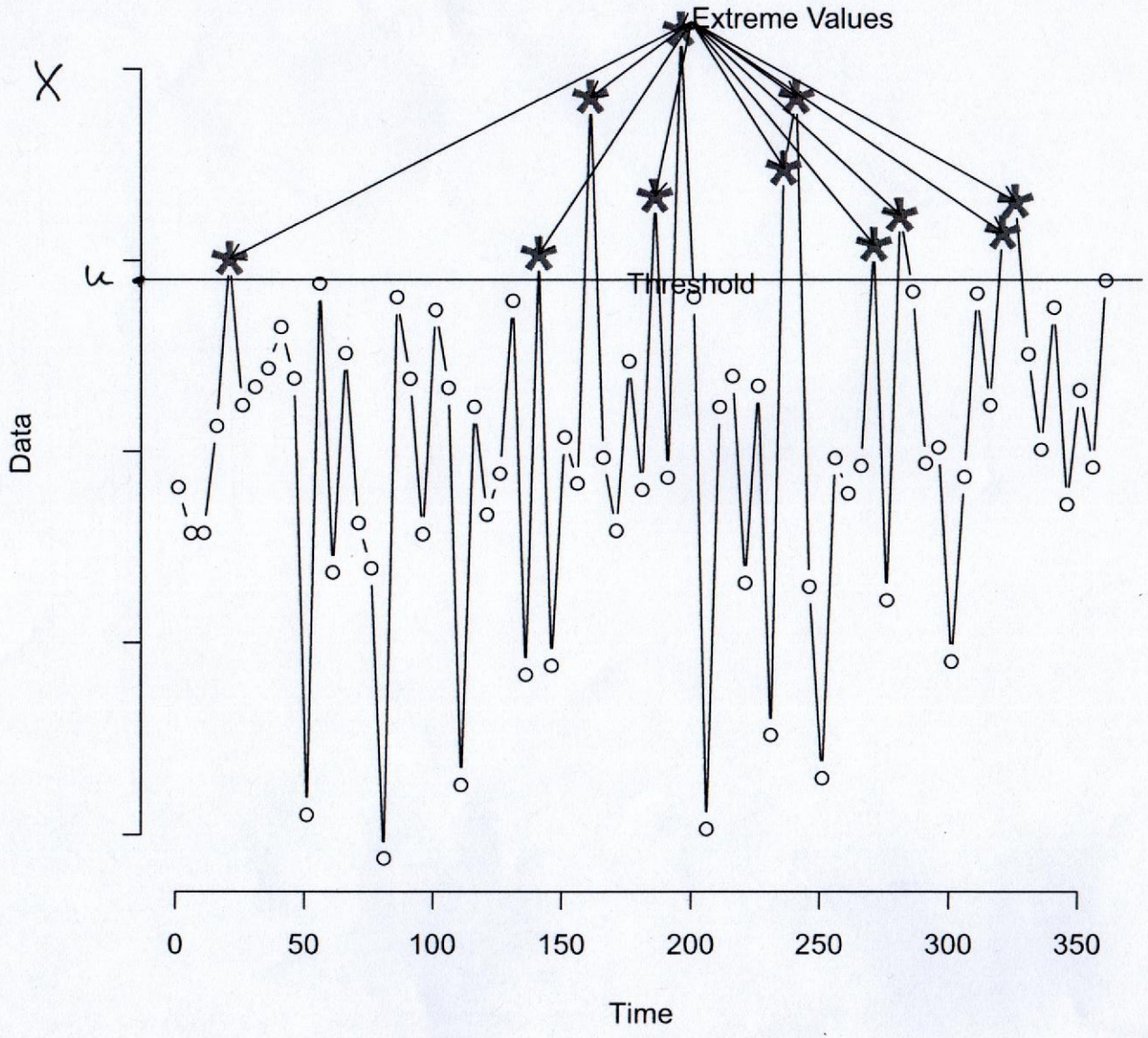
Definition 2

X = variable of interest (eg stock returns)

u = Threshold (considered high enough)

X is an extreme value if $X > u$

$X - u$ = Excess amount



Definition 2

What is the distribution of

$X - u$ (the excess) if $X > u$

(that is X is an extreme value)?

$$P(X - u < x \mid X > u) = ?$$

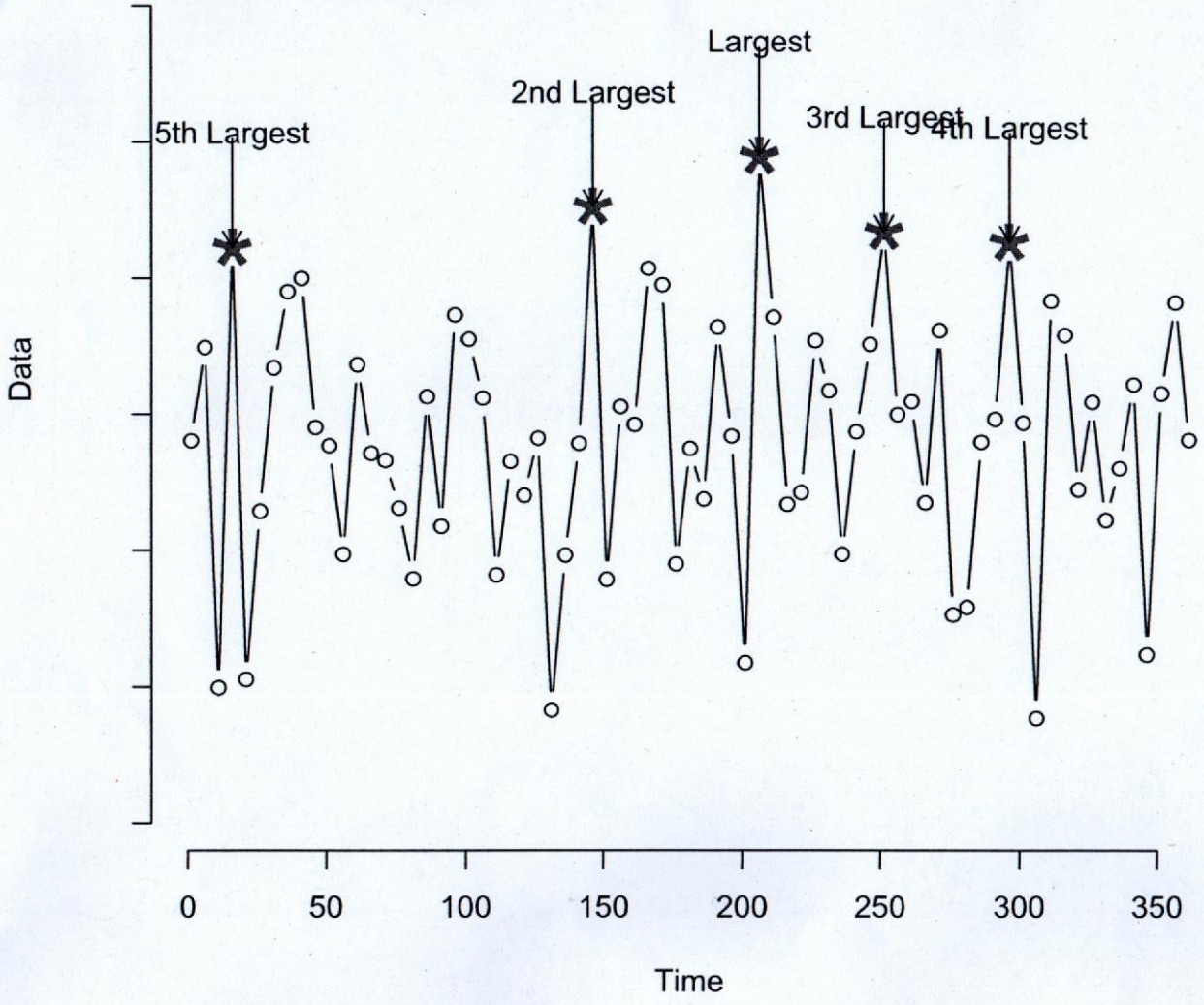
LECTURE

5 OCTOBER

10:00-11:00AM

MATH4/68181

Definition : "r largest" method



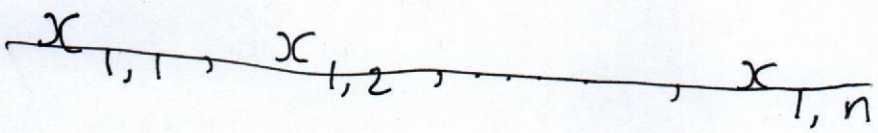
Extreme Values

Level 4 & 6 students

- Defn 1
- Defn 2
- Defn 3

Level 3 students

ML Estimation



$x_{1,1}, x_{2,1}, \ldots, x_{n,1}$

(largest obsns for n years)

$x_{1,2}, x_{2,2}, \ldots, x_{n,2}$

(2nd largest obsns for n years)

$\vdots$

$x_{1,r}, x_{2,r}, \ldots, x_{n,r}$

(r th largest obsn for n years)

Definition 3

Suppose $X_1, X_2, \ldots, X_n$ are IID observations with CDF $F(\cdot)$

Let $M_n^{(1)}$ = largest of $X_1, \ldots, X_n$

$M_n^{(2)}$ = 2nd " " "

$\vdots$

$M_n^{(r)}$ = r^{th} " " "

It can be shown under certain conditions that

$$\Pr\left(\frac{M_n^{(1)} - b_n}{a_n} < x_1, \ldots, \frac{M_n^{(r)} - b_n}{a_n} < x_r \right)$$

$$\xrightarrow[\text{as } n \to \infty]{} \sum_{s_1=0}^{1} \sum_{s_2=0}^{2-s_1} \cdots \sum_{s_{r-1}=0}^{r-1-s_1-\cdots-s_{r-2}} \frac{(\gamma_2 - \gamma_1)^{s_1}}{s_1!} \cdots \frac{(\gamma_r - \gamma_{r-1})^{s_{r-1}}}{s_{r-1}!} e^{-\gamma_r} \quad (*)$$

where $\gamma_i = \left(1 + \xi \frac{x_i - 0}{1}\right)^{-\frac{1}{\xi}}$

$= (1 + \xi x_i)^{-\frac{1}{\xi}}$, $i = 1, 2, \ldots, r$

It can be deduced from (*) that the joint PDF of $(M_n^{(1)}, M_n^{(2)}, \ldots, M_n^{(r)})$ is

$$f(x_1, x_2, \ldots, x_r)$$

$$= \sigma^{-r} e^{-\left(1+\xi\frac{x_r-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} \times e^{-\left(\frac{1}{\xi}+1\right)\sum_{i=1}^{r}\log\left(1+\frac{x_i-\mu}{\sigma}\right)}$$

where $x_1 \geq x_2 \geq \ldots \geq x_r$ and

$$1+\xi\frac{(x_i-\mu)}{\sigma} > 0, \quad i = 1, 2, \ldots, r$$

$$L(\mu, \sigma, \xi) = \prod_{i=1}^{n} f(x_{i,1}, x_{i,2}, \ldots, x_{i,r})$$

$$= \prod_{i=1}^{n} \left[\sigma^{-r} e^{-\left(1+\xi \frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} \right.$$

$$\left. e^{-\left(1+\frac{1}{\xi}\right) \sum_{j=1}^{r} \log\left(1+\frac{x_{i,j}-\mu}{\sigma}\right)} \right]$$

$$= \sigma^{-nr} e^{-\sum_{i=1}^{n} \left(1+\xi \frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$e^{-\left(1+\frac{1}{\xi}\right) \sum_{i=1}^{n} \sum_{j=1}^{r} \log\left(1+\frac{x_{i,j}-\mu}{\sigma}\right)}$$

.

$$\Rightarrow \log L(\mu, \sigma, \xi) = -nr \log \sigma$$

$$- \sum_{i=1}^{n} \left(1+\xi \frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$- \left(1+\frac{1}{\xi}\right) \sum_{i=1}^{n} \sum_{j=1}^{r} \log\left(1+\frac{x_{i,j}-\mu}{\sigma}\right).$$

MLE equations for the r largest distribution

The MLEs of μ, σ and ξ are the simultaneous solutions of

$$\begin{aligned}\frac{\partial \log L}{\partial \mu} &= -\frac{1}{\sigma}\sum_{i=1}^{n}\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &\quad -\frac{1+\xi}{\sigma}\sum_{i=1}^{n}\sum_{j=1}^{r}\left(1+\xi\frac{x_{i,j}-\mu}{\sigma}\right)^{-1} \\ &= 0, \qquad (1)\end{aligned}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \sigma} &= -\frac{nr}{\sigma}-\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_{i,r}-\mu)\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &\quad +\frac{1+\xi}{\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{r}(x_{i,j}-\mu)\left(1+\xi\frac{x_{i,j}-\mu}{\sigma}\right)^{-1} \\ &= 0, \qquad (2)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \xi} &= -\frac{1}{\xi^2}\sum_{i=1}^{n}\log\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}} \\ &\quad +\frac{1}{\xi\sigma}\sum_{i=1}^{n}(x_{i,r}-\mu)\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &\quad +\frac{1}{\xi^2}\sum_{i=1}^{n}\sum_{j=1}^{r}\log\left(1+\xi\frac{x_{i,j}-\mu}{\sigma}\right) \\ &\quad -\frac{1+\xi}{\xi\sigma}\sum_{i=1}^{n}\sum_{j=1}^{r}(x_{i,j}-\mu)\left(1+\xi\frac{x_{i,j}-\mu}{\sigma}\right)^{-1} \\ &= 0. \qquad (3)\end{aligned}$$

The MLEs of μ, σ & ξ are the simultaneous solutions of (1), (2) and (3). The R package evd can compute the MLEs μ, σ & ξ.

$e^{y} \approx 1 + y$ small y

$e^{y} = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

$\underline{\gamma(t)} > 0 \quad \forall\, t$

EXAMPLE CLASS

1 OCTOBER

12:00-13:00PM

MATH3/4/68181

Q1

$$\Lambda(x) = e^{-e^{-x}}$$

$$\Lambda'(x) = e^{-e^{-x}} (-1) e^{-x} (-1)$$

$$= e^{-e^{-x}} e^{-x}$$

Q2

$$\Phi_\alpha(x) = \begin{cases} 0 & x < 0 \\ e^{-x^{-\alpha}} & x \geq 0 \end{cases}$$

$$\Phi_\alpha'(x) = \begin{cases} 0 & x < 0 \\ e^{-x^{-\alpha}} (-1)(-\alpha) x^{-\alpha-1} & x \geq 0 \end{cases}$$

$$= \begin{cases} 0 & x < 0 \\ \alpha x^{-\alpha-1} e^{-x^{-\alpha}} & x \geq 0 \end{cases}$$

Q3

$$\Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha} & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$$\Psi_\alpha'(x) = \begin{cases} e^{-(-x)^\alpha} (-1)(-x)^{\alpha-1} \cdot \alpha \cdot (-1) & x < 0 \\ 0 & x \geq 0 \end{cases}$$

$$= \begin{cases} \alpha (-x)^{\alpha-1} e^{-(-x)^\alpha} & x < 0 \\ 0 & x \geq 0 \end{cases}$$

Q2

$$\Lambda(x) = e^{-e^{-x}}$$

$$\Lambda'(x) = e^{-x} e^{-e^{-x}}$$

$$\text{Mean} = \int_{-\infty}^{\infty} x \cdot e^{-x} e^{-e^{-x}} \, dx$$

Set $y = e^{-x} \Rightarrow x = -\log y$

$$\frac{dx}{dy} = -\frac{1}{y}$$

$$= \int_{\infty}^{0} (-\log y)\, y\, e^{-y} \left(-\frac{1}{y}\right) dy$$

$$= -\int_{0}^{\infty} \log y \; e^{-y} \, dy$$

$$\frac{d\, y^a}{da} = y^a \log y$$

$$\left.\frac{dy^a}{da}\right|_{a=0} = y^0 \log y = \log y \quad \text{— (*)}$$

$$\overset{(*)}{=} -\int_{0}^{\infty} \left.\frac{dy^a}{da}\right|_{a=0} e^{-y} \, dy$$

$$= -\frac{d}{da}\left[\int_{0}^{\infty} y^a e^{-y} \, dy\right]\Bigg|_{a=0}$$

$$= -\frac{d}{da}\,\Gamma(a+1)\Big|_{a=0}$$

$$= -\,\Gamma'(a+1)\Big|_{a=0}$$

$$= -\,\Gamma'(1)$$

$$\Gamma(a) = \int_{0}^{\infty} t^{a-1} e^{-t} \, dt$$

GAMMA FUNCTION

Q3

$$\Lambda(x) = e^{-e^{-x}}$$

$$\Lambda'(x) = e^{-x}\, e^{-e^{-x}}$$

$$\text{Variance} = E(X^2) - (E(X))^2$$

$$= E(X^2) - [\Gamma'(1)]^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, e^{-x}\, e^{-e^{-x}}\, dx$$

Set $y = e^{-x} \Rightarrow x = -\log y$

$$\Rightarrow \frac{dx}{dy} = -\frac{1}{y}$$

$$= \int_{\infty}^{0} (\log y)^2\, y\, e^{-y} \left(-\frac{1}{y}\right) dy$$

$$= \int_{0}^{\infty} (\log y)^2\, e^{-y}\, dy$$

$$\frac{dy^a}{da} = y^a \log y$$

$$\Rightarrow \frac{d^2 y^a}{da^2} = y^a (\log y)^2$$

$$\Rightarrow \left.\frac{d^2 y^a}{da^2}\right|_{a=0} = y^0 (\log y)^2 = (\log y)^2 \quad \cdots (*)$$

(*)

$$\int_0^{\infty} \left.\frac{d^2 y^a}{da^2}\right|_{a=0} e^{-y}\, dy$$

$$= \frac{d^2}{da^2}\left[\int_0^{\infty} y^a e^{-y}\, dy\right]\Bigg|_{a=0}$$

$$= \frac{d^2}{da^2}\left[\Gamma(a+1)\right]\Big|_{a=0} = \Gamma''(1)$$

$$\text{Variance} = \Gamma''(1) - [\Gamma'(1)]^2.$$

Q4

$$\Lambda^n(x) = \Lambda(\alpha_n x + \beta_n)$$

$$\Leftrightarrow \left[e^{-e^{-x}}\right]^n = e^{-e^{-\alpha_n x - \beta_n}}$$

$$\Leftrightarrow e^{-ne^{-x}} = e^{-e^{-\alpha_n x - \beta_n}}$$

Take log

$$\Leftrightarrow -ne^{-x} = -e^{-\alpha_n x - \beta_n}$$

$$\Leftrightarrow ne^{-x} = e^{-\alpha_n x - \beta_n}$$

Take log

$$\Leftrightarrow \log n - x = -\alpha_n x - \beta_n$$

$$\Leftrightarrow -1 = -\alpha_n$$

$$\& \quad \log n = -\beta_n$$

$$\Leftrightarrow \alpha_n = 1 \quad \text{and} \quad \beta_n = -\log n.$$

~~https:/~~

https://minerva.it.manchester.ac.uk/~saralees/

extremes3.html

extremes4.html

extremes6.html

Q7 Done in lectures

Q8 $F(x) = \left[1 - e^{-x}\right]^{\alpha}$

$F(x) = 1 \Rightarrow \left[1 - e^{-x}\right]^{\alpha} = 1$

$\Rightarrow 1 - e^{-x} = 1 \Rightarrow e^{-x} = 0 \Rightarrow x = \infty.$

$\Rightarrow w(F) = \infty$

$$I: \lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{1 - \left[1 - e^{-t - x\gamma(t)}\right]^{\alpha}}{1 - \left[1 - e^{-t}\right]^{\alpha}}$$

$(1-a)^{\alpha} \approx 1 - \alpha a$ as $a \to 0$

$$= \lim_{t \to \infty} \frac{1 - \left[1 - \alpha e^{-t - x\gamma(t)}\right]}{1 - \left[1 - \alpha e^{-t}\right]}$$

$$= \lim_{t \to \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t \to \infty} e^{-x\gamma(t)}$$

$= e^{-x}$ if $\gamma(t) \equiv 1$ ~~$\forall t$~~

By the ETT, there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\left[F(a_n x + b_n)\right]^n \to e^{-e^{-x}}$$

as $n \to \infty$.

EXAMPLE CLASS

2 OCTOBER

16:00-17:00PM

MATH3/4/68181

Q1

$$\Lambda(x) = e^{-e^{-x}}$$

$$\Lambda'(x) = e^{-e^{-x}}(-1)e^{-x}(-1)$$

$$= e^{-e^{-x}} e^{-x}$$

$$\Phi_\alpha(x) = \begin{cases} 0 & x < 0 \\ e^{-x^{-\alpha}} & x \geq 0 \end{cases}$$

$$\Phi_\alpha'(x) = \begin{cases} 0 & x < 0 \\ e^{-x^{-\alpha}}(-1)(-\alpha)x^{-\alpha-1} & x \geq 0 \end{cases}$$

$$= \begin{cases} 0 & x < 0 \\ \alpha x^{\alpha-1} e^{-x^{-\alpha}} & x \geq 0 \end{cases}$$

$$\Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha} & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$$\Psi_\alpha'(x) = \begin{cases} e^{-(-x)^\alpha}(-1)\alpha(-x)^{\alpha-1}(-1) & x < 0 \\ 0 & x \geq 0 \end{cases}$$

$$= \begin{cases} \alpha(-x)^{\alpha-1} e^{-(-x)^\alpha} & x < 0 \\ 0 & x \geq 0 \end{cases}$$

Q2

$$\wedge(x) = e^{-e^{-x}}$$

$$\wedge'(x) = e^{-e^{-x}} e^{-x}$$

$$\text{Mean} = \int_{-\infty}^{\infty} x \cdot e^{-e^{-x}} e^{-x} \, dx$$

Let $y = e^{-x} \Rightarrow x = -\log y$

$$\frac{dx}{dy} = -\frac{1}{y}$$

$$= \int_{\infty}^{0} (-\log y)\, e^{-y}\, y \left(-\frac{1}{y}\right) dy$$

$$= -\int_{0}^{\infty} \log y \, e^{-y} \, dy$$

$$\frac{dy^a}{da} = y^a \log y$$

$$\Rightarrow \left.\frac{dy^a}{da}\right|_{a=0} = y^0 \log y = \log y \quad \ldots (*)$$

$$= -\int_{0}^{\infty} \left.\frac{dy^a}{da}\right|_{a=0} e^{-y} \, dy$$

$$= -\frac{d}{da}\left[\int_{0}^{\infty} y^a e^{-y} \, dy\right]\Bigg|_{a=0}$$

$$\Gamma(a+1) = \int_{0}^{\infty} y^a e^{-y} \, dy$$

GAMMA FUNCTION

$$= -\frac{d}{da}\left[\Gamma(a+1)\right]\Big|_{a=0}$$

$$= -\Gamma'(1)$$

Q3 $\Lambda(x) = e^{-e^{-x}}$

$\Lambda'(x) = e^{-e^{-x}} e^{-x}$

$\text{Variance} = E(X^2) - [E(X)]^2$

$= E(X^2) - [\Gamma'(1)]^2$

$E(X^2) = \int_{-\infty}^{\infty} x^2 e^{-e^{-x}} e^{-x} \, dx$

$y = e^{-x} \Rightarrow x = -\log y \Rightarrow \frac{dx}{dy} = -\frac{1}{y}$

$= \int_{\infty}^{0} (\log y)^2 e^{-y} \, y \left(-\frac{1}{y}\right) dy$

$= \int_{0}^{\infty} (\log y)^2 e^{-y} \, dy$

$\frac{dy^a}{da} = y^a \log y$

$\Rightarrow \frac{d^2 y^a}{da^2} = \left(\frac{dy^a}{da}\right) \log y = (y^a \log y) \log y$

$= y^a (\log y)^2$

$\Rightarrow \left.\frac{d^2 y^a}{da^2}\right|_{a=0} = y^0 (\log y)^2 = (\log y)^2 \ldots (*)$

$= \int_{0}^{\infty} \left.\frac{d^2 y^a}{da^2}\right|_{a=0} e^{-y} \, dy$

$= \frac{d^2}{da^2} \left[\int_0^{\infty} y^a e^{-y} \, dy\right] \Big|_{a=0}$

$= \frac{d^2}{da^2} \Gamma(a+1) \Big|_{a=0}$

$= \Gamma''(1)$

$\text{Variance} = \Gamma''(1) - [\Gamma'(1)]^2.$

Q4 $\Lambda^n(x) = \Lambda(\alpha_n + \beta_n x)$

$\Leftrightarrow \left[e^{-e^{-x}}\right]^n = e^{-e^{-\alpha_n - \beta_n x}}$

$\Leftrightarrow e^{-ne^{-x}} = e^{-e^{-\alpha_n - \beta_n x}}$

Take log $\Leftrightarrow -ne^{-x} = -e^{-\alpha_n - \beta_n x}$

$\Leftrightarrow ne^{-x} = e^{-\alpha_n - \beta_n x}$

Take log $\Leftrightarrow \log n - x = -\alpha_n - \beta_n x$

$\Leftrightarrow$ Eq coeff of x: $-1 = -\beta_n \Rightarrow \beta_n = 1$

" constants: $\log n = -\alpha_n \Rightarrow \alpha_n = -\log n$.

Q5, **Q6** are similar

Q7 done in lectures

Q8 $F(x) = [1 - e^{-x}]^{\alpha}$, $x > 0$

$F(x) = 1 \Rightarrow [1 - e^{-x}]^{\alpha} = 1 \Rightarrow 1 - e^{-x} = 1$

$\Rightarrow e^{-x} = 0 \Rightarrow x = \infty \Rightarrow \omega(F) = \infty$

I: $$\lim_{t \to \infty} \frac{1 - [1 - e^{-t - x\gamma(t)}]^{\alpha}}{1 - [1 - e^{-t}]^{\alpha}}$$ ($e^{-t-x\gamma(t)} \to 0$, $e^{-t} \to 0$)

$(1-y)^{\alpha} \approx 1 - \alpha y$ as $y \to 0$

$$= \lim_{t \to \infty} \frac{1 - [1 - \alpha e^{-t - x\gamma(t)}]}{1 - [1 - \alpha e^{-t}]}$$

$$= \lim_{t \to \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t \to \infty} e^{-x\gamma(t)}$$

$= e^{-x}$ if $\gamma(t) \equiv 1 \quad \forall t$

Hence, cond (I) is satisfied.
By the ETT, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) \longrightarrow e^{-e^{-x}}$$

as $n \longrightarrow \infty$.

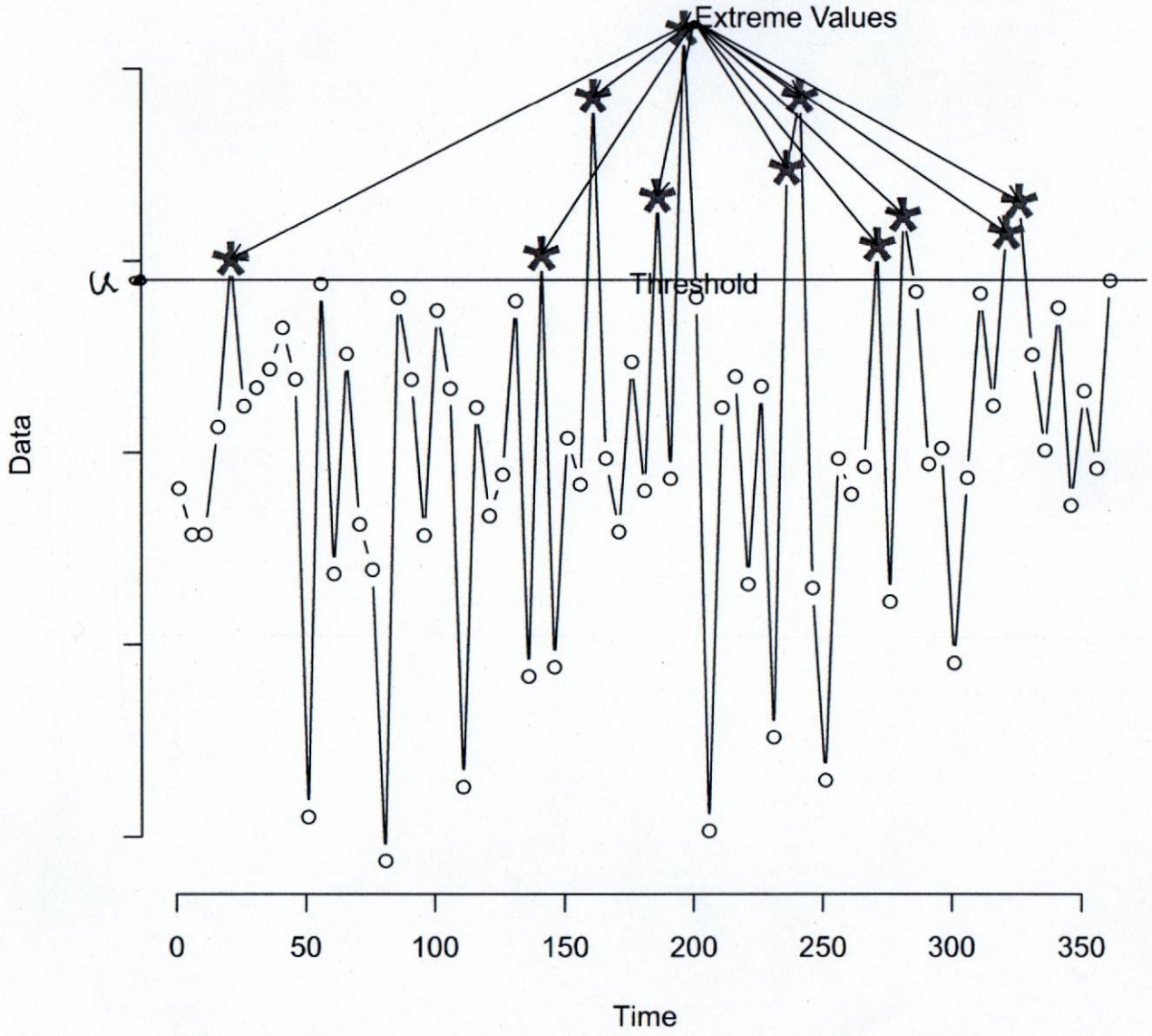
LECTURE

8 OCTOBER

9:00-10:00AM

MATH3/4/68181

Definition 2



Definition 2

X = variable of interest (eg stock returns)

u = Threshold (considered high enough)

X is an extreme value if $X > u$

$X - u$ = Excess amount

What is the distribution of

$X - u$ (the excess) if $X > u$

(that is X is an extreme value)?

$$P(X - u < x \mid X > u) = ?$$

$$= \frac{P(X - u < x, X > u)}{P(X > u)}$$

$$= \frac{P(u < X < u + x)}{P(X > u)}$$

$$= \frac{F(u + x) - F(u)}{1 - F(u)}$$

F denotes the CDF of X

$$\Downarrow$$

$$1 - \left[1 + \xi \frac{x}{\sigma}\right]^{-\frac{1}{\xi}}, \quad \sigma > 0 \text{ "scale"}, \quad -\infty < \xi < \infty \text{ "shape"}$$

as $u \longrightarrow w(F)$

This result is due to Pickands (1975).

For u large enough,

$$\frac{F(u+x)-F(u)}{1-F(u)} \approx 1-\left[1+\xi\frac{x}{\sigma}\right]^{-\frac{1}{\xi}}$$

$$\Leftrightarrow \quad \frac{1-F(u+x)}{1-F(u)} \approx \left[1+\xi\frac{x}{\sigma}\right]^{-\frac{1}{\xi}}$$

$$\Leftrightarrow \quad 1-F(u+x) \approx \left[1-F(u)\right]\left[1+\xi\frac{x}{\sigma}\right]^{-\frac{1}{\xi}}$$

$$\Leftrightarrow \quad F(\underbrace{u+x}_{\text{"}y}) \approx 1-\left[1-F(u)\right]\left[1+\xi\frac{x}{\sigma}\right]^{-\frac{1}{\xi}}$$

$$\Leftrightarrow \quad F(y) \approx 1-\left[1-F(u)\right]\left[1+\xi\frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}}$$

Let $q = 1-F(u)$

$$\Leftrightarrow \quad \boxed{F(y) \approx 1-q\left[1+\xi\frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}}}$$

Generalized Pareto (GP) Model

$-\infty < \xi < \infty$ "shape parameter"

$\sigma > 0$ "scale " "

u = threshold

q = prob that $X > u$.

Special Cases

i) $\boxed{\xi = 0}$

$$F(y) = 1 - \lim_{\xi \to 0} q\left[1 + \xi \frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}}$$

$$= 1 - q \lim_{\xi \to 0} \left[1 + \xi \frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}}$$

$$\boxed{\text{Set} \quad m = \frac{1}{\xi}}$$

$$= 1 - q \lim_{m \to \infty} \left[1 + \frac{\frac{y-u}{\sigma}}{m}\right]^{-m}$$

$$= 1 - q \lim_{m \to \infty} \left\{\left[1 + \frac{\frac{y-u}{\sigma}}{m}\right]^{m}\right\}^{-1}$$

$$\boxed{\lim_{m \to \infty} \left(1 + \frac{z}{m}\right)^{m} = e^{z}}$$

$$= 1 - q \left[e^{\frac{y-u}{\sigma}}\right]^{-1}$$

$$= 1 - q\, e^{-\frac{y-u}{\sigma}}, \quad y > u$$

Truncated exponential distribution

The domain is (v, ∞).

i) $\boxed{\xi > 0}$

$$F(y) \approx 1 - q\left[1 + \xi \frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}}, \quad y > u$$

provided

$$1 + \xi \frac{y-u}{\sigma} > 0$$

$$\Leftrightarrow \quad \frac{y-u}{\sigma} > -\frac{1}{\xi}$$

$$\Leftrightarrow \quad y > u - \frac{\sigma}{\xi}$$

So the domain of $F(y)$ is

~~$\left(u - \frac{\sigma}{\xi}, \infty\right)$~~.

(u, ∞).

iii) $\boxed{\xi < 0}$

$$F(y) \approx 1 - q\left[1 + \xi \frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}}, \quad y > u$$

provided

$$1 + \xi \frac{y-u}{\sigma} > 0$$

$$\Leftrightarrow \quad \xi \frac{y-u}{\sigma} > -1$$

$$\Leftrightarrow \quad y - u < -\frac{\sigma}{\xi}$$

$$\Leftrightarrow \quad y < u - \frac{\sigma}{\xi}$$

So the domain of $F(y)$ is

$$\left(u, u - \frac{\sigma}{\xi}\right)$$

$F(y)$ has a finite upper end point, $w(F) = u - \frac{\sigma}{\xi}$.

CDF

$$F(y) \approx 1 - q\left[1 + \xi \frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}}$$

PDF

$$f(y) \approx \frac{q}{\sigma}\left[1 + \xi \frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}-1}$$

Quantile

$$\text{s.t} \quad F(y) = p, \quad 0 < p < 1$$

$$\Leftrightarrow 1 - q\left[1 + \xi \frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}} = p$$

$$\Leftrightarrow \left[1 + \xi \frac{y-u}{\sigma}\right]^{-\frac{1}{\xi}} = \frac{1-p}{q}$$

$$\Leftrightarrow 1 + \xi \frac{y-u}{\sigma} = \left(\frac{1-p}{q}\right)^{-\xi}$$

$$\Leftrightarrow y = u + \frac{\sigma}{\xi}\left[\left(\frac{1-p}{q}\right)^{-\xi} - 1\right].$$

$$\text{Median} = u + \frac{\sigma}{\xi}\left[(2q)^{\xi} - 1\right].$$

Return Level

is the extreme value occuring on average every T years.

This corresponds to taking

$$p = 1 - \frac{1}{mT}$$

mT: average number of exceedances per year.

$\Rightarrow$ Return Level

$$= u + \frac{\sigma}{\xi}\left[(mT\zeta_u)^{\xi} - 1\right].$$

How to choose u ?

For the GP model,

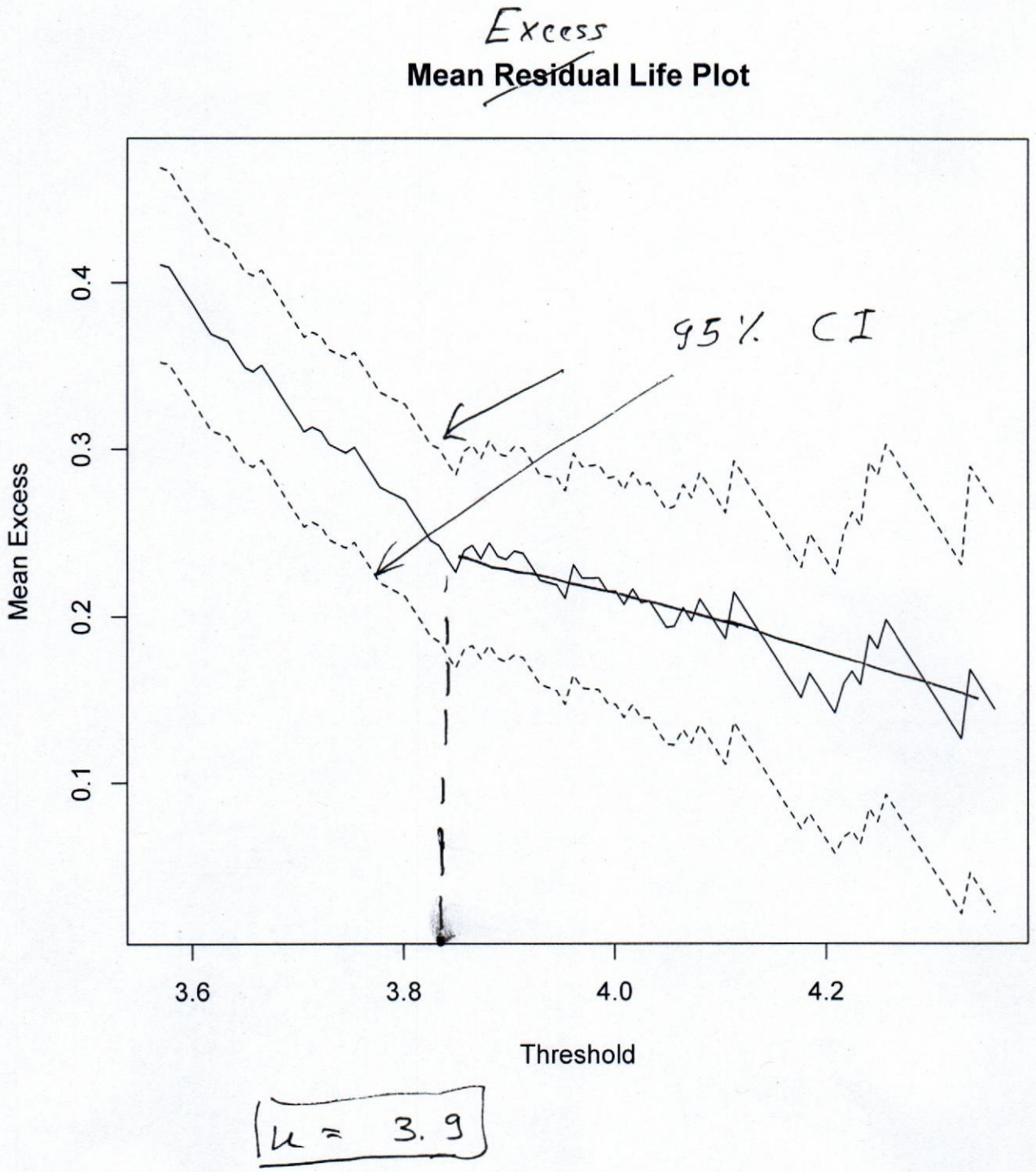
$$E(Y-u \mid Y > u) = a \cdot u + b$$

- $E(Y-u \mid Y > u)$: Mean Excess
- u: Threshold
- a, b: constants

- Plot mean excess vs u.
- Choose u so that the plot is approximately linear.

MRLplot

In R, `mrlplot` to draw the plot of mean excess vs u.


Excess
Mean Residual Life Plot
95% CI
Mean Excess
0.1
0.2
0.3
0.4
3.6
3.8
4.0
4.2
Threshold
u = 3.9

LECTURE

9 OCTOBER

9:00-10:00AM

MATH3/4/68181

ML Estimation

Suppose $x_1, x_2, \ldots, x_n$ are IID from the GP model.

$$L(\sigma, \xi) = \prod_{i=1}^{n} \frac{1}{\sigma}\left[1 + \xi \frac{x_i - \mu}{\sigma}\right]^{-\frac{1}{\xi}-1}$$

$$= \frac{1^n}{\sigma^n}\left\{\prod_{i=1}^{n}\left[1 + \xi \frac{x_i - \mu}{\sigma}\right]\right\}^{-\frac{1}{\xi}-1}$$

$$\Rightarrow \log L(\sigma, \xi) = n \log 1 - n \log \sigma$$

$$-\left(\frac{1}{\xi}+1\right)\sum_{i=1}^{n} \log\left[1 + \xi \frac{x_i - \mu}{\sigma}\right]$$

MLE equations for the GP distribution

The MLEs of σ and ξ are the simultaneous solutions of

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1+\xi}{\sigma^2} \sum_{i=1}^{n} (x_i - t) \left(1 + \xi \frac{x_i - t}{\sigma}\right)^{-1} = 0, \quad \text{... (1)}$$

and

$$\frac{\partial \log L}{\partial \xi} = \frac{1}{\xi^2} \sum_{i=1}^{n} \log \left(1 + \xi \frac{x_i - t}{\sigma}\right) - \frac{1+\xi}{\xi\sigma} \sum_{i=1}^{n} (x_i - t) \left(1 + \xi \frac{x_i - t}{\sigma}\right)^{-1} = 0. \quad \text{..... (2)}$$

The MLEs of σ and ξ are the simultaneous solutions of (1) and (2).

In R, `fpot` will compute the MLEs of σ and ξ.

Is there a way to check that ETT will hold without checking conditions (I) - (III)? Yes.

Suppose X is a discrete RV. Let $F(\cdot)$ denote the CDF of X. Then ETT will <u>not</u> <u>hold</u> if

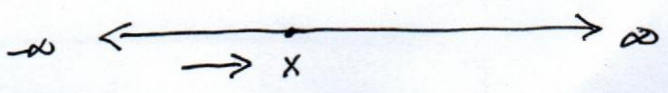
$$\frac{P(X=k)}{1-F(k-1)} \not\longrightarrow 0 \qquad (*)$$

as $k \longrightarrow w(F)$.

Suppose X is a continuous RV with CDF $F(\cdot)$. Then ETT will <u>not</u> <u>hold</u> if

$$\frac{f(x)}{1-F(x^-)} \not\longrightarrow 0$$

as $x \longrightarrow w(F)$.



Ex 3 $X \sim \text{Pois}(\lambda)$.

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

$$\omega(F) = \infty$$

$$(*) = \lim_{k\to\infty} \frac{\frac{e^{-\lambda}\lambda^k}{k!}}{1 - \sum_{i=0}^{k-1} \frac{e^{-\lambda}\lambda^i}{i!}}$$

$$= 1 - P(X \le k-1)$$

$$= P(X > k-1) = P(X \ge k)$$

$$= \lim_{k\to\infty} \frac{\frac{e^{-\lambda}\lambda^k}{k!}}{\sum_{i=k}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!}}$$

$$= \lim_{k\to\infty} \frac{\frac{\lambda^k}{k!}}{\sum_{i=k}^{\infty} \frac{\lambda^i}{i!}}$$

$$= \lim_{k\to\infty} \frac{\frac{\lambda^k}{k!}}{\frac{\lambda^k}{k!} + \sum_{i=k+1}^{\infty} \frac{\lambda^i}{i!}}$$

$$= \lim_{k \to \infty} \frac{1}{1 + \sum_{i=k+1}^{\infty} \frac{k! \lambda^{i-k}}{i!}}$$

$$\frac{k!}{i!} = \frac{1}{(k+1)(k+2)\cdots(k+i-k)}$$

$$(k+1) \le k, \quad (k+2) \le k, \quad \ldots \quad (k+i-k) \le k$$

$$< \frac{1}{k^{i-k}}$$

$$> \lim_{k \to \infty} \frac{1}{1 + \sum_{i=k+1}^{\infty} \frac{\lambda^{i-k}}{k^{i-k}}}$$

$$j = i - k$$

$$= \lim_{k \to \infty} \frac{1}{1 + \sum_{j=1}^{\infty} \left(\frac{\lambda}{k}\right)^{j}}$$

$$= \lim_{k \to \infty} \frac{1}{1 + \frac{1}{1 - \frac{\lambda}{k}} - 1} = 1$$

$$\frac{\lambda}{k} \to 0$$

$\Rightarrow \quad (*) \quad > 1$

$\Rightarrow \quad (*) \quad \neq 0$

$\Rightarrow$ ETT will not hold.

$$\sum_{n=1}^{\infty} \theta^n = \frac{1}{1-\theta} - 1$$

LECTURE

12 OCTOBER

10:00-11:00AM

MATH4/68181

Financial Ratios

Examples

1) Current ratio = $\frac{\text{Current assets (X)}}{\text{" liabilities (Y)}}$

2) Sales margin = $\frac{\text{Sales (X)} - \text{Cost (Y)}}{\text{Sales (X)}}$

3) Interest cover = $\frac{\text{Earnings (X)} + \text{Interest paid (Y)}}{\text{Earnings (X)}}$

4) Liabilities ratio = $\frac{\text{Liabilities (X)}}{\text{Equity (Y)} + \text{Liabilities (X)}}$

The most popular (also the oldest) model for income data is the Pareto distribution.

↑ Italian Economist

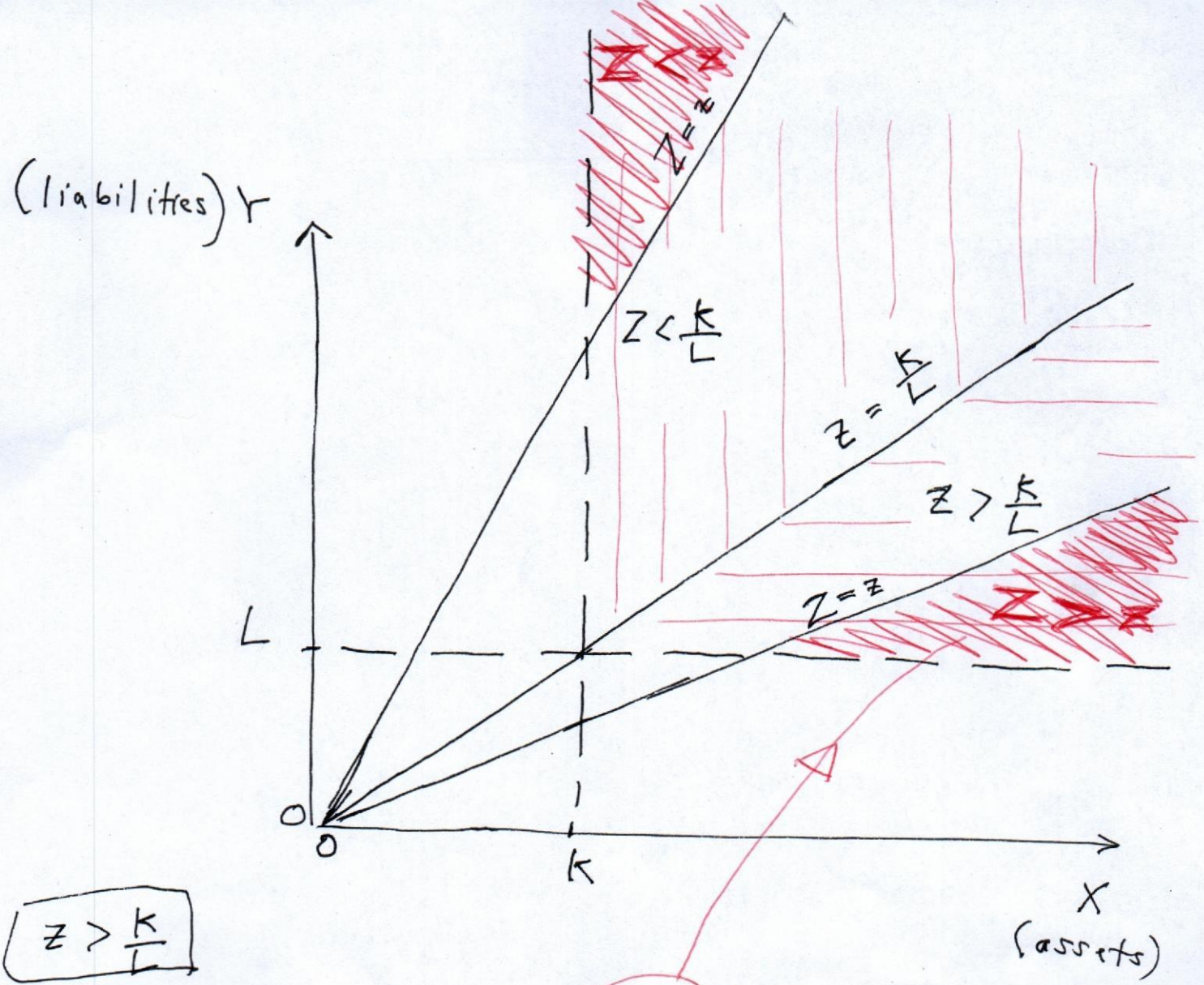
Suppose X and Y are independent Pareto RVs with CDFs

$$F_X(x) = 1 - \left(\frac{k}{x}\right)^a, \quad \begin{matrix} x \geq k \\ a > 0 \end{matrix}$$

$$F_Y(y) = 1 - \left(\frac{L}{y}\right)^b, \quad \begin{matrix} y \geq L \\ b > 0 \end{matrix}$$

What is the distribution of $Z = \frac{X}{Y}$?

↑ Current ratio



$z > \frac{K}{L}$

$$F_Z(z) = P(Z \leq z) = 1 - P(Z > z)$$

$$= 1 - \int_L^\infty \int_{yz}^\infty f_X(x) f_Y(y)\, dx\, dy$$

$$= 1 - \int_L^\infty \int_{yz}^\infty \frac{a K^a}{x^{a+1}} \frac{b L^b}{y^{b+1}}\, dx\, dy$$

$$= 1 - ab K^a L^b \int_L^\infty \frac{1}{y^{b+1}} \left[\frac{x^{-a}}{-a}\right]_{yz}^\infty dy$$

$$= 1 - ab K^a L^b \int_L^\infty \frac{1}{y^{b+1}} \left[0 - \frac{(yz)^{-a}}{-a}\right] dy$$

$$= 1 - b K^a L^b z^{-a} \int_L^\infty \frac{1}{y^{a+b+1}}\, dy$$

$$= 1 - b K^a L^b z^{-a} \left[y^{-a-b}/(-a-b)\right]_L^\infty$$

$$= 1 \cdot \frac{b}{} \; K^a L^b z^{-a} \left[0 - \frac{L^{-a-b}}{-a-b} \right]$$

$$\boxed{F_Z(z) = 1 \cdot \frac{b}{a+b} \; K^a L^{-a} z^{-a} \quad \text{if} \quad z > \frac{K}{L}}$$

$z < \frac{K}{L}$

$$F_Z(z) = P(Z < z)$$

$$= \int_K^\infty \int_{x/z}^\infty \frac{a K^a}{x^{a+1}} \frac{b L^b}{y^{b+1}} \, dy \, dx$$

$$= ab K^a L^b \int_K^\infty \frac{1}{x^{a+1}} \left[\frac{y^{-b}}{-b} \right]_{\frac{x}{z}}^\infty dx$$

$$= a b K^a L^b \int_K^\infty \frac{1}{x^{a+1}} \left[0 - \frac{(x/z)^{-b}}{-b} \right] dx$$

$$= a K^a L^b z^b \int_K^\infty \frac{1}{x^{a+b+1}} \, dx$$

$$= a K^a L^b z^b \left[\frac{x^{-a-b}}{-a-b} \right]_K^\infty$$

$$= a K^a L^b z^b \left[0 - \frac{K^{-a-b}}{-a-b} \right]$$

$$= \frac{a}{a+b} K^{-b} L^b z^b \quad \text{if} \quad z < \frac{K}{L}$$

The CDF of Z is

$$F_Z(z) = \begin{cases} 1 - \frac{b}{a+b}\left(\frac{k}{L}\right)^a z^{-a}, & z > \frac{k}{L} \\ \frac{a}{a+b}\left(\frac{L}{k}\right)^b z^b, & z \leq \frac{k}{L} \end{cases}$$

The PDF of Z is

$$f_Z(z) = \begin{cases} \frac{ab}{a+b}\left(\frac{k}{L}\right)^a z^{-a-1} & z > \frac{k}{L} \\ \frac{ab}{a+b}\left(\frac{L}{k}\right)^b z^{b-1} & z \leq \frac{k}{L} \end{cases}$$

Median (Current ratio) = ? $F_Z(z) = \frac{1}{2} \Rightarrow z = ?$

Current ratio so extreme expected once in 10 years = ?

$F_Z(z) = 1 - \frac{1}{10} \Rightarrow z = ?$

EXAMPLE CLASS

8 OCTOBER

12:00-13:00PM

MATH3/4/68181

Q9 $F(x) = x, \quad 0 < x < 1$

$\omega(F) = 1$

I: $$\lim_{t \to 1} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to 1} \frac{1 - (t + x\gamma(t))}{1 - t}$$

$$= \lim_{t \to 1} \left[1 + \frac{x\gamma(t)}{1 - t} \right]$$

$$\neq e^{-x}$$

$\Rightarrow$ cond (I) is not satisfied

III: $$\lim_{t \to 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)}$$

$$= \lim_{t \to 0} \frac{1 - (1 - tx)}{1 - (1 - t)}$$

$$= \lim_{t \to 0} \frac{tx}{t}$$

$$= x$$

$\Rightarrow$ cond (III) is satisfied with $\alpha = 1$

By the ETT, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that $[F(a_n x + b_n)]^n \longrightarrow \begin{cases} e^x & x < 0 \\ 1 & x > 0 \end{cases}$

Q10

$$F(x) = 1 - \left(\frac{k}{x}\right)^a, \quad x > k$$

$$\omega(F) = \infty.$$

I: $$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{1 - \left[1 - \left(\frac{k}{t + x\gamma(t)}\right)^a\right]}{1 - \left[1 - \left(\frac{k}{t}\right)^a\right]}$$

$$= \lim_{t \to \infty} \left(\frac{t}{t + x\gamma(t)}\right)^a$$

$$= \lim_{t \to \infty} \left(\frac{1}{1 + \frac{x\gamma(t)}{t}}\right)^a$$

$$\neq e^{-x}$$

$\Rightarrow$ Cond (I) is not satisfied.

II :

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{1 - \left[1 - \left(\frac{k}{tx}\right)^a\right]}{1 - \left[1 - \left(\frac{k}{t}\right)^a\right]}$$

$$= \lim_{t \to \infty} \frac{t^a}{(tx)^a}$$

$$= x^{-a}$$

$\Rightarrow$ Cond (II) is satisfied with $a = a$.

By the ETT, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\left[F(a_n x + b_n)\right]^n \longrightarrow \begin{cases} e^{-x^{-a}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

as $n \longrightarrow \infty$.

Q11

Two cdfs F & G.

$$f(x) = K\, g(x) [G(x)]^{a-1} [1-G(x)]^{b-1} e^{-cG(x)}$$

Show that F belongs to the same DoA as G

✓(i) G belongs to Gumbel domain
⇒ F " " " "

(ii) G belongs to Frechet domain
⇒ F " " " "

(iii) G belongs to Weibull domain
⇒ F " " " "

Proof of (i)

Assume G belongs to Gumbel domain.

$\Rightarrow \exists\ \gamma(t) > 0$ such that

$$\lim_{t \to w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x} \quad \cdots (*)$$

We need to show that

$$\lim_{t \to w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = e^{-x} \quad (**)$$

LHS of $(**)$

$$\overset{LH}{=} \lim_{t \to w(F)} \frac{-f(t + x\gamma(t))\,(1 + x\gamma'(t))}{-f(t)}$$

$$= \lim_{t \to w(F)} \frac{f(t + x\gamma(t))\,(1 + x\gamma'(t))}{f(t)}$$

$$= \lim_{t \to w(F)} \frac{g(t + x\gamma(t))\,[G(t + x\gamma(t))]^{a-1}\,[1 - G(t + x\gamma(t))]^{b-1}\,e^{-c\,G(t + x\gamma(t))}}{g(t)\,[G(t)]^{a-1}\,[1 - G(t)]^{b-1}\,e^{-c\,G(t)}} \times (1 + x\gamma'(t))$$

$w(F) = w(G)$

$$= \lim_{t \to w(G)} \frac{g(t+x\gamma(t))\,\left[1-G(t+x\gamma(t))\right]^{b-1}}{g(t)\,\left[1-G(t)\right]^{b-1}}\,(1+x\gamma'(t))$$

$$= \lim_{t \to w(G)} \frac{g(t+x\gamma(t))}{g(t)}\,(1+x\gamma'(t))\left[\frac{1-G(t+x\gamma(t))}{1-G(t)}\right]^{b-1}$$

(*)

$$= \lim_{t \to w(G)} \frac{g(t+x\gamma(t))}{g(t)}\,(1+x\gamma'(t))\,\left[e^{-x}\right]^{b-1}$$

$\overset{LH}{\underset{\text{in reverse}}{=}}$ $\lim_{t \to w(G)} \frac{1-G(t+x\gamma(t))}{1-G(t)}\,e^{-(b-1)x}$

(*)

$$= e^{-x}\,e^{-(b-1)x} = e^{-bx},$$

the SAME TYPE as e^{-x}.

Hence, F also belongs to the Gumbel domain.

EXAMPLE CLASS

9 OCTOBER

16:00-17:00PM

MATH3/4/68181

Q 9

$F(x) = x, \quad 0 < x < 1$

$w(F) = 1$

I: $\lim_{t \to 1} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$

$= \lim_{t \to 1} \frac{1 - (t + x\gamma(t))}{1 - t}$

$= \lim_{t \to 1} 1 - \frac{x\gamma(t)}{1 - t}$

$\neq e^{-x}$

$\Rightarrow$ cond I is not satisfied

III: $\lim_{t \to 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)}$

$= \lim_{t \to 0} \frac{1 - (1 - tx)}{1 - (1 - t)}$

$= \lim_{t \to 0} \frac{tx}{t}$

$= x$

$\Rightarrow$ Cond III is satisfied with $\alpha = 1$

By ETT, there exists $a_n > 0$ and $b_n \in R$ such that

$$[F(a_n x + b_n)]^n \longrightarrow \begin{cases} e^x & x < 0 \\ 1 & x \geq 1 \end{cases}$$

Q10 $F(x) = 1 - \left(\frac{k}{x}\right)^a, \quad x \geq k$

$w(F) = \infty$

I: $\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$

$= \lim_{t \to \infty} \frac{1 - \left[1 - \left(\frac{k}{t + x\gamma(t)}\right)^a\right]}{1 - \left[1 - \left(\frac{k}{t}\right)^a\right]}$

$= \lim_{t \to \infty} \left[\frac{t}{t + x\gamma(t)}\right]^a$

$= \lim_{t \to \infty} \left[\frac{1}{1 + x\frac{\gamma(t)}{t}}\right]^a$

$\neq e^{-x}$

$\Rightarrow$ cond (I) is not satisfied

II: $\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)}$

$= \lim_{t \to \infty} \frac{1 - \left[1 - \left(\frac{k}{tx}\right)^a\right]}{1 - \left[1 - \left(\frac{k}{t}\right)^a\right]}$

$= \lim_{t \to \infty} \left(\frac{t}{tx}\right)^a$

$= x^{-a}$

$\Rightarrow$ cond (II) is satisfied with $\alpha = +a$

By ETT, there exists $a_n > 0$ and $b_n \in R$ s.t.

$[F(a_n x + b_n)]^n \longrightarrow \begin{cases} e^{-x^{-a}} & x > 0 \\ 0 & x \leq 0 \end{cases}$

Q11 $f(x) = k\, g(x) [G(x)]^{a-1} [1 - G(x)]^{b-1} e^{-c\,G(x)}$

Show that F belongs to the same domain of attraction as G

(i) G belongs to the Gumbel DoA
$\Rightarrow$ F " " " " "

(ii) G belongs to the Frechet DoA
$\Rightarrow$ F " " " " "

(iii) G belongs to the Weibull DoA
F " " " " "

(i) Assume G belongs to Gumbel DoA.

$\exists\ \gamma(t) > 0$ such that

$$\lim_{t \to w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x} \quad \cdots (*)$$

We need to show that

$$\lim_{t \to w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = e^{-x} \quad \cdots (**)$$

($1 - F(t + x\gamma(t)) \to 1$, $1 - F(t) \to 1$)

LHS of (**)

$$\overset{L'H}{=} \lim_{t \to w(F)} \frac{-f(t + x\gamma(t)) \cdot (1 + x\gamma'(t))}{-f(t)}$$

$$= \lim_{t \to w(F)} \frac{f(t + x\gamma(t))}{f(t)} (1 + x\gamma'(t))$$

$$= \lim_{t \to w(F)} \frac{k\, g(t + x\gamma(t)) [G(t + x\gamma(t))]^{a-1} [1 - G(t + x\gamma(t))]^{b-1}}{k\, g(t) [G(t)]^{a-1} [1 - G(t)]^{b-1}}$$

$w(F) = w(G)$

$$\times \frac{e^{-c(G(t + x\gamma(t)))}}{e^{-c(G(t))}} (1 + x\gamma'(t))$$

$$= \lim_{t \to w(G)} \frac{g(t + x\gamma(t))}{g(t)} \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b-1} (1 + x\gamma'(t))$$

$$\overset{(*)}{=} \lim_{t \to w(G)} \frac{g(t + x\gamma(t))}{g(t)} (1 + x\gamma'(t)) \left[e^{-x}\right]^{b-1}$$

$$\overset{L'H}{=} \text{backwards} \lim_{t \to w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} e^{-(b-1)x}$$

$$\overset{(*)}{=} \lim_{t \to w(G)} e^{-x} e^{-(b-1)x}$$

$$= e^{-bx},$$ the SAME TYPE as e^{-x}

Hence, F also belongs to Gumbe DoA.

(ii) Homework

(iii) "

LECTURE

15 OCTOBER

9:00-10:00AM

MATH3/4/68181

PORTFOLIO THEORY

Portfolio is a collection of investments

Ex gold, silver, copper

Suppose there are n investments.

Let

X_1 = Loss on investment 1

X_2 = " " " 2

$\vdots$

X_n = " " " n

$X_1, X_2, \ldots, X_n$ are random variables

Variables of Interest

1) $S = X_1 + \cdots + X_n$

= "total portfolio loss"

2) $U = \max(X_1, \ldots, X_n)$

= "maximum portfolio loss"

3) $V = \min(X_1, \ldots, X_n)$

= "minimum portfolio loss"

Six Cases

i) $X_1, \ldots, X_n$ IID, n fixed

ii) $X_1, \ldots, X_N$ IID, N is a RV

iii) $X_1, \ldots, X_n$ INID, n fixed
(Independent but not identically distributed)

iv) $X_1, \ldots, X_N$ INID, N RV

Level 4 & Level 6 ONLY

v) $X_1, \ldots, X_n$ dependent RVs, n fixed

vi) $X_1, \ldots, X_n$ " ", N is a RV

Case 1 $X_1, \dots, X_n$ IID, n fixed

1) $S = X_1 + \dots + X_n$

The CDF of S is

$$F_S(s) = P(S \le s)$$

$$= P(X_1 + \dots + X_n \le s)$$

$$= \int \cdots \int_{x_1 + \dots + x_n \le s} F(s - x_1 - \dots - x_{n-1}) \prod_{j=1}^{n-1} f(x_j)$$

$$dx_{n-1}\, dx_{n-2} \cdots dx_1$$

F = CDF of $X_1, \dots, X_n$

f = PDF of $X_1, \dots, X_n$

The PDF of S is

$$f_S(s) = \frac{d}{ds} F_S(s)$$

$$= \int \cdots \int_{x_1 + \dots + x_n = s} f(s - x_1 - \dots - x_{n-1}) \prod_{j=1}^{n-1} f(x_j)$$

$$dx_{n-1}\, dx_{n-2} \cdots dx_1$$

The mth moment of S

$$E[S^m] = E[(X_1 + \dots + X_n)^m]$$

$$= \sum_{p_1 + p_2 + \dots + p_n = m} \binom{m}{p_1\, p_2 \cdots p_n} E[X_1^{p_1} X_2^{p_2} \cdots X_n^{p_n}]$$

$$\binom{m}{p_1\, p_2 \cdots p_n} = \frac{m!}{p_1!\, p_2! \cdots p_n!}$$

$$= \sum \binom{m}{p_1\, p_2 \cdots p_n} E[X_1^{p_1}]\, E[X_2^{p_2}] \cdots E[X_n^{p_n}]$$

Mean of S

$= E[X_1 + \cdots + X_n]$

$= E[X_1] + \cdots + E[X_n]$

$= n \cdot E(X)$ because of the identical assumption

Variance of S

$= Var[X_1 + \cdots + X_n]$

$= Var[X_1] + \cdots + Var[X_n]$ because of independence

$= n \cdot Var[X]$ because of the identical assumption

2) $U = \max(X_1, \ldots, X_n)$

The CDF of U is

$$F_U(u) = P(U \le u)$$

$$= P[\max(X_1, \ldots, X_n) \le u]$$

$$= P[X_1 \le u, \ldots, X_n \le u]$$

$$= P[X_1 \le u] \cdots P[X_n \le u] \quad \text{by independence}$$

$$= F(u) \cdots F(u) \quad \text{by the identical assumption}$$

$$= [F(u)]^n$$

The PDF of U is

$$f_U(u) = n[F(u)]^{n-1} f(u).$$

The mth moment of U is

$$E[U^m] = \int_{-\infty}^{\infty} u^m f_U(u)\, du$$

$$= n \int_{-\infty}^{\infty} u^m [F(u)]^{n-1} f(u)\, du.$$

3) $V = \min(X_1, \dots, X_n)$.

The CDF of V is

$$F_V(v) = P(V \le v) = 1 - P(V > v)$$

$$= 1 - P[\min(X_1, \dots, X_n) > v]$$

$$= 1 - P[X_1 > v, \dots, X_n > v]$$

$$= 1 - P[X_1 > v] \cdots P[X_n > v] \quad \text{by independence}$$

$$= 1 - (1 - P(X_1 \le v)) \cdots (1 - P(X_n \le v))$$

$$= 1 - (1 - F(v)) \cdots (1 - F(v)) \quad \text{by the identical assumption}$$

$$= 1 - [1 - F(v)]^n$$

The PDF of V is

$$f_V(v) = n[1 - F(v)]^{n-1} f(v)$$

The m^{th} moment of V is

$$E(V^m) = \int_{-\infty}^{\infty} v^m f_V(v)\, dv$$

$$= n \int_{-\infty}^{\infty} v^m [1 - F(v)]^{n-1} f(v)\, dv.$$

Ex1

Suppose $X_1, \cdots, X_n$ IID $N(\mu, \sigma^2)$

$$S = X_1 + \cdots + X_n$$

$$\sim N(n\mu, n\sigma^2)$$

Math 20802

The CDF of S is

$$F_S(s) = \Phi\left(\frac{s - n\mu}{\sqrt{n}\sigma}\right)$$

Φ: CDF of $N(0,1)$

The PDF of S is

$$f_S(s) = \frac{1}{\sqrt{2\pi}\sqrt{n}\sigma} e^{-\frac{(s-n\mu)^2}{2n\sigma^2}}$$

The mean of $S = n\mu$

The variance $S = n\sigma^2$.

LECTURE

16 OCTOBER

9:00-10:00AM

MATH3/4/68181

QUIZ 2 DUE 12:00 NOON
TUESDAY
23 OCTOBER

N is fixed
(not a RV)

Ex 2

Suppose $X_1, \ldots, X_n$ IID $Exp(a)$.

$$F_U(u) = \left[1 - e^{-au}\right]^n$$

$$f_U(u) = n\left[1 - e^{-au}\right]^{n-1} a e^{-au}$$

$$E(U^m) = na \int_0^\infty u^m e^{-au} \left[1 - e^{-au}\right]^{n-1} du$$

Set $y = e^{-au}$

$\Rightarrow u = -\frac{1}{a} \log y \Rightarrow \frac{du}{dy} = -\frac{1}{ay}$

$$= na \int_1^0 \left(-\frac{1}{a} \log y\right)^m a^{a-1} [1-y]^{n-1} \left(-\frac{1}{a}\right) dy$$

$$= n \left(-\frac{1}{a}\right)^m \int_0^1 (\log y)^m [1-y]^{n-1} dy$$

$$\frac{d^m y^\alpha}{d\alpha^m} = y^\alpha (\log y)^m$$

$$\Rightarrow \frac{d^m}{d\alpha^m} y^\alpha \Big|_{\alpha=0} = (\log y)^m \qquad (*)$$

$$\overset{(*)}{=} n \left(-\frac{1}{a}\right)^m \int_0^1 \frac{d^m}{d\alpha^m} y^\alpha \Big|_{\alpha=0} [1-y]^{n-1} dy$$

$$= n \left(-\frac{1}{a}\right)^m \frac{d^m}{d\alpha^m} \left(\int_0^1 y^\alpha [1-y]^{n-1} dy \right) \Big|_{\alpha=0}$$

$$= n \left(-\frac{1}{a}\right)^m \frac{d^m}{d\alpha^m} B(\alpha+1, n) \Big|_{\alpha=0}$$

Ex 3 Suppose $X_1, \ldots, X_n$ IID $Exp(a)$.

$$F_V(v) = 1 - [1 - [1 - e^{-av}]]^n$$

$$= 1 - e^{-nav}$$

$$\Rightarrow V \sim Exp(na)$$

$$f_V(v) = na\, e^{-nav}$$

$$E(V) = \frac{1}{na}$$

$$Var(V) = \frac{1}{(na)^2}.$$

Case ii $X_1, \ldots, X_N$ IID

N is a RV indep of $X_1, X_2, \ldots$

The CDF of $S = X_1 + \cdots + X_N$ is

$$F_S(s) = P[X_1 + \cdots + X_N \leq s]$$

$$= \sum_{n=1}^{\infty} P[X_1 + \cdots + X_n \leq s \mid N = n] \; P(N = n)$$

Total Probability Rule

$$= \sum_{n=1}^{\infty} \left[\int \cdots \int_{x_1 + \cdots + x_n \leq s} F(s - x_1 - \cdots - x_{n-1}) \prod_{j=1}^{n-1} f(x_j) \, dx_{n-1} \, dx_{n-2} \cdots dx_1 \right] P(N = n)$$

The PDF of S is

$$f_S(s) = \sum_{n=1}^{\infty} \left[\int \cdots \int_{x_1 + \cdots + x_n = s} f(s - x_1 - \cdots - x_{n-1}) \prod_{j=1}^{n-1} f(x_j) \, dx_{n-1} \, dx_{n-2} \cdots dx_1 \right] P(N = n)$$

The mth moment of S is

$$E[S^m] = \sum_{n=1}^{\infty} E[S^m \mid N = n] \; P(N = n)$$

$$= \sum_{n=1}^{\infty} \sum_{p_1 + \cdots + p_n = m} \binom{m}{p_1 \, p_2 \cdots p_n} E[X_1^{p_1}] \cdots E[X_n^{p_n}] \; P(N = n)$$

The CDF of U is

$$F_U(u) = P\left[\max(X_1, \ldots, X_N) \leq u\right]$$

$$= \sum_{n=1}^{\infty} P\left[\max(X_1, \ldots, X_N) \leq u \,|\, N = n\right] P(N = n)$$

$$= \sum_{n=1}^{\infty} \left[F(u)\right]^n P(N = n)$$

The PDF of U is

$$f_U(u) = \sum_{n=1}^{\infty} n\left[F(u)\right]^{n-1} f(u)\, P(N = n)$$

$$= f(u) \sum_{n=1}^{\infty} n\left[F(u)\right]^{n-1} P(N = n)$$

The mth moment of U is

$$E[U^m] = \sum_{n=1}^{\infty} n\, P(N = n) \int_{-\infty}^{\infty} u^m f(u) \left[F(u)\right]^{n-1} du$$

The CDF of V is

$$F_V(v) = P\left[\min(X_1, \ldots, X_N) \leq v\right]$$

$$= \sum_{n=1}^{\infty} P\left[\min(X_1, \ldots, X_n) \leq v \,\middle|\, N = n\right] P(N = n)$$

$$= \sum_{n=1}^{\infty} \left\{1 - \left[1 - F(v)\right]^n\right\} P(N = n)$$

$$= 1 - \sum_{n=1}^{\infty} \left[1 - F(v)\right]^n P(N = n)$$

The PDF of V is

$$f_V(v) = f(v) \sum_{n=1}^{\infty} n \left[1 - F(v)\right]^{n-1} P(N = n)$$

The mth moment of V is

$$E(V^m) = \sum_{n=1}^{\infty} n\, P(N = n) \int_{-\infty}^{\infty} v^m f(v) \left[1 - F(v)\right]^{n-1} dv$$

Ex 1

Suppose $X_1, X_2, \ldots$ IID $N(\mu, \sigma^2)$

Suppose also $N \sim \text{Geom}(p)$ indep of $X_1, X_2, \ldots$

The CDF of S is

$$F_S(s) = \sum_{n=1}^{\infty} P[X_1 + \cdots + X_n \leq s \mid N = n] \, P(N = n)$$

$$= \sum_{n=1}^{\infty} \Phi\left(\frac{s - n\mu}{\sqrt{n}\,\sigma}\right) p(1-p)^{n-1}$$

$$= p \sum_{n=1}^{\infty} \Phi\left(\frac{s - n\mu}{\sqrt{n}\,\sigma}\right) (1-p)^{n-1}$$

The PDF of S is

$$f_S(s) = p \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{n}\,\sigma} e^{-\frac{(s-n\mu)^2}{2n\sigma^2}} (1-p)^{n-1}$$

$$= \frac{p}{\sqrt{2\pi}\,\sigma} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} e^{-\frac{(s-n\mu)^2}{2n\sigma^2}} (1-p)^{n-1}$$

The mean of S is

$$E[S] = E[E[S \mid N]]$$

$$= E[N\mu] = \mu E[N] = \frac{\mu}{p}$$

The variance of S is

$$\text{Var}[S] = E[S^2] - (E[S])^2$$

$$= E[S^2] - \left(\frac{\mu}{p}\right)^2$$

$$= E[E[S^2 \mid N]] - \left(\frac{\mu}{p}\right)^2$$

$$= E\left[N\sigma^2 + (N\mu)^2\right] - \left(\frac{\mu}{p}\right)^2$$

$$= \sigma^2 E(N) + \mu^2 E(N^2) - \left(\frac{\mu}{p}\right)^2$$

$$= \frac{\sigma^2}{p} + \mu^2 \cdot \left(\frac{2}{p^2} - \frac{1}{p}\right) - \left(\frac{\mu}{p}\right)^2$$

$$= \frac{\sigma^2}{p} + \frac{\mu^2}{p^2} - \frac{\mu^2}{p}$$

LECTURE

19 OCTOBER

10:00-11:00AM

MATH4/68181

Six Cases

i) $X_1, \ldots, X_n$ IID, n fixed

ii) $X_1, \ldots, X_N$ IID, N is a RV

iii) $X_1, \ldots, X_n$ INID, n fixed
(Independent but not identically distributed)

iv) $X_1, \ldots, X_N$ INID, N RV

Level 4 & Level 6 ONLY

v) $X_1, \ldots, X_n$ dependent RVs, n fixed

vi) $X_1, \ldots, X_n$ " ", N is a RV

Case iii $X_1, \cdots, X_n$ are INID, n fixed

1) $S = X_1 + \cdots + X_n$
= total portfolio loss

The CDF of S is

$$F_S(s) = P(X_1 + \cdots + X_n \leq s)$$

$$= \int_{x_1+\cdots+x_n \leq s} \cdots \int F_n(s - x_1 - \cdots - x_{n-1}) \prod_{j=1}^{n-1} f_j(x_j) \, dx_{n-1} \, dx_{n-2} \cdots dx_1$$

$F_j(\cdot)$ = CDF of X_j
$f_j(\cdot)$ = PDF of X_j

The PDF of S is

$$f_S(s) = \int_{x_1+\cdots+x_n=s} \cdots \int f_n(s - x_1 - \cdots - x_{n-1}) \prod_{j=1}^{n-1} f_j(x_j) \, dx_{n-1} \, dx_{n-2} \cdots dx_1$$

The mth moment of S is

$$E[S^m] = E[(X_1 + \cdots + X_n)^m]$$

$$= \sum_{p_1+\cdots+p_n=m} \binom{m}{p_1 \, p_2 \cdots p_n} E[X_1^{p_1} X_2^{p_2} \cdots X_n^{p_n}]$$

$$= \sum_{p_1+\cdots+p_n=m} \binom{m}{p_1 \, p_2 \cdots p_n} E[X_1^{p_1}] E[X_2^{p_2}] \cdots E[X_n^{p_n}]$$

$$E(S) = E(X_1 + \cdots + X_n)$$

$$= E(X_1) + \cdots + E(X_n)$$

$$= \sum_{j=1}^{n} E(X_j)$$

$$Var(S) = Var(X_1 + \cdots + X_n)$$

$$= Var(X_1) + \cdots + Var(X_n) \quad \text{by indep}$$

$$= \sum_{j=1}^{n} Var(X_j)$$

Ex 3

Suppose $X_i \sim N(\mu_i, \sigma_i^2)$ are indep RVs.

Let $S = X_1 + \cdots + X_n$.

Then $S \sim N(\mu_1 + \cdots + \mu_n, \sigma_1^2 + \cdots + \sigma_n^2)$

The CDF of S is

$$F_S(s) = \Phi\left(\frac{s - (\mu_1 + \cdots + \mu_n)}{\sqrt{\sigma_1^2 + \cdots + \sigma_n^2}}\right)$$

Φ: CDF of $N(0,1)$

The PDF of S is

$$f_S(s) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2 + \cdots + \sigma_n^2}} e^{-\frac{[s - (\mu_1 + \cdots + \mu_n)]^2}{2(\sigma_1^2 + \cdots + \sigma_n^2)}}$$

Mean of S = $\mu_1 + \cdots + \mu_n$

Variance of S = $\sigma_1^2 + \cdots + \sigma_n^2$

2) $U = \max(X_1, \ldots, X_n)$

The CDF of U is

$$F_U(u) = P[\max(X_1, \ldots, X_n) \leq u]$$

$$= P[X_1 \leq u, \ldots, X_n \leq u]$$

$$= P[X_1 \leq u] \cdots P[X_n \leq u] \quad \text{by independence}$$

$$= F_1(u) \cdots F_n(u)$$

$$= \prod_{j=1}^{n} F_j(u)$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \sum_{k=1}^{n} f_k(u) \prod_{\substack{j=1 \\ j \neq k}}^{n} F_j(u)$$

PRODUCT RULE OF DIFFERENTIATION FOR n FUNCTION

The mth moment of U is

$$E[U^m] = \int_{-\infty}^{\infty} u^m f_U(u)\, du$$

$$= \sum_{k=1}^{n} \int_{-\infty}^{\infty} u^m f_k(u) \prod_{\substack{j=1 \\ j \neq k}}^{n} F_j(u)\, du$$

Ex 1

$n = 2$

$$F_U(u) = F_1(u)\, F_2(u)$$

$$f_U(u) = f_1(u)\, F_2(u) + F_1(u)\, f_2(u)$$

$n = 3$

$$F_U(u) = F_1(u)\, F_2(u)\, F_3(u)$$

$$f_U(u) = f_1(u)\, F_2(u)\, F_3(u) + F_1(u)\, f_2(u)\, F_3(u) + F_2(u)\, F_2(u)\, f_3(u)$$

3) $V = \min(X_1, \dots, X_n)$

The CDF of V is

$$F_V(v) = P[\min(X_1, \dots, X_n) \leq v]$$

$$= 1 - P[\min(X_1, \dots, X_n) > v]$$

$$= 1 - P[X_1 > v, \dots, X_n > v]$$

$$= 1 - P[X_1 > v] \cdots P[X_n > v] \quad \text{by independence}$$

$$= 1 - (1 - P(X_1 \leq v)) \cdots (1 - P(X_n \leq v))$$

$$= 1 - (1 - F_1(v)) \cdots (1 - F_n(v))$$

$$= 1 - \prod_{j=1}^{n} [1 - F_j(v)]$$

The PDF of V is

$$f_V(v) = \sum_{k=1}^{n} f_k(v) \prod_{\substack{j=1 \\ j \neq k}}^{n} [1 - F_j(v)]$$

The m^{th} moment of V is

$$E[V^m] = \int_{-\infty}^{\infty} v^m f_V(v)\, dv$$

$$= \sum_{k=1}^{n} \int_{-\infty}^{\infty} v^m f_k(v) \prod_{\substack{j=1 \\ j \neq k}}^{n} [1 - F_j(v)]\, dv$$

Ex 2

$n = 2$

$$F_V(v) = 1 - [1 - F_1(v)][1 - F_2(v)]$$

$$= F_1(v) + F_2(v) - F_1(v) F_2(v)$$

$$f_V(v) = f_1(v) + f_2(v) - f_1(v) F_2(v) - F_1(v) f_2(v)$$

$n = 3$

$$F_V(v) = 1 - [1 - F_1(v)][1 - F_2(v)][1 - F_3(v)]$$

$$= F_1(v) + F_2(v) + F_3(v) - F_1(v) F_2(v) - F_1(v) F_3(v) - F_2(v) F_3(v) + F_1(v) F_2(v) F_3(v)$$

$$f_V(v) = f_1(v) + f_2(v) + f_3(v) - f_1(v) F_2(v) - F_1(v) f_2(v) - f_1(v) F_3(v) - F_1(v) f_3(v) - f_2(v) F_3(v) - F_2(v) f_3(v) + f_1(v) F_2(v) F_3(v) + F_1(v) f_2(v) F_3(v) + F_1(v) F_2(v) f_3(v).$$

Case iv $X_1, \dots, X_N$ are INID, N is a RV independent of $X_1, X_2, \dots$

1) $S = X_1 + \dots + X_N$

The CDF of S is

$$F_S(s) = P[X_1 + \dots + X_N \le s]$$

$$= \sum_{n=1}^{\infty} P[X_1 + \dots + X_N \le s \mid N = n] \, P(N = n)$$

Total Probability Rule

$$= \sum_{n=1}^{\infty} \left[\int \dots \int_{x_1 + \dots + x_n \le s} F_n(s - x_1 - \dots - x_{n-1}) \prod_{j=1}^{n-1} f_j(x_j) \, dx_{n-1} \, dx_{n-2} \dots dx_1 \right] P(N = n)$$

The PDF of S is

$$f_S(s) = \sum_{n=1}^{\infty} \left[\int \dots \int_{x_1 + \dots + x_n = s} f_n(s - x_1 - \dots - x_{n-1}) \prod_{j=1}^{n-1} f_j(x_j) \, dx_{n-1} \, dx_{n-2} \dots dx_1 \right] P(N = n)$$

The mth moment of S is

$$E[S^m] = E[(X_1 + \dots + X_N)^m]$$

$$= \sum_{n=1}^{\infty} E[(X_1 + \dots + X_N)^m \mid N = n] \, P(N = n)$$

$$= \sum_{n=1}^{\infty} \sum_{p_1 + \dots + p_n = m} \binom{m}{p_1 p_2 \dots p_n} E[X_1^{p_1}] \dots E[X_n^{p_n}] \, P(N = n)$$

The mean of S is

$$E[S] = \sum_{n=1}^{\infty} E[S|N=n] \; P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[\sum_{j=1}^{n} E(X_j) \right] P(N=n)$$

The variance of S is

$$Var[S] = \sum_{n=1}^{\infty} Var[S|N=n] \; P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[\sum_{j=1}^{n} Var(X_j) \right] P(N=n)$$

EXAMPLE CLASS

15 OCTOBER

12:00-13:00PM

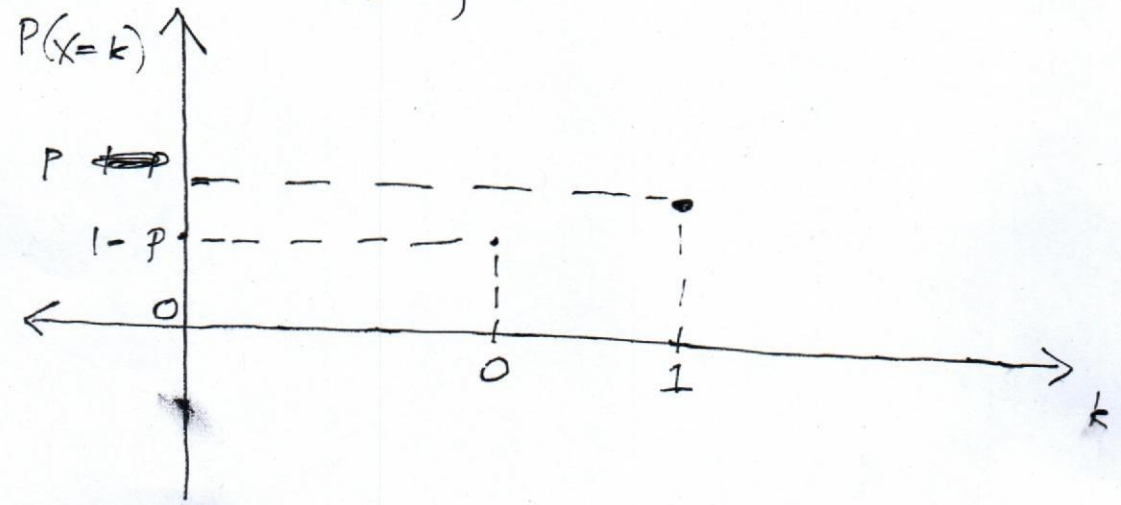
MATH3/4/68181

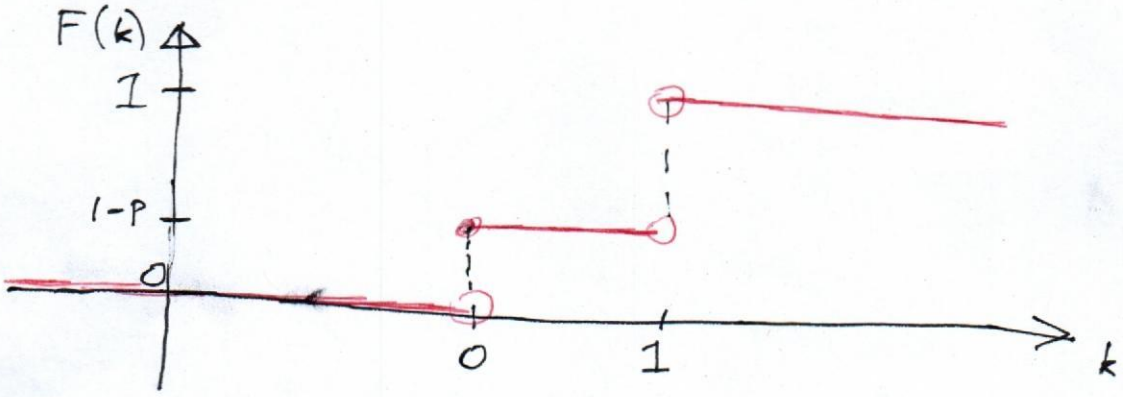
Q1

Bernoulli(p)

$$X = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{" " } 1-p \end{cases}$$

$$\lim_{k \to w(F)} \frac{P(X=k)}{1 - F(k-1)}$$



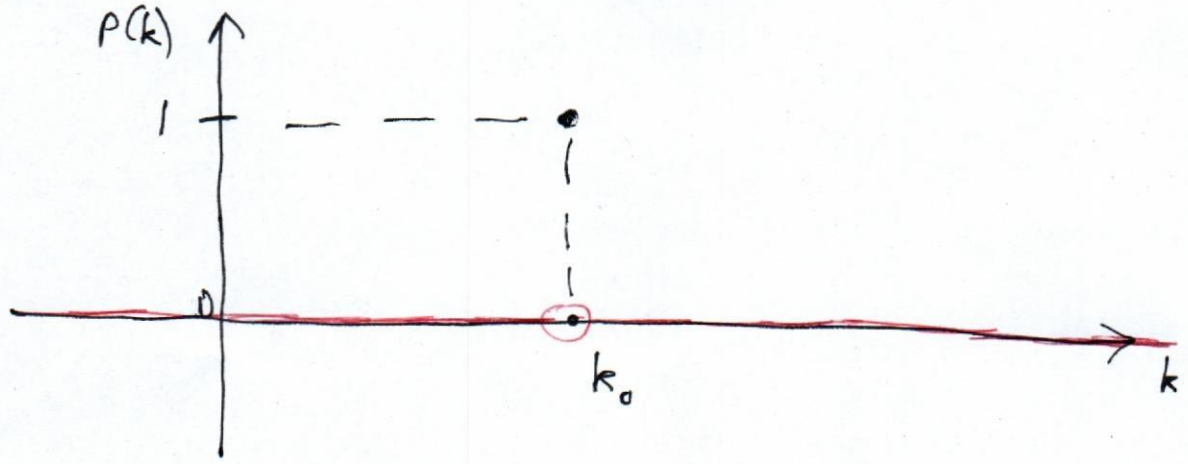


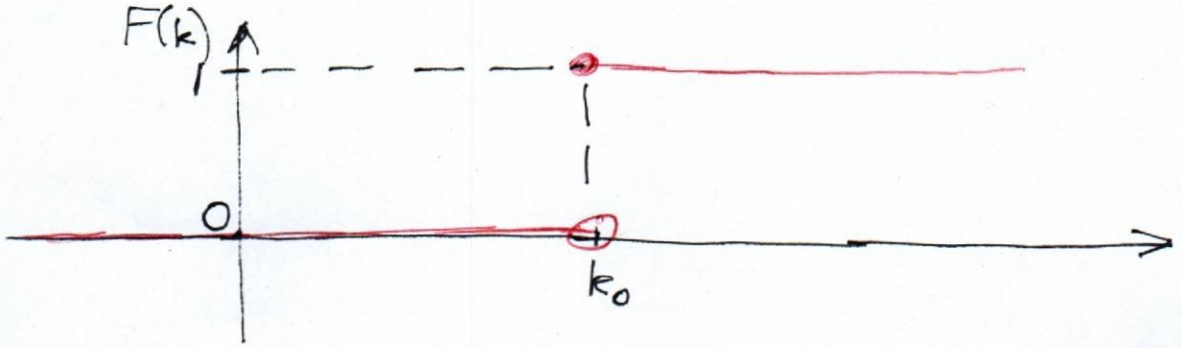
$w(F) = 1$

$$= \frac{P(X=1)}{1 - F(1-1)} = \frac{p}{1 - F(0)} = \frac{p}{1-(1-p)} = 1 \neq 0$$

$\Rightarrow$ There exist no sequences $a_n > 0$ and $b_n \in \mathbb{R}$ such that ETT holds.

Q2





$$w(F) = k_0$$

$$\lim_{k \to k_0} \frac{P(X=k)}{1-F(k-1)} = \frac{P(X=k_0)}{1-F(k_0-1)}$$

$$= \frac{1}{1-0} = 1 \neq 0$$

$\Rightarrow$ There exist no sequences $a_n > 0$ and $b_n \in \mathbb{R}$ such that ETT holds.

Q4

$$P(k) = \frac{k^{-s}}{\zeta(s)}, \quad k = 1, 2, \ldots$$

$$\omega(F) = \infty$$

$$\lim_{k \to \infty} \frac{P(X = k)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{P(X = k)}{1 - P(X \leq k-1)}$$

$$= \lim_{k \to \infty} \frac{P(X = k)}{P(X \geq k)}$$

$$= \lim_{k \to \infty} \frac{P(X = k)}{\sum_{i=k}^{\infty} P(X = i)}$$

$$= \lim_{k \to \infty} \frac{k^{-s} / \zeta(s)}{\sum_{i=(k)}^{\infty} i^{-s} / \zeta(s)}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{\sum_{i=(k)}^{\infty} i^{-s}} = \lim_{k \to \infty} \frac{k^{-s}}{\int_{k}^{\infty} x^{-s} \, dx}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{\left[\frac{x^{1-s}}{1-s} \right]_{k}^{\infty}} = \lim_{k \to \infty} \frac{k^{-s}}{0 - \frac{k^{1-s}}{1-s}} \quad \text{if } s > 1$$

$$= \lim_{k \to \infty} \frac{s-1}{k} = 0 \Rightarrow \text{ETT holds.}$$

Q5

$$P(k) = -\log_2\left[1 - \frac{1}{(k+1)^2}\right], \quad k = 1, 2, \ldots$$

$$\omega(F) = \infty.$$

$$\lim_{k\to\infty} \frac{P(X=k)}{1-F(k-1)}$$

$$= \lim_{k\to\infty} \frac{-\log_2\left[1 - \frac{1}{(k+1)^2}\right]}{\left[X - \left[X - \log_2\left[\frac{k+1}{k}\right]\right]\right]}$$

$$= \lim_{k\to\infty} \frac{-\log_2\left[\frac{k^2+2k}{(k+1)^2}\right]}{\log_2\left[\frac{k+1}{k}\right]}$$

$$= \lim_{k\to\infty} \frac{-\log_2[k^2+2k] + 2\log_2[k+1]}{\log_2[k+1] - \log_2 k}$$

$$\frac{\partial \log_2 x}{\partial x} = \frac{1}{(\log 2)\cdot x}$$

$$\overset{LH}{=} \lim_{k\to\infty} \frac{-\frac{2k+2}{(\log 2)(k^2+2k)} + \frac{2}{(\log 2)(k+1)}}{\frac{1}{(\log 2)(k+1)} - \frac{1}{(\log 2)\cdot k}}$$

$$= \lim_{k\to\infty} \frac{\frac{-(2k+2)(k+1) + 2(k^2+2k)}{(k^2+2k)(k+1)}}{-\frac{1}{(k+1)k}}$$

$$= \lim_{k\to\infty} \frac{\frac{-1}{(k^2+2k)(k+1)}}{-\frac{1}{(k+1)k}} = \lim -\frac{k}{k^2+2k} = 0$$

$\Rightarrow$ ETT holds.

EXAMPLE CLASS

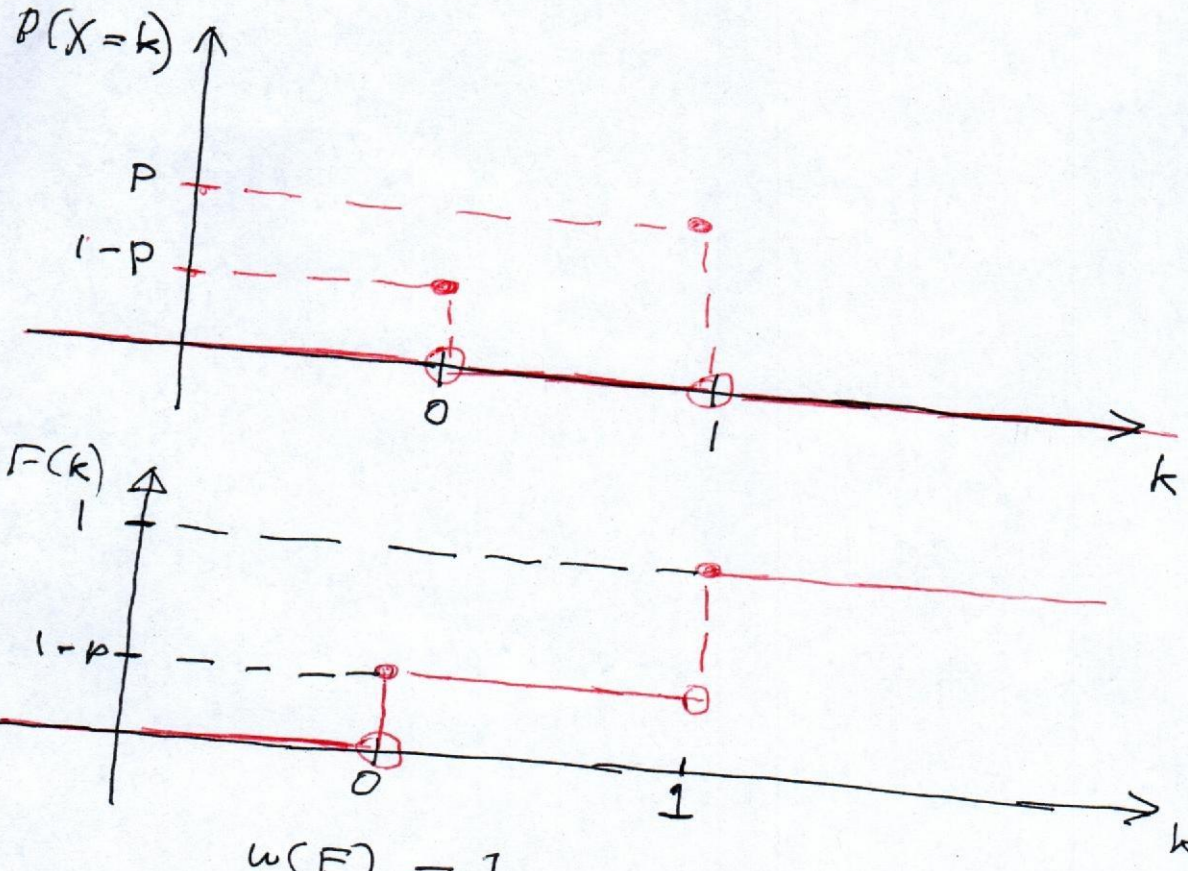
16 OCTOBER

16:00-17:00PM

MATH3/4/68181

Q1 $X \sim \text{Bernoulli}(p)$

$$X = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{" " } 1-p \end{cases}$$



$\omega(F) = 1$

$$\lim_{k \to 1} \frac{P(X=k)}{1 - F(k-1)} = \frac{P(X=1)}{1 - F(1-1)}$$

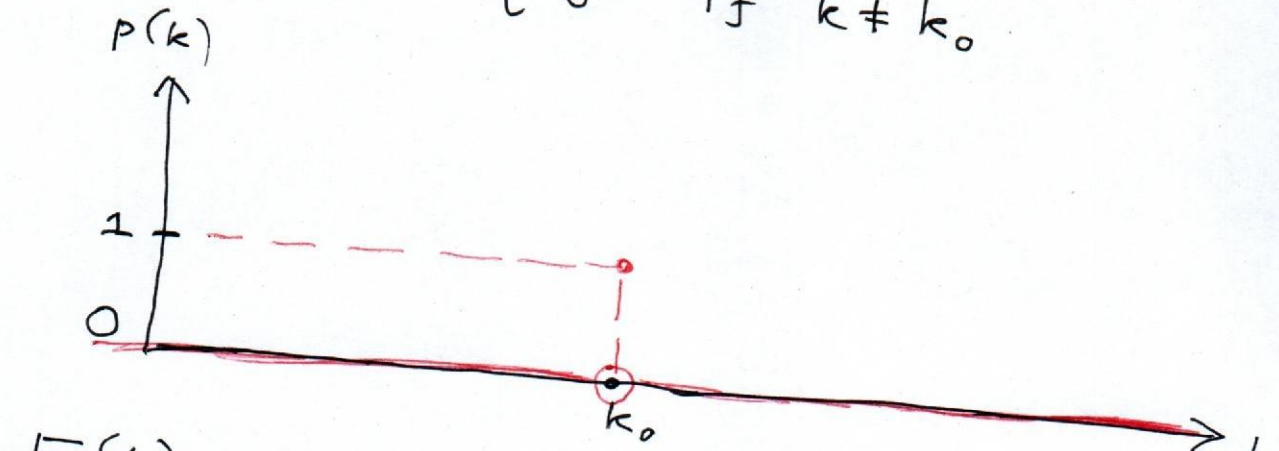
$$= \frac{p}{1 - F(0)} = \frac{p}{1-(1-p)}$$

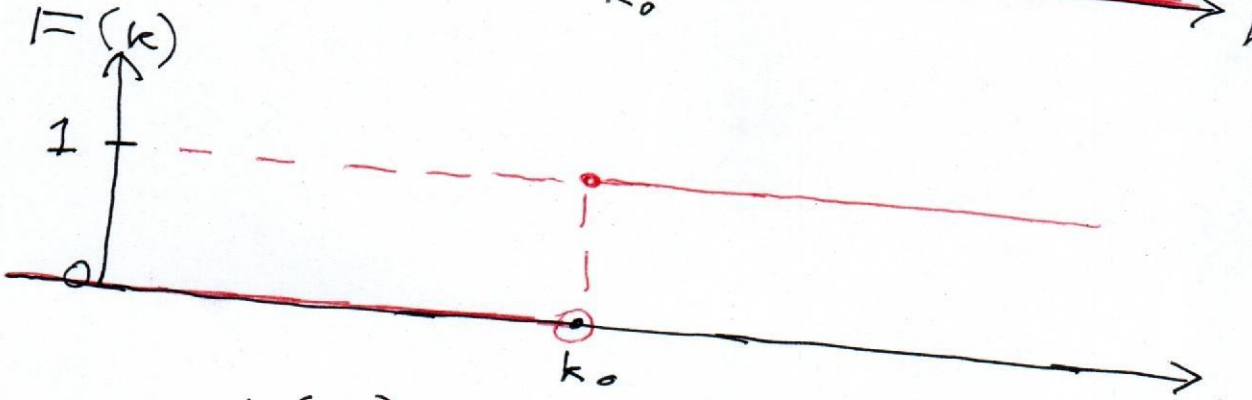
$$= \frac{p}{p} = 1 \neq 0$$

$\Rightarrow$ There exist no sequences $a_n > 0$ and $b_n \in \mathbb{R}$ such that ETT holds.

Q2

$$P(k) = \begin{cases} 1 & \text{if } k = k_0 \\ 0 & \text{if } k \neq k_0 \end{cases}$$





$$w(F) = k_0$$

$$\lim_{k \to k_0} \frac{P(X=k)}{1 - F(k-1)}$$

$$= \frac{P(X=k_0)}{1 - F(k_0 - 1)}$$

$$= \frac{1}{1-0} = 1 \neq 0$$

There exists no sequences $a_n > 0$ and $b_n \in \mathbb{R}$ such that ETT holds.

Q4

$$p(k) = \frac{k^{-s}}{\zeta(s)}, \quad k = 1, 2, \ldots$$

$$\omega(F) = \infty$$

$$\lim_{k \to \infty} \frac{P(X=k)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{P(X=k)}{1 - P(X \leq k-1)}$$

$$= \lim_{k \to \infty} \frac{P(X=k)}{P(X \geq k)}$$

$$= \lim_{k \to \infty} \frac{P(X=k)}{\sum_{i=k}^{\infty} P(X=i)}$$

$$= \lim_{k \to \infty} \frac{\frac{k^{-s}}{\zeta(s)}}{\sum_{i=k}^{\infty} \frac{i^{-s}}{\zeta(s)}} = \lim_{k \to \infty} \frac{k^{-s}}{\sum_{i=k}^{\infty} i^{-s}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{\int_{k}^{\infty} x^{-s}\,dx} = \lim_{k \to \infty} \frac{k^{-s}}{\left[\frac{x^{1-s}}{1-s}\right]_{k}^{\infty}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{0 - \frac{k^{1-s}}{1-s}} \qquad \text{provided} \quad s > 1$$

$$= \lim_{k \to \infty} \frac{s-1}{k} = 0 \implies \text{ETT will hold.}$$

Q5 $P(k) = -\log\left[1 - \frac{1}{(k+1)^2}\right]$, $k = 1, 2, \ldots$

$w(F) = \infty$

$$\lim_{k\to\infty} \frac{P(X=k)}{1-F(k-1)}$$

$$= \lim_{k\to\infty} \frac{-\log_2\left[1 - \frac{1}{(k+1)^2}\right]}{X - \left[X - \log_2\left[\frac{k+1}{k}\right]\right]}$$

$$= \lim_{k\to\infty} \frac{-\log_2\left[\frac{k^2+2k}{(k+1)^2}\right]}{\log_2\left[\frac{k+1}{k}\right]}$$

$$= \lim_{k\to\infty} \frac{-\log_2(k^2+2k) + 2\log_2(k+1)}{\log_2(k+1) - \log_2 k}$$

LH $$\lim_{k\to\infty} \frac{-\frac{2k+2}{(\log 2)\cdot(k^2+2k)} + \frac{2}{(\log 2)(k+1)}}{\frac{1}{(\log 2)(k+1)} - \frac{1}{(\log 2)k}}$$

$$= \lim_{k\to\infty} \frac{-\frac{2(k+1)}{k(k+2)} + \frac{2}{k+1}}{\frac{1}{k+1} - \frac{1}{k}}$$

$$= \lim_{k\to\infty} \frac{\frac{-2(k+1)^2 + 2k(k+2)}{k(k+2)(k+1)}}{\frac{-1}{(k+1)k}}$$

$$= \lim_{k\to\infty} \frac{\frac{-2}{k(k+2)(k+1)}}{\frac{-1}{(k+1)k}} = \lim_{k\to\infty} \frac{2}{k+2} = 0$$

$\Rightarrow$ ETT will hold

$$\frac{\partial}{\partial x}\log_2 x = \frac{1}{(\log 2)\,x}$$

Q7

$$w(F) = \infty$$

$$\lim_{k \to \infty} \frac{P(X = k)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{F(k) - F(k-1)}{1 - F(k-1)}$$

$P(X = k)$
$= F(k) - F(k-1)$

LECTURE

22 OCTOBER

9:00-10:00AM

MATH3/4/68181

2 variable case

(X_1, X_2)

The joint CDF of (X_1, X_2) is

$$F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)$$

The joint survival function of (X_1, X_2) is

$$\bar{F}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$$

The marginal CDFs are

$$P(X_1 \leq x_1) = F(x_1, \infty)$$
$$= 1 - \bar{F}(x_1, -\infty),$$
$$P(X_2 \leq x_2) = F(\infty, x_2)$$
$$= 1 - \bar{F}(-\infty, x_2)$$

The relationships between F and $\bar{F}$:

$$\bar{F}(x_1, x_2) = 1 - F(x_1, \infty) - F(\infty, x_2) + F(x_1, x_2),$$

$$F(x_1, x_2) = 1 - \bar{F}(x_1, -\infty) - \bar{F}(-\infty, x_2) + \bar{F}(x_1, x_2)$$

3 variables case

(X_1, X_2, X_3)

The joint CDF of (X_1, X_2, X_3) is

$$F(x_1, x_2, x_3) = P(X_1 \le x_1, X_2 \le x_2, X_3 \le x_3)$$

The joint survival function of (X_1, X_2, X_3) is

$$\bar{F}(x_1, x_2, x_3) = P(X_1 > x_1, X_2 > x_2, X_3 > x_3)$$

The relationships between F and $\bar{F}$ are:

$$\begin{aligned}\bar{F}(x_1, x_2, x_3) = {} & 1 - F(x_1, \infty, \infty) \\ & - F(\infty, x_2, \infty) \\ & - F(\infty, \infty, x_3) \\ & + F(x_1, x_2, \infty) \\ & + F(x_1, \infty, x_3) \\ & + F(\infty, x_2, x_3) \\ & - F(x_1, x_2, x_3),\end{aligned}$$

$$\begin{aligned}F(x_1, x_2, x_3) = {} & 1 - \bar{F}(x_1, -\infty, -\infty) \\ & - \bar{F}(-\infty, x_2, -\infty) \\ & - \bar{F}(-\infty, -\infty, x_3) \\ & + \bar{F}(x_1, x_2, -\infty) \\ & + \bar{F}(x_1, -\infty, x_3) \\ & + \bar{F}(-\infty, x_2, x_3) \\ & - \bar{F}(x_1, x_2, x_3).\end{aligned}$$

p variables case

$$(X_1, X_2, \cdots, X_p)$$

The joint CDF of $(X_1, \cdots, X_p)$ is

$$F(x_1, \cdots, x_p) = P(X_1 \leq x_1, \cdots, X_p \leq x_p)$$

The joint survival function of $(X_1, \cdots, X_p)$ is

$$\bar{F}(x_1, \cdots, x_p) = P(X_1 > x_1, \cdots, X_p > x_p).$$

The relationships between F and $\bar{F}$ are:

$$\bar{F}(x_1, \cdots, x_p) = 1 - \sum_{i=1}^{p} F(\infty, \cdots, \infty, \boxed{x_i}, \infty, \cdots, \infty)$$

(ith position)

$$+ \sum_{\substack{i=1 \\ i<j}}^{p} \sum_{j=1}^{p} F(\infty, \cdots, \infty, \boxed{x_i}, \infty, \cdots, \infty, \boxed{x_j}, \infty, \cdots, \infty)$$

(ith position, jth position)

$$\vdots$$

$$+ (-1)^p F(x_1, \cdots, x_p),$$

$$F(x_1, \cdots, x_p) = 1 - \sum_{i=1}^{p} \bar{F}(-\infty, \cdots, -\infty, \boxed{x_i}, -\infty, \cdots, -\infty)$$

(ith position)

$$+ \sum_{\substack{i=1 \\ i<j}}^{p} \sum_{j=1}^{p} \bar{F}(-\infty, \cdots, -\infty, \boxed{x_i}, -\infty, \cdots, -\infty, \boxed{x_j}, -\infty, \cdots, -\infty)$$

(ith position, jth position)

$$\vdots$$

$$+ (-1)^p \bar{F}(x_1, \cdots, x_p).$$

Case V $X_1, X_2, \ldots, X_n$ are dependent, n fixed

Let $F(x_1, x_2, \ldots, x_n)$ denote the joint CDF of $(X_1, \ldots, X_n)$. Let $\bar{F}(x_1, \ldots, x_n)$ denote the joint survival function of $(X_1, \ldots, X_n)$.

1) $S = X_1 + \cdots + X_n$

The CDF of S is

$$F_S(s) = P(S \leq s) = P(X_1 + \cdots + X_n \leq s)$$

$$= \int \cdots \int_{x_1 + \cdots + x_n \leq s} f(x_1, x_2, \ldots, x_n)\, dx_n \cdots dx_2 dx_1$$

$f(x_1, \ldots, x_n)$: Joint PDF of $(X_1, \ldots, X_n)$

$$f(x_1, \ldots, x_p) = \frac{\partial^p}{\partial x_1 \cdots \partial x_p} F(x_1, \ldots, x_p)$$

$$f(x_1, \ldots, x_p) = \frac{(-1)^p \partial^p}{\partial x_1 \cdots \partial x_p} \bar{F}(x_1, \ldots, x_p)$$

The PDF of S is

$$f_S(s) = \frac{d}{ds} F_S(s)$$

$$= \int \cdots \int_{x_1 + \cdots + x_n = s} f(x_1, x_2, \ldots, x_n)\, dx_n \cdots dx_2 dx_1$$

The mth moment of S is

$$E[S^m]$$

$$= E[(X_1 + \cdots + X_n)^m]$$

$$= \sum_{p_1 + \cdots + p_n = m} \binom{m}{p_1 p_2 \cdots p_n} E[X_1^{p_1} X_2^{p_2} \cdots X_n^{p_n}]$$

The mean of S is

$$E[S] = E(X_1) + \cdots + E(X_n)$$

$$= \sum_{i=1}^{n} E(X_i)$$

The variance of S is

$$\mathrm{Var}[S] = \mathrm{Var}(X_1 + \cdots + X_n)$$

$$\neq \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

because $X_1, \ldots, X_n$ are dependent RVs.

2) $U = \max(X_1, \cdots, X_n)$

The CDF of U is

$$F_U(u) = P[\max(X_1, \cdots, X_n) \leq u]$$

$$= P[X_1 \leq u, \cdots, X_n \leq u]$$

$$= F(u, \cdots, u)$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \frac{d}{du} F(u, \cdots, u)$$

The m^{th} moment of U is

$$E[U^m] = \int_{-\infty}^{\infty} u^m f_U(u)\, du$$

$$= \int_{-\infty}^{\infty} u^m \frac{d}{du} F(u, \cdots, u)\, du.$$

3) $V = \min(X_1, \ldots, X_n)$

The CDF of V is

$$F_V(v) = P[\min(X_1, \ldots, X_n) \leq v]$$

$$= 1 - P[\min(X_1, \ldots, X_n) > v]$$

$$= 1 - P[X_1 > v, \ldots, X_n > v]$$

$$= 1 - \bar{F}(v, \ldots, v)$$

The PDF of V is

$$f_V(v) = \frac{d}{dv} F_V(v)$$

$$= -\frac{d}{dv} \bar{F}(v, \ldots, v)$$

The m^{th} moment of V is

$$E[V^m] = \int_{-\infty}^{\infty} v^m f_V(v)\, dv$$

$$= -\int_{-\infty}^{\infty} v^m \frac{d}{dv} \bar{F}(v, \ldots, v)\, dv.$$

LECTURE

23 OCTOBER

9:00-10:00AM

MATH3/4/68181

Case vi $X_1, X_2, \ldots, X_N$ are dependent RVs, N is a RV indep of $X_1, X_2, \ldots, X_N$

1) $S = X_1 + X_2 + \cdots + X_N$

The CDF of S is

$$F_S(s) = P(X_1 + \cdots + X_N \leq s)$$

$$= \sum_{n=1}^{\infty} P(X_1 + \cdots + X_N \leq s \mid N=n)\, P(N=n)$$

Total Probability Rule

$$= \sum_{n=1}^{\infty} \left[\int_{x_1 + \cdots + x_n \leq s} \cdots \int f(x_1, \ldots, x_n)\, dx_n \cdots dx_1 \right] P(N=n)$$

The PDF of S is

$$f_S(s) = \sum_{n=1}^{\infty} \left[\int_{x_1 + \cdots + x_n = s} \cdots \int f(x_1, \ldots, x_n)\, dx_n \cdots dx_1 \right] P(N=n)$$

The mth moment of S is

$$E[S^m] = \sum_{n=1}^{\infty} E[S^m \mid N=n]\, P(N=n)$$

$$= \sum_{n=1}^{\infty} \sum_{p_1 + \cdots + p_n = m} \binom{m}{p_1\, p_2 \cdots p_n} E[X_1^{p_1} \cdots X_n^{p_n}]\, P(N=n)$$

2) $U = \max(X_1, X_2, \ldots, X_N)$

The CDF of U is

$$F_U(u) = P\left[\max(X_1, \ldots, X_N) \leq u\right]$$

$$= \sum_{n=1}^{\infty} P\left[\max(X_1, \ldots, X_N) \leq u \mid N = n\right] P(N = n)$$

$$= \sum_{n=1}^{\infty} F(\underbrace{u, \ldots, u}_{u \text{ repeated } n \text{ times}}) \; P(N = n)$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \sum_{n=1}^{\infty} \left[\frac{d}{du} F(u, \ldots, u)\right] P(N = n)$$

The mth moment of U is

$$E(U^m) = \int_{-\infty}^{\infty} v^m f_U(u)\, du$$

$$= \sum_{n=1}^{\infty} \left[\int_{-\infty}^{\infty} v^m \frac{d}{du} F(u, \ldots, u)\, du\right] P(N = n)$$

3) $V = \min(X_1, \ldots, X_N)$

The CDF of V is

$$F_V(v) = P\left[\min(X_1, \ldots, X_N) \leq v\right]$$

$$= \sum_{n=1}^{\infty} P\left[\min(X_1, \ldots, X_N) \leq u \,\middle|\, N=n\right] P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[1 - \bar{F}(\underbrace{v, \ldots, v}_{v \text{ repeated } n \text{ times}})\right] P(N=n)$$

$$= \sum_{n=1}^{\infty} P(N=n) - \sum_{n=1}^{\infty} \bar{F}(v, \ldots, v) P(N=n)$$

$$= 1 - \sum_{n=1}^{\infty} \bar{F}(v, \ldots, v) P(N=n)$$

The PDF of V is

$$f_V(v) = -\sum_{n=1}^{\infty} \left[\frac{d}{dv} \bar{F}(v, \ldots, v)\right] P(N=n)$$

The mth moment of V is

$$E[V^m] = -\sum_{n=1}^{\infty} \left\{\int_{-\infty}^{\infty} v^m \left[\frac{d}{dv} \bar{F}(v, \ldots, v)\right] dv\right\} P(N=n)$$

Ex 1 Suppose a portfolio has 2 investments. Suppose

$$F(x_1, x_2) = e^{-x_1 - x_2 - \theta x_1 x_2},$$

$$x_1 > 0$$
$$x_2 > 0$$

Find the distributions of

a) S
b) U
c) V

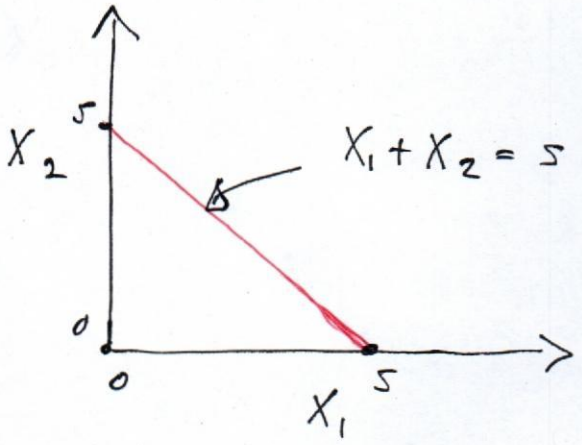
The joint PDF of (X_1, X_2) is

$$f(x_1, x_2) = \frac{(-1)^2 \partial^2}{\partial x_1 \partial x_2} e^{-x_1 - x_2 - \theta x_1 x_2}$$

$$= \frac{\partial}{\partial x_1} \left[(-1 - \theta x_1) e^{-x_1 - x_2 - \theta x_1 x_2} \right]$$

$$= (-1 - \theta x_1)(-1 - \theta x_2) e^{-x_1 - x_2 - \theta x_1 x_2} - \theta e^{-x_1 - x_2 - \theta x_1 x_2}$$

$$= \left[1 - \theta + \theta(x_1 + x_2) + \theta^2 x_1 x_2 \right] e^{-x_1 - x_2 - \theta x_1 x_2}$$



a) $$f_S(s) = \int_0^5 f(x_1, 5-x_1)\, dx_1$$

$$= \int_0^5 \left[1 - \theta + \theta 5 + \theta^2 x_1 (5 - x_1)\right] e^{-5-\theta x_1 (5-x_1)}\, dx_1$$

$$= e^{-5}(1 - \theta + \theta 5) \int_0^5 e^{-\theta x_1 (5-x_1)}\, dx_1$$

$$+ e^{-5}\theta^2 \int_0^5 x_1 (5 - x_1)\, e^{-\theta x_1 (5-x_1)}\, dx_1$$

Homework : Try working out these 2 integrals

b) The CDF of U is

$$F_U(u) = F(U, u)$$

$$= 1 - \bar{F}(u, -\infty) - \bar{F}(-\infty, u) + \bar{F}(u, u)$$

$$= 1 - e^{-u} - e^{-u} + e^{-2u-\theta u^2}$$

The PDF of U is

$$f_U(u) = 2e^{-u} - (2 - 2\theta u)e^{-2u-\theta u^2}$$

c) The CDF of V is

$$F_V(v) = 1 - \bar{F}(v, v)$$

$$= 1 - e^{-2v-\theta v^2}$$

The PDF of V is

$$f_V(v) = (2 + 2\theta v)e^{-2v-\theta v^2}$$

Ex 2 Suppose a portfolio has 3 investment. Suppose also

$$F(x_1, x_2, x_3) = \left[1 + x_1 + x_2 + x_3\right]^{-a},$$

$$a > 0$$
$$x_1 > 0$$
$$x_2 > 0$$
$$x_3 > 0$$

Find the distributions of

a) S

b) U

c) V

Homework

LECTURE

26 OCTOBER

10:00-11:00AM

MATH4/68181

Case iv $X_1, \dots, X_N$ are INID, N is a RV independent of $X_1, X_2, \dots$

1) $S = X_1 + \dots + X_N$

The CDF of S is

$$F_S(s) = P[X_1 + \dots + X_N \le s]$$

$$= \sum_{n=1}^{\infty} P[X_1 + \dots + X_N \le s \mid N = n] \, P(N = n)$$

Total Probability Rule

$$= \sum_{n=1}^{\infty} \left[\int \dots \int_{x_1 + \dots + x_n \le s} F_n(s - x_1 - \dots - x_{n-1}) \prod_{j=1}^{n-1} f_j(x_j) \, dx_{n-1} \, dx_{n-2} \dots dx_1 \right] P(N = n)$$

The PDF of S is

$$f_S(s) = \sum_{n=1}^{\infty} \left[\int \dots \int_{x_1 + \dots + x_n = s} f_n(s - x_1 - \dots - x_{n-1}) \prod_{j=1}^{n-1} f_j(x_j) \, dx_{n-1} \, dx_{n-2} \dots dx_1 \right] P(N = n)$$

The mth moment of S is

$$E[S^m] = E[(X_1 + \dots + X_N)^m]$$

$$= \sum_{n=1}^{\infty} E[(X_1 + \dots + X_N)^m \mid N = n] \, P(N = n)$$

$$= \sum_{n=1}^{\infty} \sum_{p_1 + \dots + p_n = m} \binom{m}{p_1 p_2 \dots p_n} E[X_1^{p_1}] \dots E[X_n^{p_n}] \, P(N = n)$$

The mean of S is

$$E[S] = \sum_{n=1}^{\infty} E[S|N=n] \; P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[\sum_{j=1}^{n} E(X_j) \right] P(N=n)$$

The variance of S is

$$Var[S] = \sum_{n=1}^{\infty} Var[S|N=n] \; P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[\sum_{j=1}^{n} Var(X_j) \right] P(N=n)$$

2) $U = \max(X_1, X_2, \ldots, X_N)$

The CDF of U is

$$F_U(u) = P\left[\max(X_1, \ldots, X_N) \leq u\right]$$

$$= \sum_{n=1}^{\infty} P\left[\max(X_1, \ldots, X_N) \leq u \mid N = n\right] P(N=n)$$

$$= \sum_{n=1}^{\infty} \prod_{j=1}^{n} F_j(u)\, P(N=n).$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} f_k(u) \prod_{\substack{j=1 \\ j \neq k}}^{n} F_j(u)\right] P(N=n)$$

The mth moment of U is

$$E[U^m] = \int_{-\infty}^{\infty} u^m f_U(u)\, du$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \left[\int_{-\infty}^{\infty} u^m f_k(u) \prod_{\substack{j=1 \\ j \neq k}}^{n} F_j(u)\, du\right] P(N=n).$$

3) $V = \min(X_1, X_2, \cdots, X_N)$

The CDF of V is

$$F_V(v) = P[\min(X_1, \cdots, X_N) \leq v]$$

$$= 1 - P[\min(X_1, \cdots, X_N) > v]$$

$$= 1 - \sum_{n=1}^{\infty} P[\min(X_1, \cdots, X_N) > v / N = n] \, P(N = n)$$

$$= 1 - \sum_{n=1}^{\infty} \prod_{j=1}^{n} [1 - F_j(v)] \, P(N = n) .$$

The PDF of V is

$$f_V(v) = \frac{d}{dv} F_V(v)$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_k(v) \prod_{\substack{j=1 \\ j \neq k}}^{n} [1 - F_j(v)] \, P(N = n)$$

The mth moment of V is

$$E[V^m] = \int_{-\infty}^{\infty} v^m f_V(v) \, dv$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \left[\int_{-\infty}^{\infty} v^m f_k(v) \prod_{\substack{j=1 \\ j \neq k}}^{n} [1 - F_j(v)] \, dv \right] P(N = n).$$

Ex 1 Suppose $X_i \sim Exp(a_i)$, $i = 1, 2, \ldots, n$

Find the distribution of U and V.

The CDF of U is

$$F_U(u) = \prod_{j=1}^{n} F_j(u)$$

$$= \prod_{j=1}^{n} \left[1 - e^{-a_j u}\right]$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \sum_{k=1}^{n} a_k e^{-a_k u} \prod_{\substack{j=1 \\ j \neq k}}^{n} \left[1 - e^{-a_j u}\right]$$

The CDF of V is

$$F_V(v) = 1 - \prod_{j=1}^{n} \left[1 - F_j(v)\right]$$

$$= 1 - \prod_{j=1}^{n} e^{-a_j v}$$

$$= 1 - e^{-(a_1 + a_2 + \cdots + a_n) v}$$

The PDF of V is

$$f_V(v) = (a_1 + \cdots + a_n) e^{-(a_1 + \cdots + a_n) v}$$

$$\Rightarrow \quad V \sim Exp(a_1 + \cdots + a_n).$$

Mean of V $= \dfrac{1}{a_1 + \cdots + a_n}$

Variance of V $= \dfrac{1}{(a_1 + \cdots + a_n)^2}$

Ex 2

Suppose X_i are IID Exp(a) for $i = 1, 2, \ldots, N$.

$N \sim$ Truncated Poisson(λ) independent of $X_1, X_2, \ldots$

Poisson: $P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}$, $k = 0, 1, \ldots$

Truncated Poisson: $P(X = k) = \frac{e^{-\lambda} \lambda^k}{(1 - e^{-\lambda}) k!}$, $k = 1, 2, \ldots$

Find the distributions of U and V

The CDF of U is

$$F_U(u) = \sum_{n=1}^{\infty} \left[\prod_{j=1}^{n} F_j(u) \right] P(N = n)$$

$$= \sum_{n=1}^{\infty} (1 - e^{-au})^n \frac{e^{-\lambda} \lambda^n}{(1 - e^{-\lambda}) n!}$$

$$= \frac{e^{-\lambda}}{1 - e^{-\lambda}} \sum_{n=1}^{\infty} \frac{[(1 - e^{-au}) \lambda]^n}{n!}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1$$

$$= \frac{e^{-\lambda}}{1 - e^{-\lambda}} \left\{ e^{(1 - e^{-au}) \lambda} - 1 \right\}$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \frac{1}{1-e^{-\lambda}} \; e^{-\lambda e^{-au}} \; \lambda a e^{-au}$$

$$= \frac{\lambda a}{1-e^{-\lambda}} \; e^{-\lambda e^{-au} - au}$$

The CDF of V is

$$F_V(v) = 1 - \sum_{n=1}^{\infty} \prod_{j=1}^{n} \left[1 - F_j(v)\right] \; P(N=n)$$

$$= 1 - \sum_{n=1}^{\infty} e^{-nav} \frac{e^{-\lambda} \lambda^n}{(1-e^{-\lambda})\, n!}$$

$$= 1 - \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{n=1}^{\infty} \frac{(e^{-av}\lambda)^n}{n!}$$

$$= 1 - \frac{1}{e^{\lambda}-1} \left[e^{e^{-av}\lambda} - 1 \right]$$

The PDF of V is

$$f_V(v) = \frac{d}{dv} F_V(v)$$

$$= \frac{\lambda a}{(e^{\lambda})-1} \; e^{-av} \; e^{\lambda e^{-av}}$$

After reading week...

Financial Risk Measures

1) Value at Risk
2) Expected Shortfall.

EXAMPLE CLASS

22 OCTOBER

12:00-13:00PM

MATH3/4/68181

Q1

$X_1, \ldots, X_\alpha$ IID $Exp(\lambda)$.

Let $X = \max(X_1, \ldots, X_\alpha)$.

1. The CDF of X is

$$F_X(x) = P[\max(X_1, \ldots, X_\alpha) \le x]$$

$$= P[X_1 \le x, \ldots, X_\alpha \le x]$$

$$= P[X_1 \le x] \cdots P[X_\alpha \le x] \quad \text{by indep}$$

$$= (1 - e^{-\lambda x}) \cdots (1 - e^{-\lambda x})$$

$$= (1 - e^{-\lambda x})^\alpha$$

2. The PDF of X is

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$= \alpha (1 - e^{-\lambda x})^{\alpha - 1} \lambda e^{-\lambda x}$$

$$= \alpha \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha - 1}$$

3.

$$E(X^n) = \int_0^\infty x^n \alpha\lambda e^{-\lambda x}\left[1 - e^{-\lambda x}\right]^{\alpha-1} dx$$

$$= \alpha\lambda \int_0^\infty x^n e^{-\lambda x}\left[1 - e^{-\lambda x}\right]^{\alpha-1} dx$$

Set $y = e^{-\lambda x}$

$$x = -\frac{\log y}{\lambda}$$

$$\frac{dx}{dy} = -\frac{1}{y\lambda}$$

$$= \alpha\lambda \int_1^0 \left(-\frac{\log y}{\lambda}\right)^n y \left[1 - y\right]^{\alpha-1} \left(-\frac{1}{y\lambda}\right) dy$$

$$= \alpha \left(-\frac{1}{\lambda}\right)^n \int_0^1 (\log y)^n \left[1 - y\right]^{\alpha-1} dy$$

$$\frac{d^n y^\beta}{d\beta^n} = y^\beta (\log y)^n$$

$$\Rightarrow \frac{d^n}{d\beta^n} y^\beta \Big|_{\beta=0} = y^0 (\log y)^n = (\log y)^n \quad \therefore (*)$$

$$\circledast \quad \alpha \left(-\frac{1}{\lambda}\right)^n \int_0^1 \frac{d^n}{d\beta^n} y^\beta \Big|_{\beta=0} \left[1 - y\right]^{\alpha-1} dy$$

$$= \alpha \left(-\frac{1}{\lambda}\right)^n \frac{d^n}{d\beta^n} \left[\int_0^1 y^\beta \left[1 - y\right]^{\alpha-1} dy\right] \Big|_{\beta=0}$$

$$= \alpha \left(-\frac{1}{\lambda}\right)^n \frac{d^n}{d\beta^n} B(\beta+1, \alpha) \Big|_{\beta=0}$$

4. $$\text{Mean} = -\frac{\alpha}{\lambda} \frac{d}{d\beta} B(\beta+1, \alpha) \Big|_{\beta=0}$$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$= -\frac{\alpha}{\lambda} \frac{d}{d\beta} \frac{\Gamma(\beta+1)\Gamma(\alpha)}{\Gamma(\beta+1+\alpha)} \Big|_{\beta=0}$$

$$= -\frac{\alpha\,\Gamma(\alpha)}{\lambda} \frac{\Gamma(\beta+1+\alpha)\,\Gamma'(\beta+1) - \Gamma(\beta+1)\,\Gamma'(\beta+1+\alpha)}{[\Gamma(\beta+1+\alpha)]^2} \Big|_{\beta=0}$$

$$= -\frac{\alpha\,\Gamma(\alpha)}{\lambda} \frac{\Gamma(1+\alpha)\,\Gamma'(1) - \Gamma'(1+\alpha)}{[\Gamma(1+\alpha)]^2}.$$

5. $$\text{Variance} = E(X^2) - [E(X)]^2.$$

$$E(X^2) = +\frac{\alpha}{\lambda^2} \frac{d^2}{d\beta^2} B(\beta+1, \alpha) \Big|_{\beta=0}$$

$$= \frac{\alpha}{\lambda^2} \frac{d^2}{d\beta^2} \frac{\Gamma(\beta+1)\,\Gamma(\alpha)}{\Gamma(\beta+1+\alpha)} \Big|_{\beta=0}$$

$$= \frac{\alpha\,\Gamma(\alpha)}{\lambda^2} \frac{d^2}{d\beta^2} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)} \Big|_{\beta=0}$$

6 & 7. Need to wait until after the Reading week.

8. Suppose there is only 1 observation on x. The likelihood function is

$$L(\alpha, \lambda) = \alpha\lambda e^{-\lambda x}\left[1 - e^{-\lambda x}\right]^{\alpha-1}$$

$$\Rightarrow \log L(\alpha, \lambda) = \log(\alpha\lambda) - \lambda x + (\alpha - 1)\log\left[1 - e^{-\lambda x}\right]$$

$$\Rightarrow \frac{\partial \log L}{\partial \alpha} = \frac{1}{\alpha} + \log\left[1 - e^{-\lambda x}\right], \quad = 0 \quad -(1)$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{1}{\lambda} - x + (\alpha - 1)\frac{x e^{-\lambda x}}{1 - e^{-\lambda x}} = 0 \quad -(2)$$

$$(1) \Rightarrow 1 - e^{-\lambda x} = e^{-\frac{1}{\alpha}}$$

$$\Rightarrow e^{-\lambda x} = 1 - e^{-\frac{1}{\alpha}}$$

$$\Rightarrow \lambda = -\frac{1}{x}\log\left(1 - e^{-\frac{1}{\alpha}}\right) \quad -(4)$$

Sub these into (2) gives

$$-\frac{x}{\log\left(1 - e^{-\frac{1}{\alpha}}\right)} - x + (\alpha - 1)\frac{x\left(1 - e^{-\frac{1}{\alpha}}\right)}{e^{-\frac{1}{\alpha}}} = 0 \quad -(3)$$

(3) depends ONLY on α. Solve it for α. (4) will give the MLE for λ.

EXAMPLE CLASS

23 OCTOBER

16:00-17:00PM

MATH3/4/68181

$X_1, X_2, \ldots, X_\alpha$ IID $Exp(\lambda)$.

$X = \max(X_1, X_2, \ldots, X_\alpha)$

1) The CDF of X is

$$F_X(x) = P(X \le x)$$

$$= P[\max(X_1, \ldots, X_\alpha) \le x]$$

$$= P[X_1 \le x, \ldots, X_\alpha \le x]$$

$$= P[X_1 \le x] \cdots P[X_\alpha \le x] \quad \text{independence}$$

$$= (1 - e^{-\lambda x}) \cdots (1 - e^{-\lambda x})$$

$$= (1 - e^{-\lambda x})^\alpha$$

2) $f_X(x) = \frac{d}{dx} F_X(x)$

$$= \alpha (1 - e^{-\lambda x})^{\alpha - 1} \lambda e^{-\lambda x}$$

$$= \alpha \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha - 1}$$

3) $$E(X^n) = \int_0^\infty x^n \, \alpha\lambda e^{-\lambda x}[1-e^{-\lambda x}]^{\alpha-1} dx$$

$$= \alpha\lambda \int_0^\infty x^n e^{-\lambda x}[1-e^{-\lambda x}]^{\alpha-1} dx$$

Set $y = e^{-\lambda x} \Rightarrow x = -\frac{1}{\lambda}\log y$

$\Rightarrow \frac{dx}{dy} = -\frac{1}{\lambda y}$

$$= \alpha\lambda \int_1^0 \left(-\tfrac{1}{\lambda}\log y\right)^n y[1-y]^{\alpha-1}\left(-\tfrac{1}{\lambda y}\right) dy$$

$$= \alpha \int_0^1 \left(-\tfrac{1}{\lambda}\log y\right)^n [1-y]^{\alpha-1} dy$$

$$= \alpha \left(-\tfrac{1}{\lambda}\right)^n \int_0^1 (\log y)^n [1-y]^{\alpha-1} dy$$

$$\frac{d^n y^\beta}{d\beta^n} = y^\beta (\log y)^n$$

$$\Rightarrow \frac{d^n}{d\beta^n} y^\beta \Big|_{\beta=0} = y^0 (\log y)^n = (\log y)^n \quad (*)$$

$$\overset{(*)}{=} \alpha \left(-\tfrac{1}{\lambda}\right)^n \int_0^1 \frac{d^n}{d\beta^n} y^\beta \Big|_{\beta=0} [1-y]^{\alpha-1} dy$$

$$= \alpha \left(-\tfrac{1}{\lambda}\right)^n \frac{d^n}{d\beta^n} \left[\int_0^1 y^\beta [1-y]^{\alpha-1} dy\right]\Big|_{\beta=0}$$

$$= \alpha \left(-\tfrac{1}{\lambda}\right)^n \frac{d^n}{d\beta^n} B(\beta+1, \alpha)\Big|_{\beta=0}$$

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

Beta function

4) $$\text{Mean} = -\frac{\alpha}{\lambda} \frac{d}{d\beta} B(\beta+1, \alpha)\Big|_{\beta=0}$$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$= -\frac{\alpha}{\lambda} \frac{d}{d\beta} \frac{\Gamma(\beta+1)\Gamma(\alpha)}{\Gamma(\beta+1+\alpha)}\Big|_{\beta=0}$$

$$= -\frac{\alpha\Gamma(\alpha)}{\lambda} \frac{d}{d\beta} \cdot \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}\Big|_{\beta=0}$$

$$= -\frac{\alpha\Gamma(\alpha)}{\lambda} \frac{\Gamma(\beta+1+\alpha)\Gamma'(\beta+1) - \Gamma(\beta+1)\Gamma'(\beta+1+\alpha)}{[\Gamma(\beta+1+\alpha)]^2}\Big|_{\beta=0}$$

$$= -\frac{\alpha\Gamma(\alpha)}{\lambda} \frac{\Gamma(1+\alpha)\Gamma'(1) - \Gamma'(1+\alpha)}{[\Gamma(1+\alpha)]^2}.$$

5) $$\text{Variance} = E[X^2] - [E(X)]^2$$

$$= \frac{\alpha}{\lambda^2} \frac{d^2}{d\beta^2} B(\beta+1, \alpha)\Big|_{\beta=0} - [E(X)]^2$$

$$= \frac{\alpha}{\lambda^2} \frac{d^2}{d\beta^2} \frac{\Gamma(\beta+1)\Gamma(\alpha)}{\Gamma(\beta+1+\alpha)}\Big|_{\beta=0} - [E(X)]^2$$

$$= \frac{\alpha\Gamma(\alpha)}{\lambda^2} \frac{d^2}{d\beta^2} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}\Big|_{\beta=0} - [E(X)]^2$$

Simplify this by yourself.

6) & 7) You need to wait till the week after Reading week

8) Suppose there is only 1 observation on X. Then

$$L(\alpha, \lambda) = \alpha \lambda e^{-\lambda x} \left[1 - e^{-\lambda x}\right]^{\alpha - 1}$$

$$\Rightarrow \log L(\alpha, \lambda) = \log \alpha + \log \lambda - \lambda x + (\alpha - 1) \log \left[1 - e^{-\lambda x}\right]$$

$$\Rightarrow \frac{\partial \log L}{\partial \alpha} = \frac{1}{\alpha} + \log \left[1 - e^{-\lambda x}\right] = 0 \quad \text{— (1)}$$

$$\frac{\partial \log L}{\partial \lambda} = -x + \frac{1}{\lambda} + (\alpha - 1) \frac{x e^{-\lambda x}}{1 - e^{-\lambda x}} = 0 \quad \text{— (2)}$$

$$(1) \Rightarrow \alpha = -\frac{1}{\log \left[1 - e^{-\lambda x}\right]} \quad \text{——————— (3)}$$

Sub (3) into (2) $\Rightarrow$

$$-x + \frac{1}{\lambda} - \left(\frac{1}{\log \left[1 - e^{-\lambda x}\right]} + 1 \right) \frac{x e^{-\lambda x}}{1 - e^{-\lambda x}} = 0 \quad \text{— (4)}$$

Solve (4) for λ, say $\hat{\lambda}$.

$$\hat{\alpha} = -\frac{1}{\log \left[1 - e^{-\hat{\lambda} x}\right]}.$$

LECTURE

5 NOVEMBER

9:00-10:00AM

MATH3/4/68181

Quiz 3

i) Assume G belongs to Gumbel domain and show that F also belongs to Gumbel domain

ii) — Fréchet —

iii) — Weibull —

Sheet, Q11, Q12

Six past exam papers - each has a question similar to quiz 3.

By definition,

$$G(w(G)) = 1, \iff G(t) \to 1 \text{ as } t \to w(G)$$

I: $G(t + x\,\gamma(t)) \to 1$ as $t \to w(G)$

($x \to 0$, $\gamma(t) \to 0$)

II: $G(tx) \longrightarrow 1$ as $t \to w(G)$

III: $G(w(G) - tx) \to 1$ as $t \to 0$

In-class test

- Tues 13 Nov
- 9 - 9:40 am
- Roscoe B
  5 Rutherford

FINANCIAL RISK MEASURES

What is a risk measure?

It gives probabilities associated with a loss.

Let X = Loss.

$$P(X > £1 \text{ million}) > 0.9$$

$\Rightarrow$ no to investment

$$P(X > £1 \text{ million}) < 10^{-6}$$

$\Rightarrow$ ok to invest

Business
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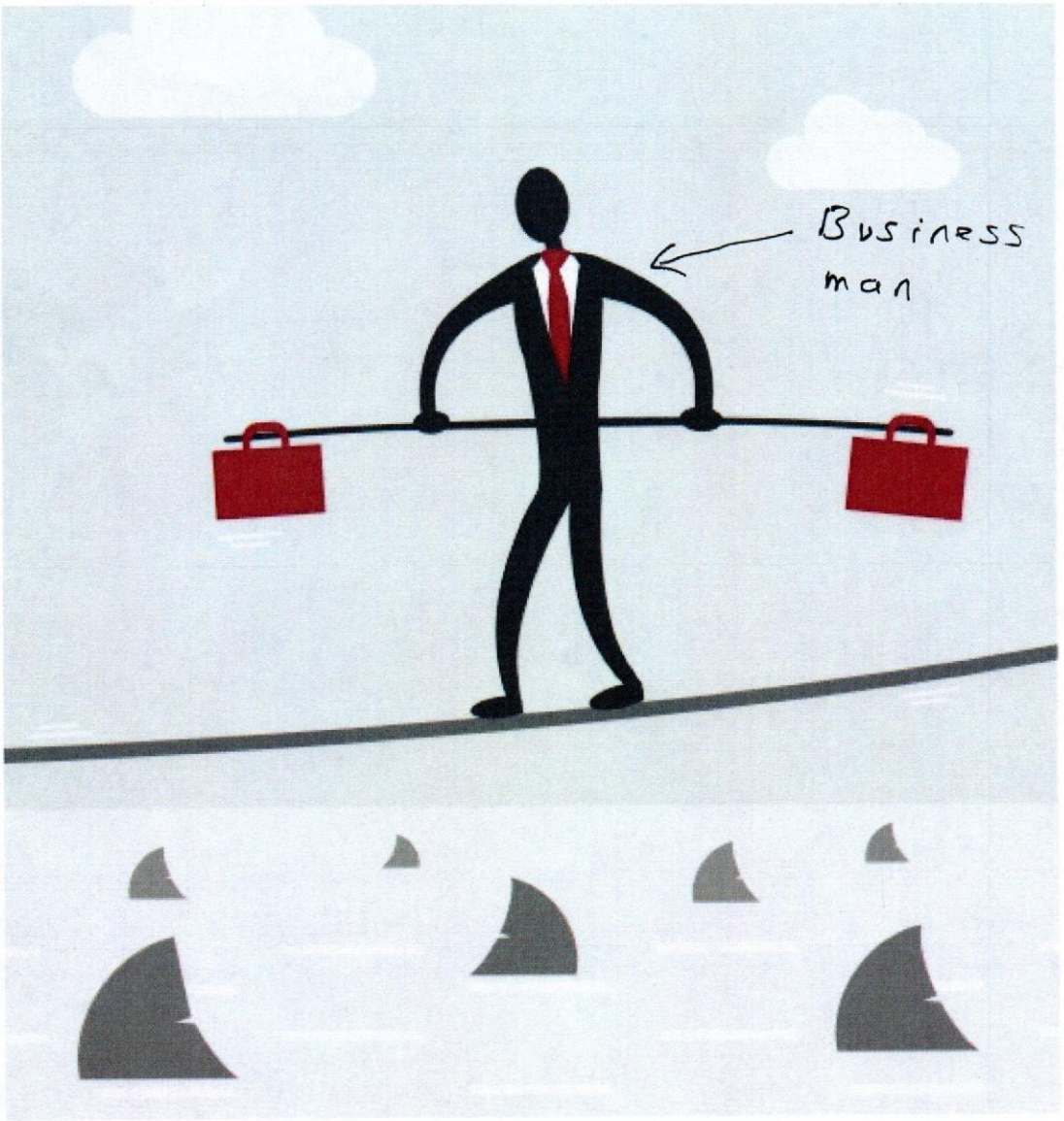
How to Avoid Unnecessary Financial Risk When Your Business is Your Biggest Asset

4 years ago



Written By: **WJ Rossi**

One of the greatest mistakes many business owners make is overestimating their personal levels of diversification while underestimating their risk exposures.

By personality type, entrepreneurs tend toward risk, but they may not realize that investing

NGO Financial Risk Management: Balancing Risks and Controls

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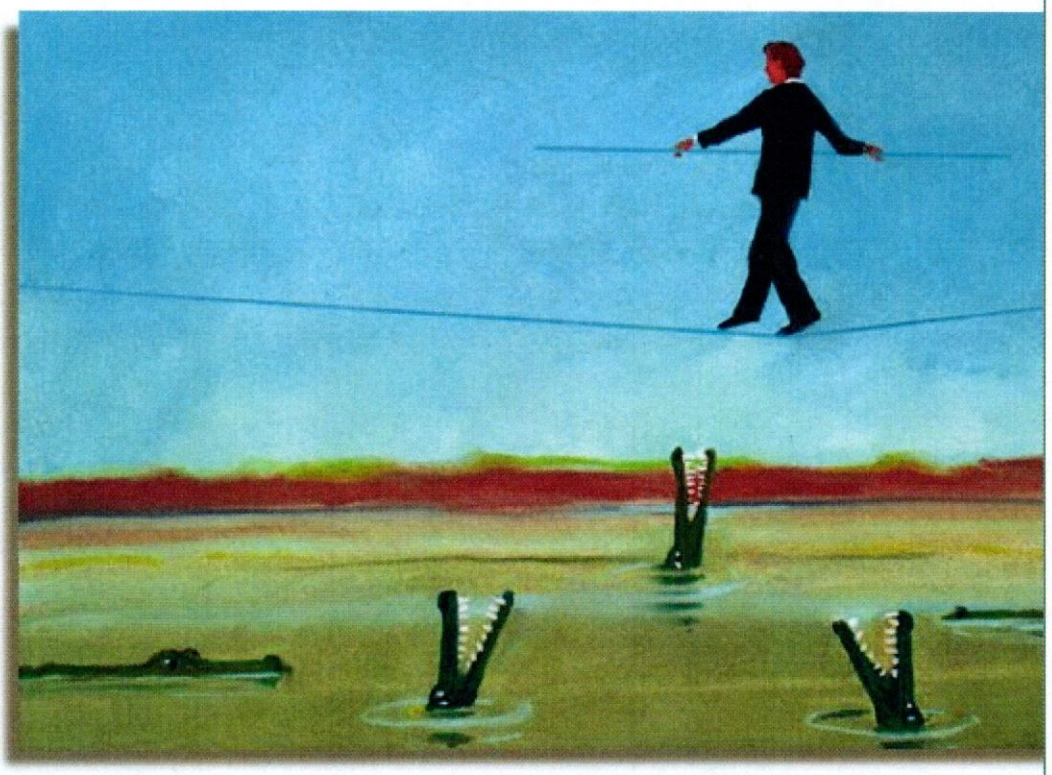
A tip of the hat to you!

I would like to welcome everyone to my Frankenstein project for Fall 2009's Finance Function at the Monterey Institute of International Studies! Enjoy browsing! Thanks for passing by!

Why this website?



The intention for creating this website is to help those in the non-profit sector understand the implications of financial risk management for their given organization paying special attention to the ideas of risks vs. controls, illiquidity and conflict of interest. There are a couple major ideas that I would like to explore. One idea that will be examined is that of controls. Is there such a thing as having too many controls in an organization? I want to stress the importance of organizational liquidity as well as explore a hypothetical financial conflict of interest situation. This is not a page about general risk management but instead focuses on financial risk in non-profits. I hope that this

Definition of a risk measure

Let $\rho : \text{Loss} \longrightarrow (0, \infty)$. $\rho(\cdot)$ is a risk measure if

(i) $\rho(0) = 0$ "normalised property"

(ii) $\rho(X + c) = \rho(X) + c$ "translative property"

c = constant

X = Loss

(iii) if $X \leq Y$ then $\rho(X) \leq \rho(Y)$ "monotone property"

What is a good risk measure?

Definition of a coherent risk measure

ρ : Loss $\longrightarrow$ $(0, \infty)$ is a coherent risk measure if it satisfies (i)-(iii) and

(iv) $\rho(cX) = c\rho(X)$ "positive homogeneity"

$c > 0$ = constant

X = Loss

(v) $\rho(X+Y) \leq \rho(X) + \rho(Y)$ "sub additive property"

Two Most Popular Risk Measures

1) Value at Risk (VaR)

was developed by J. P. Morgan in the 1970s.

Let X = Loss

Suppose X is a continuous RV.

Then

$$VaR_p(X) = F^{-1}(p), \quad 0 < p < 1$$

where F denotes the CDF of X.

Ex

$VaR_{0.9}(X)$ = Loss exceeded with probability 0.1

$VaR_{0.99}(X)$ = Loss exceeded with probability 0.01

2) Expected Shortfall

due to Artzner et al (1998)

Let X = Loss

Suppose X is a continuous RV.

Then

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt.$$

$$= \text{Average Loss given } X \leq F^{-1}(p)$$

Ex

$ES_{0.9}(X)$ = Average loss given the loss $\leq F^{-1}(0.9)$

$ES_{0.99}(X)$ = Average loss given the loss $\leq F^{-1}(0.99)$.

Are these measures ~~of~~ coherent?

		VaR	ES
(i)	Normalised property	✓	✓
(ii)	Translative "	✓	✓
(iii)	Monotone "	✓	✓
(iv)	Pos homogeneity "	✓	✓
(v)	Sub additive "	X	✓

VaR is not a coherent risk measure

ES is a " " "

Cash & Liquidity Management (https://www.treasury-management.com/category/1/cash-%26-liquidity-management.html)

VaR and its Role in the Credit Crisis

by Mark Kirkland, VP Treasury, Bombardier Transportation

The causes of the credit crisis of 2009 will be discussed by many for numerous years to come, although probably for fewer years than we now think. People have a unique ability to forget, perhaps black out, the worst episodes. I have sat down on a number of occasions and tried to think, what were the possible causes of the crisis? An inherent weakness in accounting of results, large numbers of over the counter derivatives with large fair values, weak governance by regulatory bodies or even that bankers were paid too much? In the end, I believe that none of the above was a key contributor to the crisis. In my mind there are two unrelated causes.

It is now clear that very few shareholders of banks understood the risks thtat some banks were in fact taking.

The first is the mode of compensation in the financial industry. Not the amounts. Most bankers receive a kind of option pay out. If the firm makes a large profit (based on the mark to market of future uncertain cash flows), the employees receive large cash bonuses. If the firm makes a loss, in the worst case, staff may receive no bonus. Clearly, for a betting man, this gives carte blanche to load up the company with significant risk. Since most bonuses are not discussed with the owners of the company (the shareholders) but set by a compensation committee, often chaired by senior employees, there is a tendency to overpay since this justifies the compensation of the very people making the decisions. I will not dwell on this cause much longer – except to stress that the whole model encourages large risk taking.

The second is the point of this article. Risk was and still is, very badly understood, managed and reported. It is now clear that very few shareholders of banks understood the risks that some banks were in fact taking. In part, this is because disclosure of risk is unclear. A more fundamental issue, however, is that it appears that some of the banks did not fully comprehend the risk and actually outsourced much of their risk assessment to the rating agencies and then used flawed measures such as Value at Risk (VaR) not only to manage risk but also to report to management and shareholders alike.

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Home

Did Value at Risk cause the crisis it was meant to avert?

What were the causes of the crisis of 2008? New research by Oxford Mathematicians Doyne Farmer, Christoph Aymanns, Vincent W.C. Tan and colleague Fabio Caccioli from University College London shows that managing risk using the procedure recommended by Basel II (the worldwide recommendations on banking regulation), which is called Value at Risk, may have played a central role.

The team made a very simple model for the banking system that captured the key elements of risk management under Value at Risk. Providing the banks only take modest risks, the financial system remains stable. But if they take higher risks, or if the banking sector gets larger, the market begins to spontaneously oscillate, in a way that resembles the period leading up to and including the Global Financial Crisis. For about 10 - 15 years prices and leverage slowly rise while volatility slowly falls, then prices and leverage suddenly crash and volatility spikes, as they did in the crisis.

The key problem is that Value at Risk manages risk as if each bank existed in its own universe. But if all banks follow it, the buying and selling necessary to maintain individual risk targets can destabilise the market.

The team then investigated alternative methods of managing risk and demonstrated that it is possible to do much better. The best policy depends on the size of the banking sector in relation to the rest of the market and how much risk the banks take. While the model does not show that the financial crisis and the period leading up to it were due to the use of Value at Risk, it does suggest that they could have been caused by it, and that the housing bubble may have just been the spark that triggered the crisis.

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VaR: The number that killed us

By Pablo Triana

December 1, 2010 • Reprints

FROM THE ARCHIVES

On Sept. 10, 2009 former trader and bestselling author Nassim Taleb did something that he very seldom does: he wore a tie. Taleb has oftentimes publicly expressed his distaste for the blood-constraining artifacts, as well as for those who tend to don them, so the Lebanese-American let the world know that was a very special day for him by betraying a sacred personal disposition.

So what prompted the composer of "The Black Swan" to button his shirt all the way up on that fall date? He had been invited to a very solemn venue by very distinguished hosts. And that was an invitation that Taleb had every intention of accepting. In fact, he had been waiting and expecting it for more than a decade. The raison d'etre of the event for which his company was now being required had been close to Taleb´s heart for most of his professional and intellectual life. It represented a central theme in his actions and ideas, close to an obsession. He had through the years incessantly warned as to the havoc that might be wreaked should others massively act in a manner counter to his convictions. Such concerns typically went unheeded (to the detriment, it turned out, of society), but now he was being offered a pulpit that seemed irresistible. This time, the world would have no option but to listen attentively.

As Taleb entered the Rayburn Building of the U.S. House of Representatives on Capitol Hill that September morning, he must have felt vindication. As he approached the sober room where several men and women awaited the start of the House Committee on Science and Technology ´s hearing on the responsibility of mathematical model Value at Risk (VaR) for the terrible economic and financial crisis that had caused so much misery, Taleb probably reflected proudly on all those times when, indefatigably and in the face of harsh opposition, he alerted us of the lethal threat to the system posed by the widespread use of VaR in finance. Now that the damage wrought by VaR seemed so inescapably obvious that lawmakers had been motivated into investigating the device, Taleb no longer seemed like a lone wolf howling at the moon.

What is so wrong about VaR, and why was Taleb so concerned about its impact? More importantly, why should VaR be held responsible for the crisis? VaR is a number that purports to estimate future losses derived from a portfolio of financial assets, and presents two major problems: 1) it is doomed to being a very wrong estimate, because of its analytical foundations and the realities of real-life markets; 2) in spite of such (well-known) deficiencies, it has for the past two decades become an ubiquitously influential force in the financial world, capable of directing decision-making inside the most important banks. In other words, by letting trading activity be guided by VaR, we have essentially exposed our economic fate to a deeply flawed mechanism. Such flawedness, as was the case not only in this crisis but also before, can yield untold malaise.

One dimension in a 3D world

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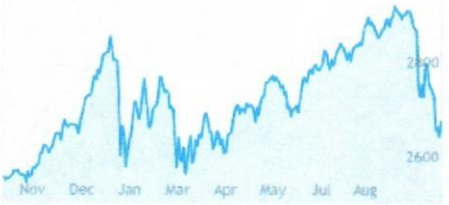

Symbol	Last	Change (%)
Emini	2697.00	+34.75 (+1.31%)
Euro	1.14225	-0.00245 (-0.21%)
Gold	1233.3	-2.1 (-0.17%)
Oil	67.15	-0.47 (-0.70%)
Gas	3.133	-0.069 (-2.15%)
Corn	367'2	-1'2 (-0.34%)

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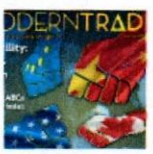

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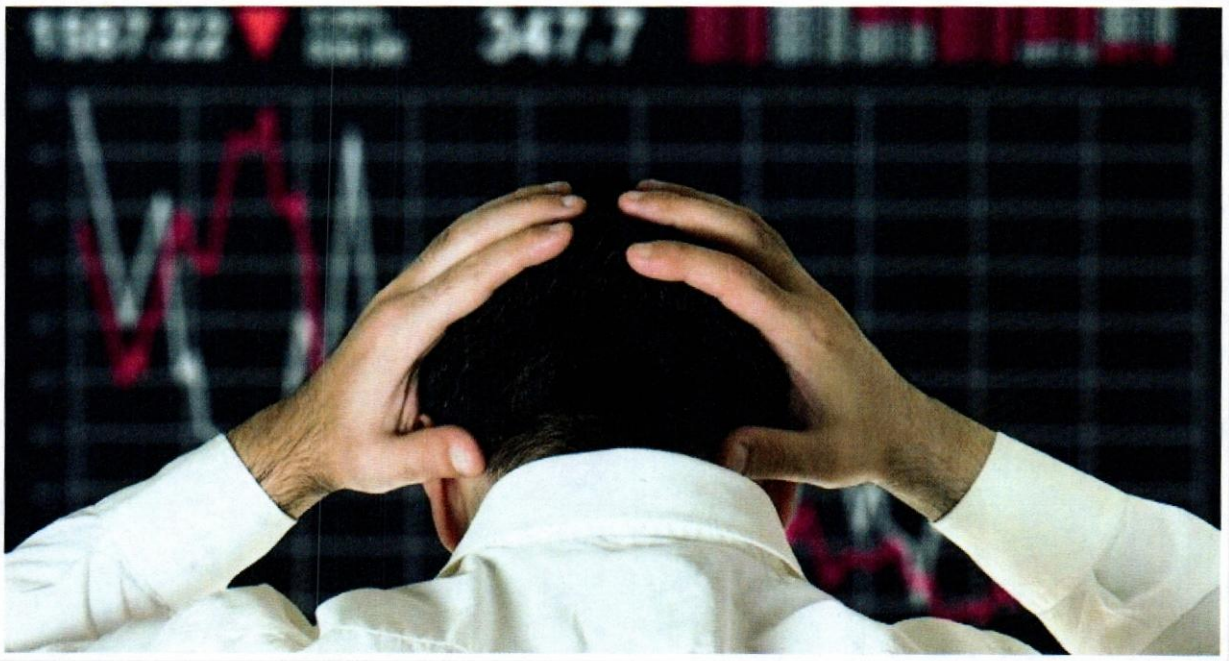
7

News - Did Value at Risk cause the crisis it was meant to avert?



News

Did Value at Risk cause the crisis it was meant to avert?

12 May 2016

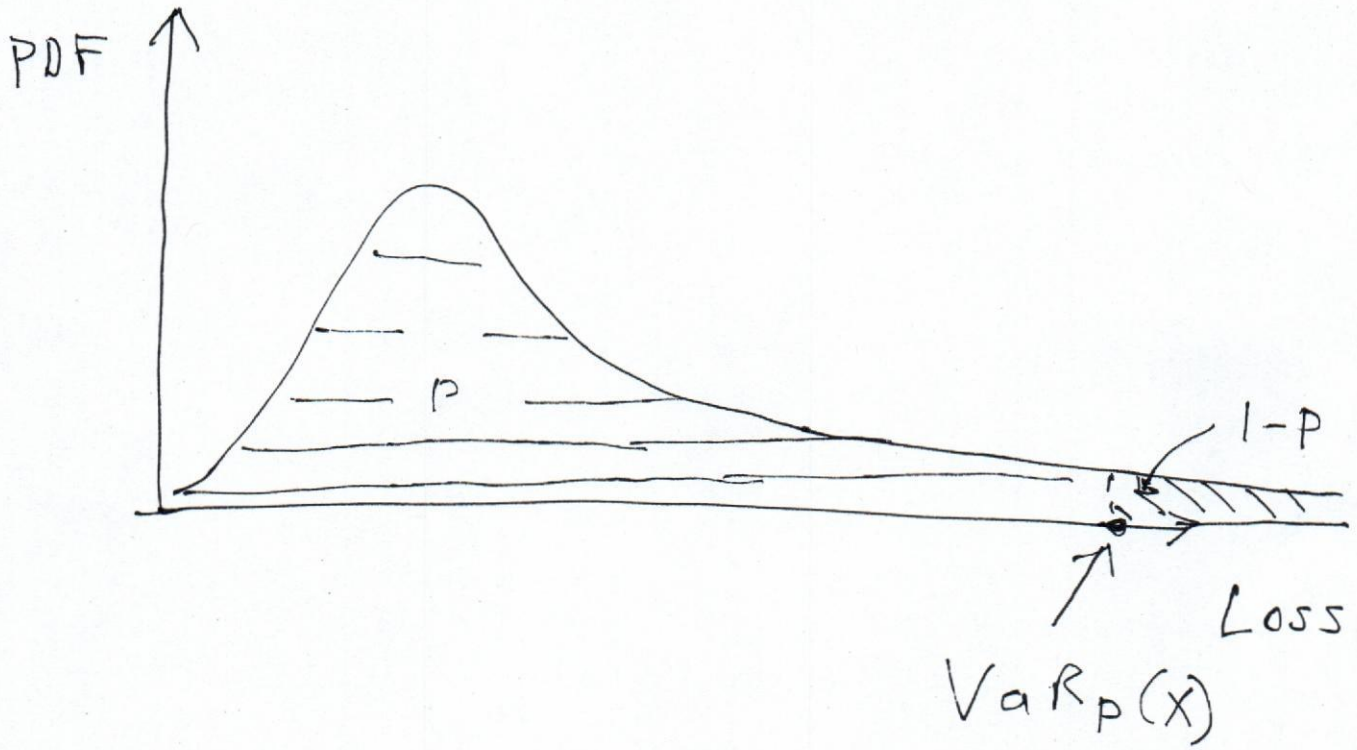
What were the causes of the crisis of 2008? We show that managing risk using the procedure recommended by Basel II, which is called *Value at Risk*, may have played a central role. We make a very simple model for the banking system that captures the key elements of risk management under Value at Risk. Providing the banks' only take modest risks, the financial system remains stable. But if they take higher risks, or if the banking sector gets larger, the market begins to spontaneously oscillate, in a way that resembles the period leading up to and including the Global Financial Crisis. For about 10 - 15 years prices and leverage slowly rise while volatility slowly falls, then prices and leverage suddenly crash and volatility

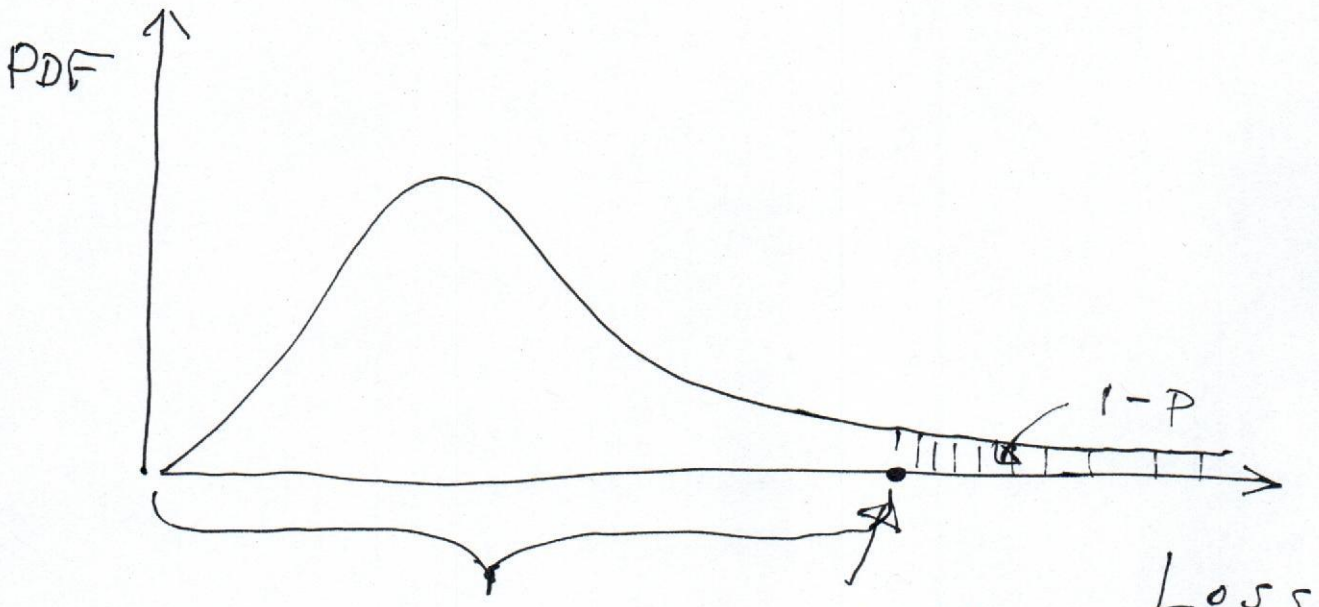
LECTURE

6 NOVEMBER

9:00-10:00AM

MATH3/4/68181


PDF
P
1-P
Loss
$VaR_p(X)$



Average of $X \leq VaR_p(X)$

$VaR_p(X)$

$=$

$ES_p(X)$

Ex 1

Suppose $X \sim N(\mu, \sigma^2)$

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) = p$$

$$\Rightarrow \quad \frac{x-\mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow \quad x = \mu + \sigma \Phi^{-1}(p)$$

$$\Rightarrow \quad VaR_p(X) = \mu + \sigma \Phi^{-1}(p)$$

$$ES_p(X) = \frac{1}{p} \int_0^p \left[\mu + \sigma \Phi^{-1}(t)\right] dt$$

$$= \mu + \frac{\sigma}{p} \int_0^p \Phi^{-1}(t)\, dt$$

Ex 2

$$X \sim \text{Uni}[a, b]$$

$$F(x) = \frac{x-a}{b-a} = p$$

$$\Rightarrow \quad x = a + p(b-a)$$

$$\Rightarrow \quad \text{VaR}_p(X) = a + p(b-a)$$

$$ES_p(X) = \frac{1}{p}\int_0^p [a + t(b-a)]\, dt$$

$$= \frac{1}{p}\left[at + \frac{t^2}{2}(b-a)\right]_0^p$$

$$= a + \frac{p}{2}(b-a)$$

Ex 3

Suppose X has CDF $F(x) = x^a$, $0 < x < 1$

$$F(x) = x^a = p$$

$$\Rightarrow x = p^{\frac{1}{a}}$$

$$\Rightarrow VaR_p(X) = p^{\frac{1}{a}}$$

$$ES_p(X) = \frac{1}{p} \int_0^p t^{\frac{1}{a}} dt$$

$$= \frac{1}{p} \left[\frac{t^{\frac{1}{a}+1}}{\frac{1}{a}+1} \right]_0^p$$

$$= \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1} .$$

Proof of Properties of VaR

Translative property

$$VaR_p(X+c) = VaR_p(X) + c \qquad (*)$$

Proof. Suppose $(*)$ holds. Then

$$F^{-1}_{X+c}(p) = F^{-1}_{X}(p) + c$$

$$\Rightarrow F_{X+c}\left(F^{-1}_{X+c}(p)\right) = F_{X+c}\left(F_X^{-1}(p) + c\right)$$

$$\Rightarrow p = F_{X+c}\left(F_X^{-1}(p) + c\right)$$

$$\Rightarrow p = Pr\left(X+c \leq F_X^{-1}(p) + c\right)$$

$$\Rightarrow p = Pr\left(X \leq F_X^{-1}(p)\right)$$

$$\Rightarrow p = F_X\left(F_X^{-1}(p)\right)$$

$$\Rightarrow p = p.$$

Hence $(*)$ must hold.

Monotone property

$$X \leq Y \Rightarrow VaR_p(X) \leq VaR_p(Y)$$

Proof $X \leq Y$

$$\Rightarrow F_X^{-1}(p) \leq F_Y^{-1}(p) \quad \forall p$$

$$\Rightarrow VaR_p(X) \leq VaR_p(Y) \quad \forall p.$$

Positive homogeneity

$$VaR_p(cX) = c\,VaR_p(X) \quad \ldots (*)$$

Proof Suppose $(*)$ holds. Then

$$F_{cX}^{-1}(p) = c\,F_X^{-1}(p)$$

$$\Rightarrow F_{cX}\left(F_{cX}^{-1}(p)\right) = F_{cX}\left(cF_X^{-1}(p)\right)$$

$$\Rightarrow p = F_{cX}\left(cF_X^{-1}(p)\right)$$

$$\Rightarrow p = Pr\left(cX \leq cF_X^{-1}(p)\right)$$

$$\Rightarrow p = F_X\left(F_X^{-1}(p)\right)$$

$$\Rightarrow p = p.$$

Hence $(*)$ must hold.

Estimation methods for VaR

a) Parametric estimation methods

b) Non-parametric " "

c) Semi-parametric " "

Level 3 – a) and b) only

Levels 4, 6 – a), b) and c)

a) Parametric estimation methods

1) Normal distribution

Suppose $X \sim N(\mu, \sigma^2)$. Then

$$VaR_p(X) = \mu + \sigma \Phi^{-1}(p).$$

Suppose $x_1, x_2, \dots, x_n$ is a random sample on X. Then the MLEs of μ and σ are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}.$$

Hence, the MLE of $VaR_p(X)$ is

$$\widehat{VaR}_p(X) = \bar{x} + \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}\, \Phi^{-1}(p)$$

Suppose $\hat{\theta}$ is an estimator of θ.

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

$\hat{\theta}$ is unbiased if $\text{Bias}(\hat{\theta}) = 0$

$$MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$$

↑
Mean Squared Error

$\hat{\theta}$ is consistent if $MSE(\hat{\theta}) \to 0$ as $n \to \infty$.

$\hat{\theta}$ is asymptotically unbiased if $\text{Bias}(\hat{\theta}) \to 0$ as $n \to \infty$

$\widehat{Var}_p(X)$ is a biased estimator.

But it is consistent.

2) Uniform distribution method

Suppose $X \sim Uni[a, b]$. Then

$$VaR_p(X) = a + p(b-a)$$

Suppose $x_1, x_2, \dots, x_n$ is a random sample on X. Then the MLEs of a and b are

$$\hat{a} = \min(X_1, \dots, X_n)$$

$$\hat{b} = \max(X_1, \dots, X_n).$$

Hence the MLE of $VaR_p(X)$ is

$$\widehat{VaR}_p(X) = \min(X_1, \dots, X_n) + p\left[\max(X_1, \dots, X_n) - \min(X_1, \dots, X_n)\right].$$

Homework: $\widehat{VaR}_p(X)$ is a biased estimator but consistent

LECTURE

9 NOVEMBER

10:00-11:00AM

MATH4/68181

Estimation methods for VaR

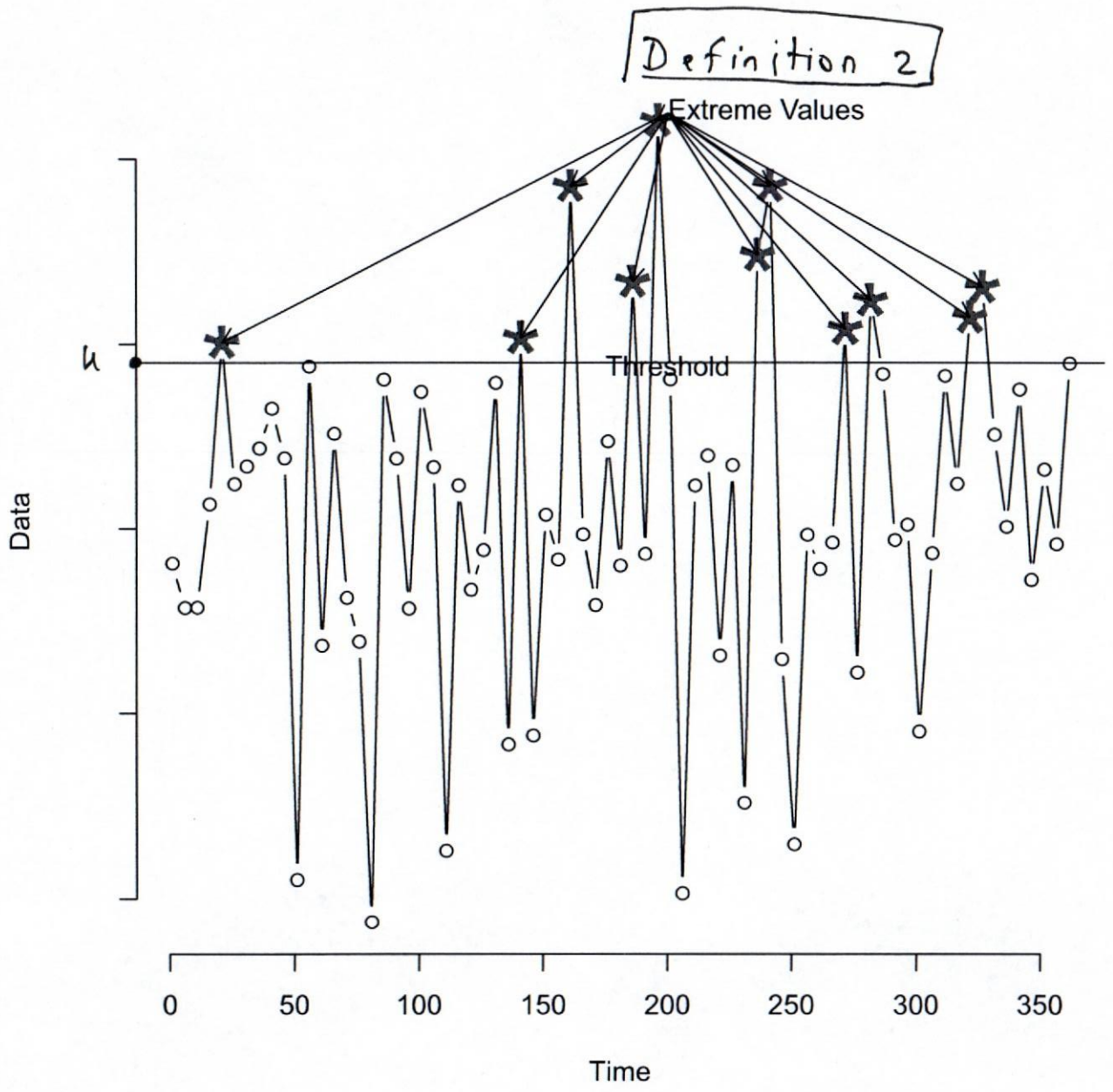
a) Parametric estimation methods

b) Non-parametric " "

c) Semi-parametric " "

a) b) Level 3

a) b) c) Levels 4, 6


Definition 2
Extreme Values
u
Threshold
Data
0
50
100
150
200
250
300
350
Time

c) Semi-parametric estimation methods

1) Extreme value method

Suppose Definition 1 of extreme value.

Let $M_n = \max(X_1, X_2, \ldots, X_n)$.

By the EVT, we can write

$$P(M_n < x) = e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

where $-\infty < \xi < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$.

To find VaR, set

$$P(M_n < x) = p$$

$$\Rightarrow e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} = p$$

$$\Rightarrow -\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} = \log p$$

$$\Rightarrow 1+\xi\frac{x-\mu}{\sigma} = (-\log p)^{-\xi}$$

$$\Rightarrow x = \mu + \frac{\sigma}{\xi}\left[(-\log p)^{-\xi} - 1\right]$$

$$\Rightarrow VaR_p = \mu + \frac{\sigma}{\xi}\left[(-\log p)^{-\xi} - 1\right]$$

How to estimate VaR_p?

Estimation of ξ

Suppose $X_1, X_2, \ldots$ are IID observations. Then two possible estimators of ξ are

$$\hat{\xi} = \frac{1}{k} \sum_{i=1}^{k} \log \frac{X_{(i)}}{X_{(i+1)}} \quad \ldots \quad (*)$$

and

$$\hat{\xi} = \frac{1}{\log 2} \log \frac{X_{(k+1)} - X_{(2k+1)}}{X_{(2k+1)} - X_{(4k+1)}} \quad \ldots (**)$$

where $k \geq 1$ is an integer and

$X_{(1)} \geq X_{(2)} \geq \ldots$ are descending order statistics

(*) is due to Hill (1975)

(**) is due to Pickands (1975)

(*) and (**) involve just the order statistics. They do not suppose any distribution for the data. Hence, (*) and (**) are non-parametric.

Estimation of μ and σ

Let L denote the likelihood for $X_1, X_2, \ldots, X_n$ following the GEV distribution. Then the MLEs of μ and σ say $\hat{\mu}$ and $\hat{\sigma}$ are the simultaneous solutions of

$$\frac{\partial \log L}{\partial \mu} = 0$$

and

$$\frac{\partial \log L}{\partial \sigma} = 0 .$$

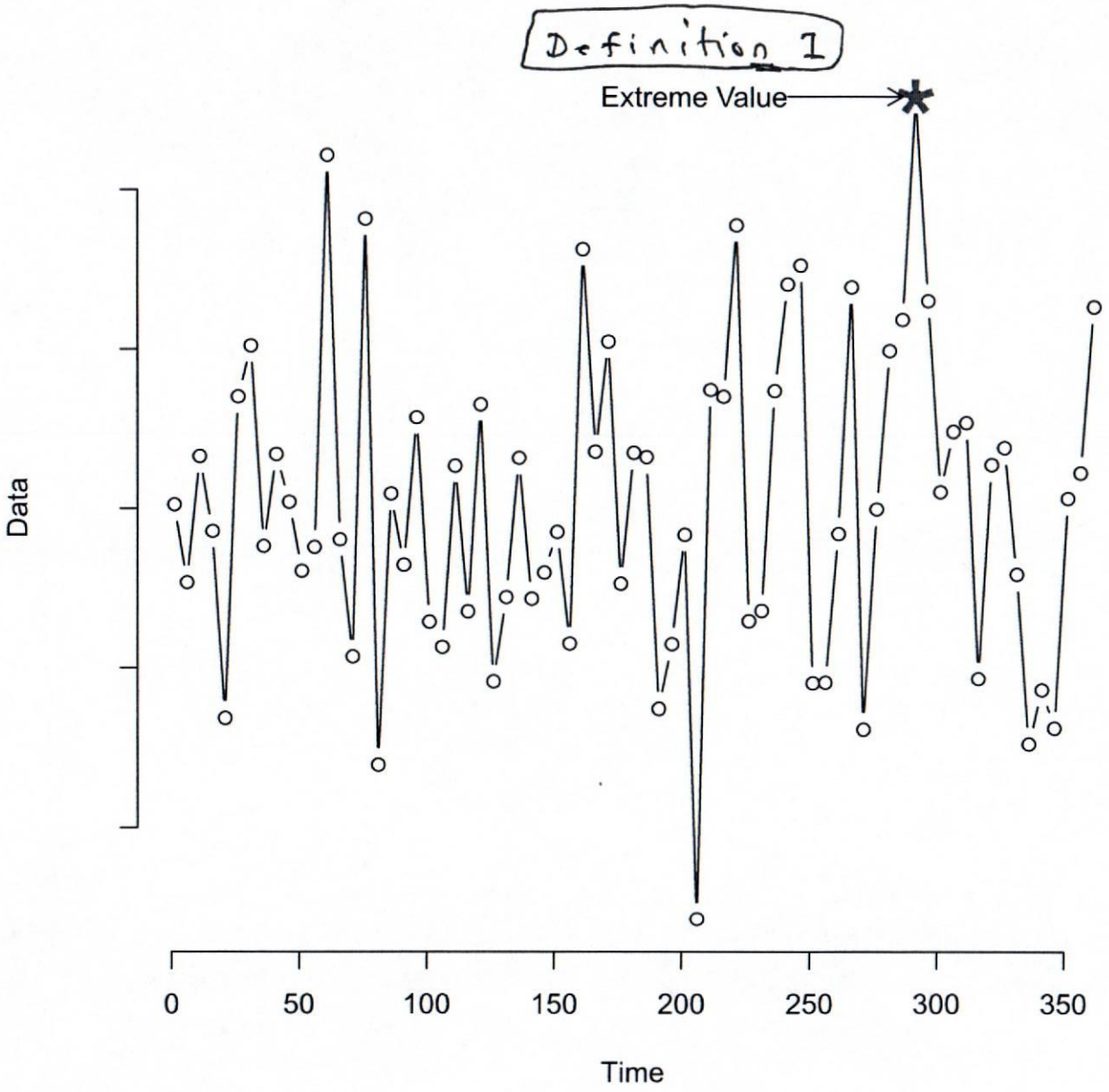
Hence, the estimator of VaR_p is

$$\widehat{VaR}_p = \hat{\mu} + \frac{\hat{\sigma}}{\hat{\xi}} \left[(-\log p)^{-\hat{\xi}} - 1 \right]$$

Parametric estimators

Non-parametric estimator

Hence, $\widehat{VaR}_p$ is a semi-parametric estimator.


Definition 1
Extreme Value
Data
0
50
100
150
200
250
300
350
Time

2. Generalized Pareto method

Suppose definition 2 of extreme values.

Let X = variable of interest

u = threshold

F = CDF of X.

From what we covered before,

$$F(x) = 1 - q\left(1 + \frac{x-u}{\sigma}\right)^{-\frac{1}{\xi}}$$

where $q = P(X > u)$, $\sigma > 0$ and $-\infty < \xi < \infty$.

To find VaR, set

$$F(x) = p$$

$$\Rightarrow 1 - q\left(1 + \frac{x-u}{\sigma}\right)^{-\frac{1}{\xi}} = p$$

$$\Rightarrow \left(1 + \frac{x-u}{\sigma}\right)^{-\frac{1}{\xi}} = \frac{1-p}{q}$$

$$\Rightarrow 1 + \frac{x-u}{\sigma} = \left(\frac{1-p}{q}\right)^{-\xi}$$

$$\Rightarrow x = u + \sigma\left[\left(\frac{1-p}{q}\right)^{-\xi} - 1\right].$$

Hence,

$$VaR_p(X) = u + \sigma\left[\left(\frac{1-p}{q}\right)^{-\xi} - 1\right].$$

Take $\hat{\xi}$ as the estimators due to Hill (1975) and Pickands (1975). These are non-parametric estimators.

Take $\hat{\sigma}$ as the MLE of σ. This is a parametric estimator.

Estimate q_r by

$$\hat{q} = \frac{1}{n} \sum_{i=1}^{n} I\{X_i > u\}$$

(proportion of data exceeding u).

Hence, the estimator of VaR_p is

$$\widehat{VaR}_p(X) = u + \hat{\sigma}\left[\left(\frac{1-p}{\hat{q}}\right)^{-\hat{\xi}} - 1\right].$$

Clearly, this is a semi-parametric estimator.

EXAMPLE CLASS

6 NOVEMBER

12:00-13:00PM

MATH3/4/68181

Q1

$$F(x) = 1 - e^{-\lambda x} = p$$

$$VaR_p(X) = F^{-1}(p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$1 - e^{-\lambda x} = p$$

$$\Rightarrow \quad e^{-\lambda x} = 1 - p$$

$$\Rightarrow \quad \lambda x = -\log(1-p)$$

$$\Rightarrow \quad x = -\frac{1}{\lambda}\log(1-p)$$

$$\Rightarrow \quad VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p -\frac{1}{\lambda}\log(1-t)\,dt$$

$$= -\frac{1}{\lambda p}\int_0^p \log(1-t)\,dt$$

$$= -\frac{1}{\lambda p}\left\{\left[t\cdot\log(1-t)\right]_0^p - \int_0^p t\cdot\frac{(-1)}{1-t}\,dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\log(1-p) - 0 + \int_0^p \frac{(t-1)+1}{1-t}\,dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{ p \log(1-p) + \int_0^p \left(-1 + \frac{1}{1-t}\right) dt \right\}$$

$$= -\frac{1}{\lambda p}\left\{ p \log(1-p) + \left[-t - \log(1-t)\right]_0^p \right\}$$

$$= -\frac{1}{\lambda p}\left\{ p \log(1-p) - p - \log(1-p) \right\}$$

$$\Rightarrow \quad ES_p(X) = -\frac{1}{\lambda p}\left\{ (p-1)\log(1-p) - p \right\}.$$

Q2

$$F(x) = x^{\alpha} = p$$

$$\Rightarrow \quad x = p^{\frac{1}{\alpha}}$$

$$\Rightarrow \quad VaR_p(X) = p^{\frac{1}{\alpha}}$$

$$ES_p(X) = \frac{1}{p} \int_0^p t^{\frac{1}{\alpha}} \, dt$$

$$= \frac{1}{p} \left[\frac{t^{\frac{1}{\alpha}+1}}{\frac{1}{\alpha}+1} \right]_0^p$$

$$= \frac{p^{\frac{1}{\alpha}}}{\frac{1}{\alpha}+1} .$$

Q3

$$F(x) = \frac{x-a}{b-a} = p$$

$$\Rightarrow \quad x = a + (b-a)p$$

$$\Rightarrow \quad VaR_p(X) = a + (b-a)p$$

$$ES_p(X) = \frac{1}{p}\int_0^p [a + (b-a)t]\, dt$$

$$= \frac{1}{p}\left[at + (b-a)\frac{t^2}{2}\right]_0^p$$

$$= \frac{1}{p}\left[ap + (b-a)\frac{p^2}{2} - 0\right]$$

$$= a + (b-a)\frac{p}{2}.$$

Q4

$$F(x) = 1 - \left(\frac{k}{x}\right)^a, \quad x > k$$

$$= p$$

$$\Rightarrow \quad \left(\frac{k}{x}\right)^a = 1 - p$$

$$\Rightarrow \quad \frac{k}{x} = (1-p)^{\frac{1}{a}}$$

$$\Rightarrow \quad x = k(1-p)^{-\frac{1}{a}} = V_a R_p(X)$$

$$ES_p(X) = \frac{1}{p} \int_0^p k(1-t)^{-\frac{1}{a}} \, dt$$

$$= \frac{k}{p} \left[\frac{(1-t)^{1-\frac{1}{a}}}{\left(1-\frac{1}{a}\right)(-1)} \right]_0^p$$

$$= \frac{k}{p\left(\frac{1}{a}-1\right)} \left[(1-p)^{1-\frac{1}{a}} - 1 \right].$$

Q5

$$F(x) = \Phi(x)$$

↖ CDF of $N(0,1)$

$$= p$$

$$\Rightarrow \quad x = \Phi^{-1}(p) = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p} \int_0^p \Phi^{-1}(t)\, dt \,.$$

Q6

$$F(x) = \frac{1}{1+\left(\frac{x}{a}\right)^{-b}} = p$$

$$\Rightarrow \quad 1+\left(\frac{x}{a}\right)^{-b} = \frac{1}{p}$$

$$\Rightarrow \quad \left(\frac{x}{a}\right)^{-b} = \frac{1-p}{p}$$

$$\Rightarrow \quad x = a\left(\frac{1-p}{p}\right)^{-\frac{1}{b}} = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p a\cdot\left(\frac{1-t}{t}\right)^{-\frac{1}{b}} dt$$

$$= \frac{a}{p}\int_0^p t^{\frac{1}{b}}(1-t)^{-\frac{1}{b}} dt$$

$$= \frac{a}{p} B_p\left(1+\frac{1}{b}, 1-\frac{1}{b}\right)$$

$$B_p(\alpha,\beta) = \int_0^p t^{\alpha-1}(1-t)^{\beta-1} dt$$

Incomplete beta function

Q7

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = p$$

$$\Rightarrow \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = 1 - p$$

$$\Rightarrow 1 + \frac{x}{\lambda} = (1-p)^{-\frac{1}{\alpha}}$$

$$\Rightarrow x = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right] = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \lambda\left[(1-t)^{-\frac{1}{\alpha}} - 1\right] dt$$

$$= \frac{\lambda}{p}\left\{\frac{(1-t)^{1-\frac{1}{\alpha}}}{\left(1-\frac{1}{\alpha}\right)(-1)} - t\right\}_0^p$$

$$= \frac{\lambda}{p}\left\{\frac{(1-p)^{1-\frac{1}{\alpha}}}{-\left(1-\frac{1}{\alpha}\right)} - p + \frac{1}{1-\frac{1}{\alpha}}\right\}.$$

Q8

$$F(x) = e^{-\left(\frac{\sigma}{x}\right)^{\alpha}} = p$$

$$\Rightarrow -\left(\frac{\sigma}{x}\right)^{\alpha} = \log p$$

$$\Rightarrow \frac{\sigma}{x} = (-\log p)^{\frac{1}{\alpha}}$$

$$\Rightarrow x = \sigma (-\log p)^{-\frac{1}{\alpha}} = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \sigma(-\log t)^{-\frac{1}{\alpha}}\, dt$$

$$= \frac{\sigma}{p}\int_0^p (-\log t)^{-\frac{1}{\alpha}}\, dt$$

Set $y = -\log t \Rightarrow t = e^{-y} \Rightarrow \frac{dt}{dy} = -e^{-y}$

$$= \frac{\sigma}{p}\int_{+\infty}^{-\log p} y^{-\frac{1}{\alpha}}(-e^{-y})\, dy$$

$$= -\frac{\sigma}{p}\int_{+\infty}^{-\log p} y^{-\frac{1}{\alpha}} e^{-y}\, dy = \frac{\sigma}{p}\int_{-\log p}^{\infty} y^{-\frac{1}{\alpha}} e^{-y}\, dy$$

$$= \frac{\sigma}{p}\Gamma\left(1-\frac{1}{\alpha}, -\log p\right)$$

$$\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t}\, dt$$

Incomp gamma Function.

EXAMPLE CLASS

6 NOVEMBER

16:00-17:00PM

MATH3/4/68181

Q1

$$F(x) = 1 - e^{-\lambda x} = p$$

$$\Rightarrow \quad e^{-\lambda x} = 1 - p$$

$$\Rightarrow \quad -\lambda x = \log(1-p)$$

$$\Rightarrow \quad x = -\frac{1}{\lambda} \log(1-p)$$

$$\Rightarrow \quad VaR_p(X) = -\frac{1}{\lambda} \log(1-p)$$

$$ES_p(X) = \frac{1}{p} \int_0^p \left(-\frac{1}{\lambda}\right) \log(1-t)\, dt$$

$$= -\frac{1}{\lambda p} \int_0^p \log(1-t)\, dt$$

$$= -\frac{1}{\lambda p} \left\{ \left[t \cdot \log(1-t) \right]_0^p - \int_0^p t \, \frac{(-1)}{1-t}\, dt \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1-p) - 0 + \int_0^p \frac{t-1+1}{1-t}\, dt \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1-p) + \left[-t - \log(1-t) \right]_0^p \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1-p) - p - \log(1-p) \right\}$$

Q4

$$F(x) = 1 - \left(\frac{k}{x}\right)^a, \quad x > k$$

$$= p$$

$$\Rightarrow \left(\frac{k}{x}\right)^a = 1 - p$$

$$\Rightarrow \frac{k}{x} = (1-p)^{\frac{1}{a}}$$

$$\Rightarrow x = k(1-p)^{-\frac{1}{a}} = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p k(1-t)^{-\frac{1}{a}}\,dt$$

$$= \frac{k}{p}\left[\frac{(1-t)^{1-\frac{1}{a}}}{\left(1-\frac{1}{a}\right)(-1)}\right]_0^p$$

$$= \frac{k}{p}\,\frac{(1-p)^{1-\frac{1}{a}} - 1}{\frac{1}{a} - 1}$$

Q6

$$F(x) = \frac{1}{1 + \left(\frac{x}{a}\right)^{-b}} = p$$

$$\Rightarrow \quad 1 + \left(\frac{x}{a}\right)^{-b} = \frac{1}{p}$$

$$\Rightarrow \quad \left(\frac{x}{a}\right)^{-b} = \frac{1-p}{p}$$

$$\Rightarrow \quad x = a\left(\frac{1-p}{p}\right)^{-\frac{1}{b}} = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p a\left(\frac{1-t}{t}\right)^{-\frac{1}{b}} dt$$

$$= \frac{a}{p}\int_0^p t^{\frac{1}{b}}(1-t)^{-\frac{1}{b}}\, dt$$

$$= \frac{a}{p}\, B_p\left(1+\frac{1}{b},\ 1-\frac{1}{b}\right)$$

$$B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\, dt$$

Incomplete Beta Function

Q7

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = p$$

$$\Rightarrow \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = 1 - p$$

$$\Rightarrow 1 + \frac{x}{\lambda} = (1-p)^{-\frac{1}{\alpha}}$$

$$\Rightarrow x = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right] = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \lambda\left[(1-t)^{-\frac{1}{\alpha}} - 1\right] dt$$

$$= \frac{\lambda}{p}\left[\frac{(1-t)^{1-\frac{1}{\alpha}}}{\left(1-\frac{1}{\alpha}\right)(-1)} - t\right]_0^p$$

$$= \frac{\lambda}{p}\left[\frac{(1-p)^{1-\frac{1}{\alpha}}}{\left(\frac{1}{\alpha}-1\right)} - p - \frac{1}{\left(\frac{1}{\alpha}-1\right)}\right].$$

Q 8

$$F(x) = e^{-\left(\frac{\sigma}{x}\right)^{\alpha}} = p$$

$$\Rightarrow -\left(\frac{\sigma}{x}\right)^{\alpha} = \log p$$

$$\Rightarrow \frac{\sigma}{x} = (-\log p)^{\frac{1}{\alpha}}$$

$$\Rightarrow x = \sigma (-\log p)^{-\frac{1}{\alpha}}$$

$$ES_p(X) = \frac{1}{p} \int_0^p \sigma (-\log t)^{-\frac{1}{\alpha}} dt$$

$$= \frac{\sigma}{p} \int_0^p (-\log t)^{-\frac{1}{\alpha}} dt$$

Set $y = -\log t \Rightarrow t = e^{-y} \Rightarrow \frac{dt}{dy} = -e^{-y}$

$$= \frac{\sigma}{p} \int_{\infty}^{-\log p} y^{-\frac{1}{\alpha}} (-e^{-y}) dy$$

$$= \frac{\sigma}{p} \int_{-\log p}^{\infty} y^{-\frac{1}{\alpha}} e^{-y} dy$$

$$= \frac{\sigma}{p} \Gamma\left(1 - \frac{1}{\alpha}, -\log p\right)$$

$\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t} dt$

Lower Incomplete gamma function

Q9

$$F(x) = 1 - e^{-\left(\frac{x}{\sigma}\right)^{\alpha}} = p$$

$$\Rightarrow \quad x = \sigma \left[-\log(1-p) \right]^{\frac{1}{\alpha}}$$

$$= VaR_p(X)$$

$$ES_p(X) = \frac{\sigma}{p} \int_0^p \left[-\log(1-t) \right]^{\frac{1}{\alpha}} dt$$

Set $y = -\log(1-t) \Rightarrow 1-t = e^{-y}$

$\Rightarrow t = 1 - e^{-y}$

$\Rightarrow \frac{dt}{dy} = e^{-y}$

$$= \frac{\sigma}{p} \int_0^{-\log(1-p)} y^{\frac{1}{\alpha}} e^{-y} \, dy$$

$$= \frac{\sigma}{p} \gamma\left(1 + \frac{1}{\alpha}, -\log(1-p)\right)$$

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} \, dt$$

Upper incomplete gamma function

LECTURE

12 NOVEMBER

9:00-10:00AM

MATH3/4/68181

Estimation methods for VaR

a) Parametric estimation methods

b) Non-parametric " "

c) Semi-parametric " "

Level 3 – a) and b) only

Levels 4, 6 – a), b) and c)

a) Parametric estimation methods

1) Normal distribution

Suppose $X \sim N(\mu, \sigma^2)$. Then

$$VaR_p(X) = \mu + \sigma \Phi^{-1}(p).$$

Suppose $x_1, x_2, \dots, x_n$ is a random sample on X. Then the MLEs of μ and σ are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}.$$

Hence, the MLE of $VaR_p(X)$ is

$$\widehat{VaR}_p(X) = \bar{x} + \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}\, \Phi^{-1}(p)$$

Suppose $\hat{\theta}$ is an estimator of ϑ.

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \vartheta$$

$\hat{\theta}$ is unbiased if $\text{Bias}(\hat{\theta}) = 0$

$$\text{MSE}(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$$

↑
Mean Squared Error

$\hat{\theta}$ is consistent if $\text{MSE}(\hat{\theta}) \to 0$ as $n \to \infty$.

$\hat{\theta}$ is asymptotically unbiased if $\text{Bias}(\hat{\theta}) \to 0$ as $n \to \infty$

$\widehat{\text{Var}}_p(X)$ is a biased estimator.

But it is consistent.

2) Uniform distribution method

Suppose $X \sim Uni[a, b]$. Then

$$VaR_p(X) = a + p(b-a)$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. Then the MLEs of a and b are

$$\hat{a} = \min(X_1, \ldots, X_n)$$

$$\hat{b} = \max(X_1, \ldots, X_n).$$

Hence the MLE of $VaR_p(X)$ is

$$\widehat{VaR}_p(X) = \min(X_1, \ldots, X_n) + p\left[\max(X_1, \ldots, X_n) - \min(X_1, \ldots, X_n)\right].$$

Homework: $\widehat{VaR}_p(X)$ is a biased estimator but consistent

3) Power function distribution method

Suppose X has CDF $F(x) = x^a$

Then

$$VaR_p(X) = p^{\frac{1}{a}}$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. The likelihood function of a is

$$L(a) = \prod_{i=1}^{n} \left[a x_i^{a-1} \right]$$

$$= a^n \left(\prod_{i=1}^{n} x_i \right)^{a-1}$$

The log-likelihood function is

$$\log L(a) = n \log a + (a-1) \sum_{i=1}^{n} \log x_i .$$

The derivative wrt a is

$$\frac{d \log L(a)}{da} = \frac{n}{a} + \sum_{i=1}^{n} \log x_i .$$

Setting this to zero, we obtain

$$\hat{a} = - \frac{n}{\sum_{i=1}^{n} \log x_i} .$$

This is an MLE since

$$\frac{d^2 \log L(a)}{da^2} = - \frac{n}{a^2} < 0 \quad \forall a$$

Hence, $\hat{a}$ is the MLE of a. The MLE of $VaR_p(X)$ is

$$\widehat{VaR_p(X)} = p^{- \frac{\sum_{i=1}^{n} \log x_i}{n}}$$

4) Variance - Covariance method (indep case)

Let

$$X = \sum_{j=1}^{p} w_j X_j$$

- X: weighted portfolio loss
- p: no of investments
- w_j: weight for jth investment
- X_j: loss for jth investment

Suppose $X_j \sim N(\mu_j, \sigma_j^2)$ independently.

Then

$$X \sim N\left(\sum_{j=1}^{p} w_j \mu_j \;,\; \sum_{j=1}^{p} w_j^2 \sigma_j^2 \right).$$

So,

$$VaR_p(X) = \sum_{j=1}^{p} w_j \mu_j + \Phi^{-1}(p) \sqrt{\sum_{j=1}^{p} w_j^2 \sigma_j^2}.$$

Suppose $x_{j1}, x_{j2}, \dots, x_{jn}$ is a random sample on X_j. Then the MLEs of μ_j and σ_j^2 are

$$\hat{\mu}_j = \frac{1}{n} \sum_{k=1}^{n} x_{jk}$$

and

$$\hat{\sigma}_j^2 = \frac{1}{n} \sum_{k=1}^{n} (x_{jk} - \hat{\mu}_j)^2.$$

Hence, the MLE of $VaR_p(X)$ is

$$\widehat{VaR}_p(X) = \sum_{j=1}^{p} w_j \hat{\mu}_j + \Phi^{-1}(p) \sqrt{\sum_{j=1}^{p} w_j^2 \hat{\sigma}_j^2}$$

5) Weibull distribution method

Suppose X has the CDF

$$F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}}, \quad x > 0.$$

To find VaR, set

$$F(x) = p$$

$$\Rightarrow \quad 1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}} = p$$

$$\Rightarrow \quad e^{-\left(\frac{x}{\theta}\right)^{\beta}} = 1-p$$

$$\Rightarrow \quad \left(\frac{x}{\theta}\right)^{\beta} = -\log(1-p)$$

$$\Rightarrow \quad x = \theta\left[-\log(1-p)\right]^{\frac{1}{\beta}}$$

$$\Rightarrow \quad VaR_p(X) = \theta\left[-\log(1-p)\right]^{\frac{1}{\beta}}$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. It can be shown that the estimates of θ and β are the solutions of

$$\frac{\bar{x}^2}{s^2} = \frac{\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2}{\Gamma\left(1+\frac{2}{\beta}\right) - \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2}$$

and

$$\hat{\theta} = \frac{\bar{x}}{\Gamma\left(1+\frac{1}{\hat{\beta}}\right)}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$.

Hence the estimate of VaR is

$$\widehat{VaR}_p(X) = \hat{\theta}\left[-\log(1-p)\right]^{\frac{1}{\hat{\beta}}}$$

b) Non-parametric estimation methods

1) Historical method

Data – $x_1, x_2, \ldots, x_n$

Order the data from smallest to largest

$$\boxed{x_{(1)}} \leq x_{(2)} \leq \cdots \leq \boxed{x_{(n)}}$$

$x_{(1)}$ smallest, $x_{(n)}$ largest

Then

$$\widehat{VaR}_p(X) = x_{(i)}$$

if $p \in \left(\frac{i-1}{n}, \frac{i}{n} \right]$.

Ex X = Losses

Data on losses as

8 −2 2 0 5

Order the data

−2	0	2	5	8
$x_{(1)}$	$x_{(2)}$	$x_{(3)}$	$x_{(4)}$	$x_{(5)}$

$$\widehat{VaR}_{0.2}(X) = x_{(1)} = -2$$

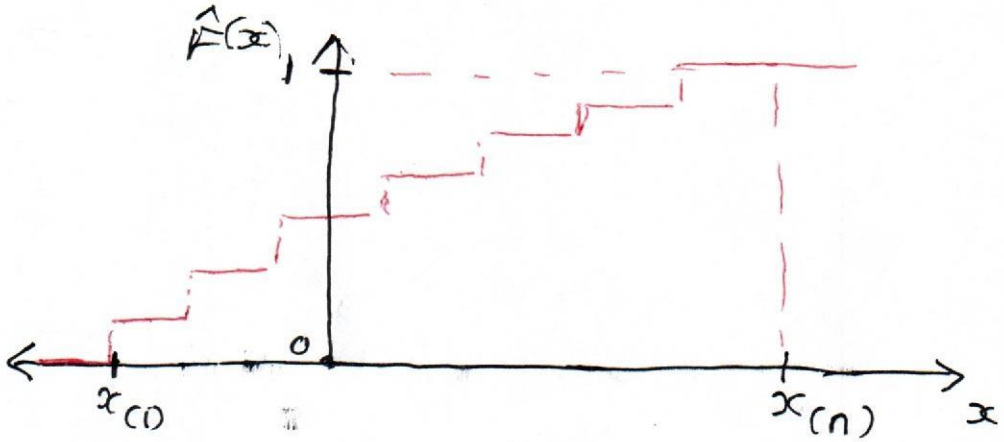
$$\widehat{VaR}_{0.9}(X) = x_{(5)} = 8$$

2) Bootstrap Method due to Efron

Data $x_1, x_2, \ldots, x_n$

Compute the empirical CDF

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} I\{x_i \leq x\}$$



i) simulate B samples from $\hat{F}$

ii) estimate VaR_p for each of the samples by the historical method, yielding $\widehat{VaR}_p^{(j)}$, $j = 1, 2, \ldots, B$

iii) Compute $\widehat{VaR}_p$ as either

$$\widehat{VaR}_p = \text{Mean}\left(\widehat{VaR}_p^{(1)}, \ldots, \widehat{VaR}_p^{(B)}\right)$$

or

$$\widehat{VaR}_p = \text{Median}\left(\widehat{VaR}_p^{(1)}, \ldots, \widehat{VaR}_p^{(B)}\right)$$

3) Jackknife method

Data $x_1, x_2, \cdots, x_n$

(i) estimate VaR_p by the historical method for $x_2, x_3, \cdots, x_n$.
Call this $\widehat{VaR}_p^{(1)}$.

(ii) estimate VaR_p by the historical method for $x_1, x_3, \cdots, x_n$.
Call this $\widehat{VaR}_p^{(2)}$.

.
.
.

(iii) estimate VaR_p by the historical method for $x_1, x_2, \cdots, x_{n-1}$.
Call this $\widehat{VaR}_p^{(n)}$

(iv) Compute $\widehat{VaR}_p$ as either

$$\widehat{VaR}_p = \text{Mean}\left(\widehat{VaR}_p^{(1)}, \cdots, \widehat{VaR}_p^{(n)}\right)$$

or

$$\widehat{VaR}_p = \text{Median}\left(\widehat{VaR}_p^{(1)}, \cdots, \widehat{VaR}_p^{(n)}\right).$$

4) Kernel method

LECTURE

16 NOVEMBER

10:00-11:00AM

MATH4/68181

Estimation Methods for Expected Shortfall

a) Parametric estimation methods

b) Non-parametric estimation methods

c) Semi-parametric estimation methods

a) and b) for Level 3

a), b) and c) for Levels 4, 6

c) Semi-parametric estimation methods

1) Heavy tail method

Suppose X = returns of losses.

Assume that

$$P(X < -x) \sim x^{-\alpha} L(x)$$

as $x \to \infty$, where $\alpha > 0$ and $L(\cdot)$ is such that

$$\frac{L(tx)}{L(t)} \longrightarrow 1$$

as $t \to \infty$.

"$\sim$" means "behaves as"

Embrechts et al (2005) (Professor at ETH) showed that

$$\widehat{ES}_p(X) = \frac{1}{p} \int_0^p \exp\left[\left(\frac{k_{n,p}}{nq} \right)^{\frac{1}{\hat{\alpha}_{k_{n,p}}}} x_{k_{n,p}} \right] dq - 1$$

where $x_1, x_2, \ldots, x_n$ is a random sample on X,

$$k_{n,p} = [n(p + 0.05)],$$

$$\hat{\alpha}_{k} = \left[\frac{1}{k} \sum_{i=1}^{k} \log\left(\frac{x_{(i)}}{x_{(k)}} \right) \right]^{-1}$$

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$$

$$\widehat{ES}_p(X) = \frac{1}{p}\int_0^p \exp\left[\left(\frac{l_{n,p}}{nq}\right)^{\widehat{\alpha}_{n,l_{n,p}}} x_{l_{n,p}}\right] dq - 1$$

— (1)

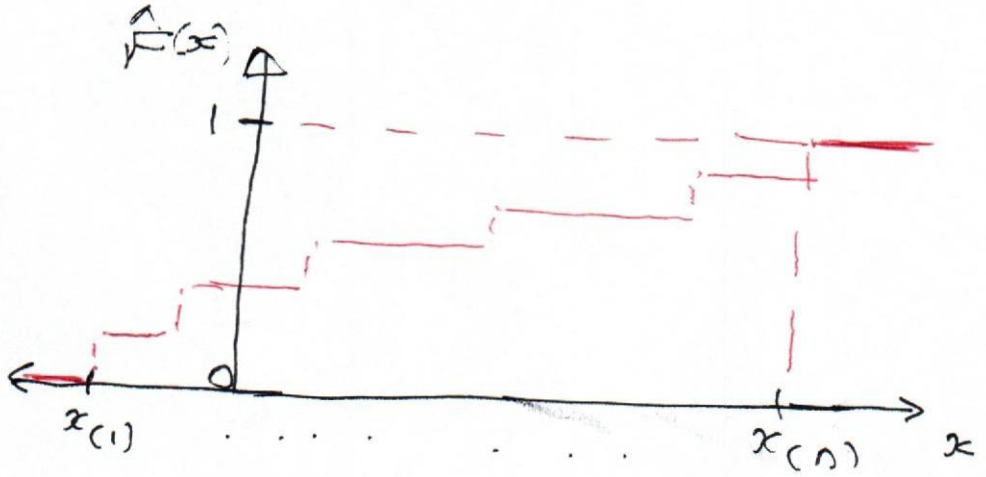
2) Method due to Necir et al (2010)

Data : $x_1, x_2, \ldots, x_n$

Ordered data : $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$

The empirical CDF

$$\hat{F}(x) = \frac{1}{n}\sum_{i=1}^{n} I\{x_i \leq x\}$$



The estimator of ES_p is

$$\widehat{ES}_p = \frac{1}{p}\int_{\frac{k}{n}}^{p} \hat{F}^{\leftarrow}(t)\,dt + \frac{k\,x_{(n-k)}}{np(1-\widehat{\alpha})}$$

— (2)

where $1 \leq k \leq n$ and

$$\widehat{\alpha} = \left[\frac{1}{k}\sum_{i=1}^{k} \log\left(\frac{x_{(i)}}{x_{(k)}}\right)\right]^{-1}.$$

(1) is semi-parametric since α is a parameter. The remaining components in (1) are based only on the data.

Similar reasons for (2).

Suppose

X_1 = stock returns for company A

X_2 = " " " " B

What is the probability $X_1 < X_2$?

Suppose

X_1 = stock returns for company A_1

X_2 = " " " " A_2

$\vdots$

X_k = " " " " A_k

What is the probability $P(X_1 < X_2 < \cdots < X_k)$

Two variables case

If X_1 and X_2 are independent RVs

$$P(X_1 < X_2) = \int_{-\infty}^{\infty} F_{X_1}(x) f_{X_2}(x) \, dx$$

where $F_{X_1}(\cdot)$ denotes the CDF of X_1 and $f_{X_2}(\cdot)$ denotes the PDF of X_2.

If X_1 and X_2 are dependent RVs

$$P(X_1 < X_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2$$

where f_{X_1, X_2} denotes the joint PDF of (X_1, X_2).

Ex 1

Suppose $X_1 \sim N(\mu_1, \sigma_1^2)$

and $X_2 \sim N(\mu_2, \sigma_2^2)$

are independent RVs.

$$P(X_1 < X_2)$$

$$= P(X_1 - X_2 < 0)$$

$$= P(N(\mu_1, \sigma_1^2) - N(\mu_2, \sigma_2^2) < 0)$$

$$= P(N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2) < 0)$$

$$= P\left(\frac{N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}} < \frac{0 - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right)$$

$$= P\left(N(0,1) < \frac{\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right)$$

$$= \underline{\Phi}\left(\frac{\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right)$$

Suppose $x_{1,1}, x_{1,2}, \cdots, x_{1,n}$ is a random sample on X_1

$x_{2,1}, x_{2,2}, \cdots, x_{2,n}$ is a random sample on X_2

The MLEs of μ_1, μ_2, σ_1^2 and σ_2^2 are

$$\hat{\mu}_1 = \frac{1}{n} \sum_{i=1}^{n} x_{1,i}$$

$$\hat{\mu}_2 = \frac{1}{n} \sum_{i=1}^{n} x_{2,i}$$

$$\hat{\sigma}_1^2 = \frac{1}{n} \sum_{i=1}^{n} (x_{1,i} - \hat{\mu}_1)^2$$

$$\hat{\sigma}_2^2 = \frac{1}{n} \sum_{i=1}^{n} (x_{2,i} - \hat{\mu}_2)^2 .$$

Hence,

$$\widehat{P(X_1 < X_2)} = \Phi\left(\frac{\hat{\mu}_2 - \hat{\mu}_1}{\sqrt{\hat{\sigma}_1^2 + \hat{\sigma}_2^2}} \right) .$$

EXAMPLE CLASS

12 NOVEMBER

12:00-13:00PM

MATH3/4/68181

Q1

$$X \sim Exp(\lambda)$$

$$VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

$$ES_p(X) = -\frac{1}{p\lambda}\left\{p\cdot\log(1-p) - p - \log(1-p)\right\}$$

From example class last week

Suppose $x_1, x_2, \ldots, x_n$ is a random sample from $Exp(\lambda)$.

The MLE of

$$\hat{\lambda} = \frac{1}{\bar{x}} \quad \text{(from Math20802)}.$$

Hence MLEs of VaR_p and ES_p are

$$\widehat{VaR}_p(X) = -\bar{x}\cdot\log(1-p)$$

and $$\widehat{ES}_p(X) = -\frac{\bar{x}}{p}\left\{p\cdot\log(1-p) - p - \log(1-p)\right\}$$

Q2

X has PDF $a x^{a-1}$

$$VaR_p(X) = p^{\frac{1}{a}}$$

$$ES_p(X) = \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1}$$

from Example class last week.

Suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. The MLE of a is

$$\hat{a} = -\frac{n}{\sum_{i=1}^{n} \log x_i}$$ (from lecture this morning)

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{VaR_p(X)} = p^{-\frac{\sum_{i=1}^{n} \log x_i}{n}}$$

and

$$\widehat{ES_p(X)} = \frac{p^{-\frac{\sum_{i=1}^{n} \log x_i}{n}}}{-\frac{\sum_{i=1}^{n} \log x_i}{n} + 1}$$

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{VaR}_p(X) = e^{\hat{\mu} + \hat{\sigma}\,\Phi^{-1}(p)}$$

and

$$\widehat{ES}_p(X) = \frac{e^{\hat{\mu}}}{p} \int_0^p e^{\hat{\sigma}\,\Phi^{-1}(t)}\, dt.$$

Q3 covered in lectures

Q4

$X \sim LN(\mu, \sigma^2)$

The CDF of X is

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right) = p$$

$$\Rightarrow \frac{\log x - \mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow \log x = \mu + \sigma \Phi^{-1}(p)$$

$$\Rightarrow x = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$\Rightarrow \boxed{VaR_p(X) = e^{\mu + \sigma \Phi^{-1}(p)}}$$

$$ES_p(X) = \frac{1}{p} \int_0^p e^{\mu + \sigma \Phi^{-1}(t)} \, dt$$

$$= \boxed{\frac{e^{\mu}}{p} \int_0^p e^{\sigma \Phi^{-1}(t)} \, dt}$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample from $LN(\mu, \sigma^2)$. The MLEs of μ and σ are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \log x_i$$

and

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\log x_i - \hat{\mu})^2}$$

(from Math20802)

Q5

X has PDF $f(x) = \theta_2 x^{\theta_2 - 1} \theta_1^{-\theta_2}$,

$0 < x < \theta_1$

The CDF of X is

$$F(x) = \theta_2 \theta_1^{-\theta_2} \int_0^x y^{\theta_2 - 1} \, dy$$

$$= \theta_2 \theta_1^{-\theta_2} \left[\frac{y^{\theta_2}}{\theta_2} \right]_0^x$$

$$= \left(\frac{x}{\theta_1} \right)^{\theta_2}$$

To find VaR, set

$$\left(\frac{x}{\theta_1} \right)^{\theta_2} = p$$

$$\Rightarrow x = \theta_1 p^{\frac{1}{\theta_2}}$$

$$\Rightarrow \text{VaR}_p(X) = \boxed{\theta_1 p^{\frac{1}{\theta_2}}}$$

$$\Rightarrow ES_p(X) = \frac{\theta_1}{p} \int_0^p t^{\frac{1}{\theta_2}} \, dt$$

$$= \frac{\theta_1}{p} \left[\frac{t^{\frac{1}{\theta_2}+1}}{\frac{1}{\theta_2}+1} \right]_0^p$$

$$= \boxed{\frac{\theta_1 \theta_2}{1+\theta_2} p^{1/\theta_2}}$$

Suppose $x_1, x_2, \cdots, x_n$ is a random sample on X. The MLEs of θ_1 and θ_2 are

$$\hat{\theta}_1 = \max(x_1, x_2, \cdots, x_n)$$

and $$\hat{\theta}_2 = \frac{n}{n \log \hat{\theta}_1 - \sum_{i=1}^{n} \log x_i}.$$

(from Math20802)

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{VaR}_p(X) = \hat{\theta}_1 \, p^{\frac{1}{\hat{\theta}_2}}$$

and $$\widehat{ES}_p(X) = \frac{\hat{\theta}_1 \hat{\theta}_2}{1 + \hat{\theta}_2} \, p^{\frac{1}{\hat{\theta}_2}}.$$

Q6

$$X \sim \text{Uni}[\mu-\delta,\ \mu+\delta]$$

$$F(x) = \frac{x-\mu+\delta}{2\delta} = p$$

$$\Rightarrow \quad VaR_p(X) = \mu - \delta + 2\delta p$$

$$\Rightarrow \quad ES_p(X) = \mu - \delta + \delta p$$

From Math 20802, the MLEs of μ and σ are

$$\hat{\mu} = \frac{\max x_i + \min x_i}{2}$$

$$\hat{\delta} = \frac{\max x_i - \min x_i}{2}.$$

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{VaR}_p(X) = \hat{\mu} - \hat{\delta} + 2\hat{\delta} p,$$

$$\widehat{ES}_p(X) = \hat{\mu} - \hat{\delta} + \hat{\delta} p.$$

EXAMPLE CLASS

13 NOVEMBER

16:00-17:00PM

MATH3/4/68181

Q1

$X \sim Exp(\lambda)$

From example class last week,

$$VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

and

$$ES_p(X) = -\frac{1}{p\lambda}\left[p\log(1-p) - p - \log(1-p)\right].$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample from $Exp(\lambda)$. The MLE of λ is

$$\hat{\lambda} = \frac{1}{\bar{x}} \quad \text{(from Math 20802)}.$$

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{VaR}_p(X) = -\bar{x}\log(1-p)$$

and

$$\widehat{ES}_p(X) = -\frac{\bar{x}}{\lambda}\left[p\log(1-p) - p - \log(1-p)\right]$$

Q2

X has PDF $f(x) = a x^{a-1}$

The CDF $F(x) = x^a$, $0 < x < 1$

From Monday's lecture,

$$VaR_p(X) = p^{\frac{1}{a}},$$

and

$$ES_p(X) = \frac{p^{\frac{1}{a}}}{\frac{1}{a} + 1}.$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. The MLE of a is

$$\hat{a} = -\frac{n}{\sum_{i=1}^{n} \log x_i}.$$

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{VaR}_p(X) = p^{\frac{1}{\hat{a}}}$$

and

$$\widehat{ES}_p(X) = \frac{p^{\frac{1}{\hat{a}}}}{\frac{1}{\hat{a}} + 1}.$$

Q3 covered in lectures

Q4

$$X \sim LN(\mu, \sigma^2)$$

The CDF of X is

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right) = p$$

$$\Rightarrow \frac{\log x - \mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow \log x = \mu + \sigma \Phi^{-1}(p)$$

$$\Rightarrow x = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$\Rightarrow VaR_p(X) = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$\Rightarrow ES_p(X) = \frac{1}{p} \int_0^p e^{\mu + \sigma \Phi^{-1}(t)} dt$$

$$= \frac{e^{\mu}}{p} \int_0^p e^{\sigma \Phi^{-1}(t)} dt$$

From Math 20802, the MLEs of μ and σ are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \log x_i$$

and $$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\log x_i - \hat{\mu})^2}$$

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{VaR}_p(X) = e^{\hat{\mu} + \hat{\sigma} \Phi^{-1}(p)},$$

$$\widehat{ES}_p(X) = \frac{e^{\hat{\mu}}}{p} \int_0^p e^{\hat{\sigma} \Phi^{-1}(t)} dt$$

Q5

X has PDF $f(x) = \theta_2 x^{\theta_2 - 1} \theta_1^{-\theta_2}$,

$0 < x < \theta_1$

The CDF $F(x) = \left(\frac{x}{\theta_1}\right)^{\theta_2} = p$

$$\Rightarrow x = \theta_1 p^{\frac{1}{\theta_2}}$$

$$\Rightarrow VaR_p(X) = \theta_1 p^{\frac{1}{\theta_2}}$$

$$\Rightarrow ES_p(X) = \frac{\theta_1}{p} \int_0^p t^{\frac{1}{\theta_2}} dt$$

$$= \frac{\theta_1}{1 + \frac{1}{\theta_2}} p^{\frac{1}{\theta_2}}$$

From Math 20802, the MLEs of θ_1 and θ_2 are

$$\hat{\theta}_1 = \max(x_1, x_2, \ldots, x_n)$$

and $$\hat{\theta}_2 = \frac{n}{n \log \hat{\theta}_1 - \sum_{i=1}^{n} \log x_i}.$$

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{VaR}_p = \hat{\theta}_1 p^{\frac{1}{\hat{\theta}_2}},$$

$$\widehat{ES}_p = \frac{\hat{\theta}_1 \hat{\theta}_2}{\hat{\theta}_2 + 1} p^{\frac{1}{\hat{\theta}_2}}$$

LECTURE

19 NOVEMBER

9:00-10:00AM

MATH3/4/68181

b) Non-parametric estimation methods

1) Historical method

Data – $x_1, x_2, \ldots, x_n$

Order the data from smallest to largest

$$\boxed{x_{(1)}} \leq x_{(2)} \leq \cdots \leq \boxed{x_{(n)}}$$

$x_{(1)}$: smallest; $x_{(n)}$: largest

Then

$$\widehat{VaR}_p(X) = x_{(i)}$$

if $p \in \left(\frac{i-1}{n}, \frac{i}{n} \right]$.

Ex X = Losses

Data on losses as

8 −2 2 0 5

Order the data

−2	0	2	5	8
$x_{(1)}$	$x_{(2)}$	$x_{(3)}$	$x_{(4)}$	$x_{(5)}$

$$\widehat{VaR}_{0.2}(X) = x_{(1)} = -2$$

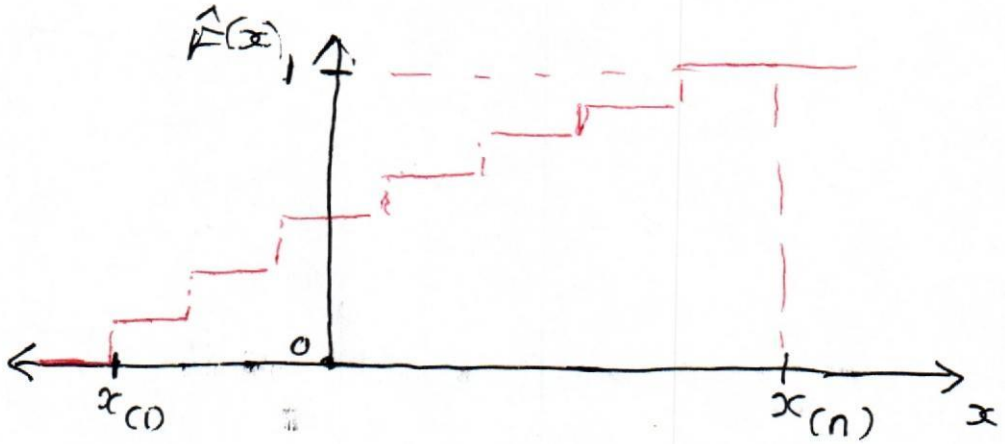
$$\widehat{VaR}_{0.9}(X) = x_{(5)} = 8$$

2) Bootstrap method due to Efron

Data $x_1, x_2, \ldots, x_n$

Compute the empirical CDF

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} I\{x_i \leq x\}$$



i) simulate B samples from $\hat{F}$

ii) estimate VaR_p for each of the samples by the historical method, yielding $\widehat{VaR}_p^{(j)}$, $j = 1, 2, \ldots, B$

iii) Compute $\widehat{VaR}_p$ as either

$$\widehat{VaR}_p = \text{Mean}\left(\widehat{VaR}_p^{(1)}, \ldots, \widehat{VaR}_p^{(B)}\right)$$

or

$$\widehat{VaR}_p = \text{Median}\left(\widehat{VaR}_p^{(1)}, \ldots, \widehat{VaR}_p^{(B)}\right)$$

3) Jackknife method

Data $x_1, x_2, \cdots, x_n$

(i) estimate VaR_p by the historical method for $x_2, x_3, \cdots, x_n$.
Call this $\widehat{VaR}_p^{(1)}$.

(ii) estimate VaR_p by the historical method for $x_1, x_3, \cdots, x_n$.
Call this $\widehat{VaR}_p^{(2)}$.

⋮

(iii) estimate VaR_p by the historical method for $x_1, x_2, \cdots, x_{n-1}$.
Call this $\widehat{VaR}_p^{(n)}$

(iv) Compute $\widehat{VaR}_p$ as either

$$\widehat{VaR}_p = \text{Mean}\left(\widehat{VaR}_p^{(1)}, \cdots, \widehat{VaR}_p^{(n)}\right)$$

or

$$\widehat{VaR}_p = \text{Median}\left(\widehat{VaR}_p^{(1)}, \cdots, \widehat{VaR}_p^{(n)}\right).$$

4) Kernel method

Let $F(\cdot)$ denote the CDF of $X = \text{Loss}$.
The kernel estimate of $F(\cdot)$ is

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} G\left(\frac{x - x_i}{h}\right)$$

where $x_1, x_2, \ldots, x_n$ is a random sample on X

h = bandwidth

$$G(x) = \int_{-\infty}^{x} K(u)\, du$$

$K(\cdot)$ = kernel PDF
usually chosen as the PDF of $N(0,1)$

The estimate of $VaR_p(X)$ can be computed as

- solve $\hat{F}(x) = p$ for x
- $$\hat{VaR}_p(X) = \frac{\sum_{i=1}^{n} \hat{F}\left(\frac{i-\frac{1}{2}}{n} - p\right) x_{(i)}}{\sum_{i=1}^{n} \hat{F}\left(\frac{i-\frac{1}{2}}{n} - p\right)}$$

where $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$ are the ordered data.

Expected shortfall

If X is a continuous RV

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

Properties

(i) $ES_p(0) = 0$

(ii) $X < Y \Rightarrow ES_p(X) < ES_p(Y)$

(iii) $ES_p(X+c) = ES_p(X) + c$

(iv) $ES_p(cX) = c\,ES_p(X)$

(v) $ES_p(X+Y) \leq ES_p(X) + ES_p(Y)$

Proof of (ii)

We know if $X < Y$ then

$$VaR_p(X) < VaR_p(Y) \quad \forall\, p$$

$$\Rightarrow \int_0^p VaR_t(X)\,dt < \int_0^p VaR_t(Y)\,dt \quad \forall\, p$$

$$\Rightarrow \frac{1}{p}\int_0^p VaR_t(X)\,dt < \frac{1}{p}\int_0^p VaR_t(Y)\,dt \quad \forall\, p$$

$$\Rightarrow ES_p(X) < ES_p(Y) \quad \forall\, p$$

Proof of (iii)

We know

$$VaR_p(X+c) = VaR_p(X) + c \quad \forall p$$

$$\Rightarrow \int_0^p VaR_t(X+c)\,dt = \int_0^p [VaR_t(X)+c]\,dt \quad \forall p$$

$$= \int_0^p VaR_t(X)\,dt + cp \quad \forall p$$

$$\Rightarrow \frac{1}{p}\int_0^p VaR_t(X+c)\,dt = \frac{1}{p}\int_0^p VaR_t(X)\,dt + c \quad \forall p$$

$$\Rightarrow ES_p(X+c) = ES_p(X) + c \quad \forall p.$$

Proof of (iv)

We know

$$VaR_p(cX) = c\,VaR_p(X) \quad \forall p$$

$$\Rightarrow \int_0^p VaR_t(cX)\,dt = c\int_0^p VaR_t(X)\,dt \quad \forall p$$

$$\Rightarrow \frac{1}{p}\int_0^p VaR_t(cX)\,dt = \frac{c}{p}\int_0^p VaR_t(X)\,dt \quad \forall p$$

$$\Rightarrow ES_p(cX) = c\,ES_p(X).$$

Estimation methods for expected shortfall

a) Parametric estimation methods

b) Non-parametric " "

c) Semi-parametric " "

a), b) for Level 3

a), b), c) for Levels 4, 6

a) Parametric estimation methods

1) Normal distribution method

Suppose $X = \text{Loss} \sim N(\mu, \sigma^2)$.

$$ES_p(X) = \mu + \frac{\sigma}{p}\int_0^p \Phi^{-1}(t)\,dt$$

If $x_1, x_2, \ldots, x_n$ is a random sample on X then

$$\hat{ES}_p(X) = \bar{x} + \frac{\hat{\sigma}}{p}\int_0^p \Phi^{-1}(t)\,dt$$

where $$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}.$$

$\hat{ES}_p(X)$ is a biased estimator but consistent.

2) Uniform distribution method

Suppose $X = \text{Loss} \sim \text{Uni}[a, b]$

$$ES_p(X) = a + \frac{p}{2}(b - a)$$

If $x_1, x_2, \cdots, x_n$ is a random sample on X then

$$\widehat{ES}_p(X) = \min(x_1, \cdots, x_n) + \frac{p}{2}\left[\max(x_1, \cdots, x_n) - \min(x_1, \cdots, x_n)\right]$$

$\widehat{ES}_p(X)$ is biased but consistent for $ES_p(X)$.

3) Power function distribution method

Suppose $X = \text{Loss}$ has PDF $f(x) = a x^{a-1}$, $0 < x < 1$.

$$ES_p(X) = \frac{p^{\frac{1}{a}}}{\frac{1}{a} + 1}$$

If $x_1, x_2, \ldots, x_n$ is a random sample on X then

$$\widehat{ES_p}(X) = \frac{p^{\frac{1}{\hat{a}}}}{\frac{1}{\hat{a}} + 1}$$

where $\hat{a} = -\frac{n}{\sum_{i=1}^{n} \log x_i}$.

4) Weibull distribution method

Suppose X = Loss has CDF $F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}}$.

We know

$$VaR_p(X) = \theta\left[-\log(1-p)\right]^{\frac{1}{\beta}}.$$

So,

$$ES_p(X) = \frac{\theta}{p}\int_0^p \left[-\log(1-t)\right]^{\frac{1}{\beta}} dt$$

Set $y = -\log(1-t)$

$t = 1 - e^{-y}$

$\frac{dt}{dy} = e^{-y}$

$$= \frac{\theta}{p}\int_0^{-\log(1-p)} y^{\frac{1}{\beta}} e^{-y}\, dy$$

$$= \frac{\theta}{p}\,\gamma\left(1+\frac{1}{\beta}, -\log(1-p)\right)$$

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-t}\, dt$$

Incomplete gamma function

Hence,

$$\widehat{ES}_p(X) = \frac{\hat{\theta}}{p}\,\gamma\left(1+\frac{1}{\hat{\beta}}, -\log(1-p)\right)$$

where $\hat{\theta}$ and $\hat{\beta}$ are the solutions of

$$\hat{\theta} = \frac{\bar{x}}{\Gamma\left(1 + \frac{1}{\hat{\beta}}\right)}$$

and

$$\frac{(\bar{x})^2}{s^2} = \frac{\left[\Gamma\left(1 + \frac{1}{\hat{\beta}}\right)\right]^2}{\Gamma\left(1 + \frac{2}{\hat{\beta}}\right) - \left[\Gamma\left(1 + \frac{1}{\hat{\beta}}\right)\right]^2}$$

where

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2.$$

b) Non-parametric estimation methods

1) Historical method

Data : $x_1, x_2, \dots, x_n$

Order the data as

$$x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$$

Then

$$\widehat{ES}_p(X) = \frac{1}{[np]} \sum_{i=1}^{[np]} x_{(i)}$$

where $[x]$ denotes the largest integer less than or equal to x.

Ex $[4.4] = 4, \quad [5.1] = 5$

Ex

−2	8	9	−10	1

−10	−2	1	8	9
=	=	=	=	=
$x_{(1)}$	$x_{(2)}$	$x_{(3)}$	$x_{(4)}$	$x_{(5)}$

$$\widehat{ES}_{0.4}(X) = \frac{1}{2} \sum_{i=1}^{2} x_{(i)}$$

$$= \frac{1}{2} (-10 - 2)$$

$$= -6.$$

LECTURE

20 NOVEMBER

9:00-10:00AM

MATH3/4/68181

b) Non-parametric estimation methods

1) Historical method

Data : $x_1, x_2, \dots, x_n$

Order the data as

$$x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$$

Then

$$\widehat{ES}_p(X) = \frac{1}{[np]} \sum_{i=1}^{[np]} x_{(i)}$$

where $[x]$ denotes the largest integer less than or equal to x.

Ex $[4.4] = 4, \quad [5.1] = 5$

Ex

−2	8	9	−10	1

−10	−2	1	8	9
=	=	=	=	=
$x_{(1)}$	$x_{(2)}$	$x_{(3)}$	$x_{(4)}$	$x_{(5)}$

$$\widehat{ES}_{0.4}(X) = \frac{1}{2} \sum_{i=1}^{2} x_{(i)}$$

$$= \frac{1}{2} (-10 - 2)$$

$$= -6.$$

2) Bootstrap method

Data $x_1, x_2, \ldots, x_n$

(i) Compute the empirical CDF

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} I\{x_i \leq x\}$$

(ii) Simulate B samples each of size n from $\hat{F}$.

(iii) Compute $\widehat{ES}_p$ for each of the B samples by the historical method, resulting $\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \ldots, \widehat{ES}_p^{(B)}$.

(iv) Compute $\widehat{ES}_p$ as

$$\text{mean}\left(\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \ldots, \widehat{ES}_p^{(B)}\right)$$

or

$$\text{median}\left(\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \ldots, \widehat{ES}_p^{(B)}\right).$$

3) Jackknife method

Data $x_1, x_2, \cdots, x_n$

(i) Compute $\widehat{ES}_p$ by the historical method for $x_2, x_3, \cdots, x_n$, resulting in $\widehat{ES}_p^{(1)}$.

(ii) Compute $\widehat{ES}_p$ by the historical method for $x_1, x_3, \cdots, x_n$, resulting in $\widehat{ES}_p^{(2)}$

⋮

(iii) Compute $\widehat{ES}_p$ by the historical method for $x_1, x_2, \cdots, x_{n-1}$, resulting in $\widehat{ES}_p^{(n)}$.

(iv) Compute $\widehat{ES}_p$ as

$$\text{Mean}\left(\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \cdots, \widehat{ES}_p^{(n)}\right)$$

or

$$\text{Median}\left(\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \cdots, \widehat{ES}_p^{(n)}\right).$$

4) Kernel method

Data $x_1, x_2, \ldots, x_n$

Order the data as $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$

The kernel estimator of $ES_p(X)$ is

$$\widehat{ES}_p(X) = \frac{1}{np} \sum_{i=1}^{n} x_i A_h\left(\hat{q}(p) - x_i\right)$$

where

h = bandwidth

$$\hat{q}(p) = \sum_{i=1}^{n} \left[\int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(t-p)\, dt \right] x_{(i)}$$

$$K_h(u) = \frac{1}{h} K\left(\frac{u}{h}\right)$$

$$A_h(u) = A\left(\frac{u}{h}\right)$$

$$A(x) = \int_{-\infty}^{x} K(u)\, du$$

$K(\cdot)$ = kernel PDF usually chosen as the PDF of $N(0, 1)$

5) Richardson's method (based on Richardson (1911))

Richardson was a professor of applied math in Manchester. His picture is on the front door of G207, ground floor of ATB.

Suppose $x_1, x_2, \ldots, x_n$ are the data

(i) Compute the empirical CDF

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} I\{x_i \leq x\}$$

(ii) simulate $x_1, x_2, \ldots, x_N$ from $\hat{F}(\cdot)$

(iii) compute $\widehat{ES}_p$ by the historical method for the data simulated in (ii)

(iv) Repeat steps (ii) and (iii) 1000 times

(v) Let m_N = the average of the 1000 estimates of ES_p

(vi) Let $S_k = m_{N_k}$, $k = 1, 2, \ldots, \ell+1$

for some ℓ and $N_1, N_2, \ldots, N_{\ell+1}$

(vii) Compute $\widehat{ES}_p$ as

$$\widehat{ES}_p = \frac{\begin{vmatrix} S_1 & S_2 & \cdots & S_{\ell+1} \\ 1 & \frac{1}{2} & \cdots & \frac{1}{\ell+1} \\ \vdots & & & \\ 1^{\ell} & \left(\frac{1}{2}\right)^{\ell} & \cdots & \left(\frac{1}{\ell+1}\right)^{\ell} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \frac{1}{2} & \cdots & \frac{1}{\ell+1} \\ \vdots & \vdots & & \\ 1^{\ell} & \left(\frac{1}{2}\right)^{\ell} & \cdots & \left(\frac{1}{\ell+1}\right)^{\ell} \end{vmatrix}}$$

Ex

$\ell = 1$

$N_1 = 100$

$N_2 = 1000$

$$\widehat{ES}_p = \frac{\begin{vmatrix} m_{100} & m_{1000} \\ 1 & \frac{1}{2} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{vmatrix}}$$

$$= \frac{\frac{m_{100}}{2} - m_{1000}}{\frac{1}{2} - 1}$$

Models for Stock Returns

1) Model based on varying volatility

(Model 1)

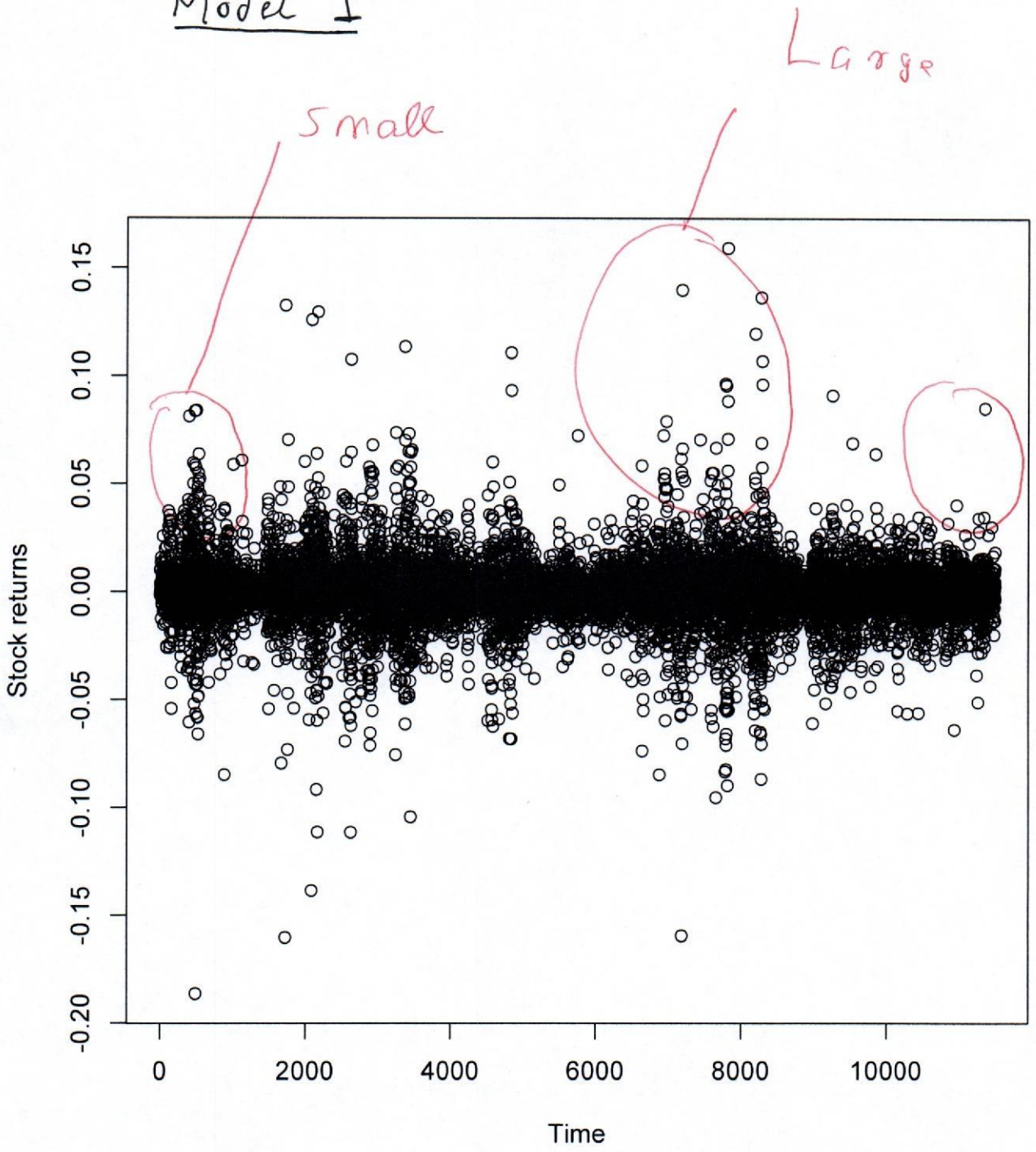
2) Model based on additive random walk

(Model 2)

3) Model based on multiplicative random walk

(Model 3)

Model I


Large
Small
Stock returns
0.15
0.10
0.05
0.00
-0.05
-0.10
-0.15
-0.20
0
2000
4000
6000
8000
10000
Time

Model I

X = stock returns

V = variability (volatility)

V is a random variable itself

V = unobservable

$X \mid V$ = observable

Hence, the PDF of X is

$$f_X(x) = \int_0^\infty \underbrace{f_{X|V}(x|v)}_{\text{conditional PDF of } X|V} \underbrace{f_V(v)}_{\text{PDF of } V} dv$$

The CDF of X is

$$F_X(x) = \int_0^\infty F_{X|V}(x|v) f_V(v) dv$$

LECTURE

23 NOVEMBER

10:00-11:00AM

MATH4/68181

Copulas

A copula is a function $C : \underbrace{[0,1] \times [0,1]}_{\text{unit square}} \to [0,1]$

which describes the dependence between two variables, say X and Y.

Egs

1) X = gold price
   Y = silver "

2) X = gold price
   Y = diamond "

3) X = petrol price
   Y = car price

Let (X, Y) be a random vector with joint CDF $F(x, y)$. Then

$$F_X(x) = F(x, \infty)$$

$$F_Y(y) = F(\infty, y).$$

For every $F(x, y)$ there is a corresponding copula

$$F(x, y) = P(X \leq x, Y \leq y)$$

$$= P(F_X(X) \leq F_X(x), F_Y(Y) \leq F_Y(y))$$

probability integral transform →

$$= P(U \leq F_X(x), V \leq F_Y(y))$$

$U \sim \text{Uni}[0, 1]$
$V \sim \text{Uni}[0, 1]$

$0 \leq \leq 1 \quad 1 \leq 0 \leq 1$

$$= F_{U,V}(F_X(x), F_Y(y))$$

$$= C(F_X(x), F_Y(y))$$

$C : [0, 1] \times [0, 1] \to [0, 1]$

For every copula there is a corresponding $F(x, y)$

$$C(u, v) = P(U \leq u, V \leq v)$$

$$= P(F_X^{-1}(U) \leq F_X^{-1}(u), F_Y^{-1}(V) \leq F_Y^{-1}(v))$$

probability integral transform

$$= P(X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(v))$$

$$= F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v))$$

Two extreme cases of copula

1) Complete independent copula

$$C(u, v) = u \cdot v$$

corresponds to

$$F_{X,Y}(x, y) = F_X(x)\, F_Y(y)$$

$\Leftrightarrow$ X and Y are indep RVs

2) Complete dependent copula

$$C(u, v) = \min(u, v)$$

corresponds to

$$F_{X,Y}(x, y) = \min\left[F_X(x), F_Y(y)\right]$$

$$= P(X \le x, Y \le y)$$

$$= P(X \le x, X \le y)$$

$$= P(X \le \min(x, y))$$

$$= F_X(\min(x, y))$$

$$= \min\left[F_X(x), F_X(y)\right]$$

Two most popular models for copulas

1) Normal (Gaussian) copula

$$C(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \times e^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}} dy\, dx$$

where $-1 < \rho < 1$ (correlation coefficient) and $\Phi(\cdot)$ is the CDF of $N(0,1)$.

Gaussian copula is not a good model to describe dependence between financial variables.

Please read "The formula that killed Wall Street".

2) t copula

$$C(u,v) = \int_{-\infty}^{T_a^{-1}(u)} \int_{-\infty}^{T_a^{-1}(v)} \frac{\Gamma\left(\frac{a}{2}+1\right)}{\pi a \Gamma\left(\frac{a}{2}\right)\sqrt{1-\rho^2}} \times \left[1 + \frac{x^2+y^2-2\rho xy}{a(1-\rho^2)}\right]^{-\frac{a+2}{2}} dy\, dx$$

where $-1 < \rho < 1$ (correlation coefficient)

a = degree of freedom

$T_a(\cdot)$ = CDF of the univariate Student's t distribution with degree of freedom a.

t copula is a popular model to describe dependence between financial variables.

Formal definition of a copula

A function $C : [0,1] \times [0,1] \to [0,1]$ is a copula if it satisfies the following:

i) $C(u, 0) = 0$

ii) $C(0, v) = 0$

iii) $C(u, 1) = u$

iv) $C(1, v) = v$

v) $\frac{\partial}{\partial u} C(u, v) \geq 0 \quad \forall\, u, v$

vi) $\frac{\partial}{\partial v} C(u, v) \geq 0 \quad \forall\, u, v$

Ex 1

$$C(u_1, u_2) = u_1 u_2$$

i) $C(u, 0) = u \cdot 0 = 0$ ✓

ii) $C(0, v) = 0 \cdot v = 0$ ✓

iii) $C(u, 1) = u \cdot 1 = u$ ✓

iv) $C(1, v) = 1 \cdot v = v$ ✓

v) $\frac{\partial}{\partial u} C(u, v) = v \geq 0$ ✓

vi) $\frac{\partial}{\partial v} C(u, v) = u \geq 0$ ✓

So, $C(u, v) = u \cdot v$ is a copula.

Ex 2

$$C(u_1, u_2) = u_1 u_2 e^{-\theta (\log u_1)(\log v_2)}$$

$$= u_1 u_2 u_2^{-\theta \log u_1} = u_1 u_2^{1-\theta \log u_1}$$

i) $C(u, 0) = u \cdot 0^{1-\theta \log u_1} = 0$

ii) $C(0, v) = 0 \cdot u_2^{\infty} = 0$

iii) $C(u, 1) = u \cdot 1 \cdot e^{-0} = u$

iv) $C(1, v) = 1 \cdot v \cdot e^{-0} = v$

v) $\frac{\partial}{\partial u_1} C(u_1, u_2) = u_2 e^{-\theta (\log u_1)(\log u_2)}$

$$+ u_1 u_2 e^{-\theta (\log u_1)(\log u_2)} \left(-\frac{\theta}{u_1}\right)$$

$$= u_2 e^{-\theta (\log u_1)(\log u_2)} [1 - \theta]$$

EXAMPLE CLASS

19 NOVEMBER

12:00-13:00PM

MATH3/4/68181

Q1

$$L(\sigma) = \prod_{i=1}^{n} \left[\frac{1}{\sigma} e^{-\frac{x_i}{\sigma}} e^{-e^{-\frac{x_i}{\sigma}}} \right]$$

$$= \frac{1}{\sigma^n} e^{-\frac{1}{\sigma}\sum_{i=1}^{n} x_i} e^{-\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}}$$

$$\log L(\sigma) = -n \log \sigma - \frac{1}{\sigma}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}$$

$$\frac{d \log L(\sigma)}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{x_i}{\sigma^2} e^{-\frac{x_i}{\sigma}} = 0$$

$$\Rightarrow n\sigma + \sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}} = \sum_{i=1}^{n} x_i$$

must be solved numerically for the MLE of σ.

Q2

$$L(\lambda,\sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{\lambda} x_i^{-\lambda-1} e^{-\sigma^{\lambda} x_i^{-\lambda}} \right]$$

$$\log L(\lambda,\sigma) = n \log \lambda + n\lambda \log \sigma - (\lambda+1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n \log \sigma - \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda} \log\left(\frac{\sigma}{x_i}\right) = 0 \quad \text{— (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} - \sum_{i=1}^{n} \frac{\lambda \sigma^{\lambda-1}}{x_i^{\lambda}} = 0 \quad \text{— (2)}$$

$$(2) \Rightarrow \sigma^{-\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i^{-\lambda}$$

$$\Rightarrow \sigma = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{-\lambda} \right)^{-\frac{1}{\lambda}} \quad \text{— (3)}$$

Sub (3) into (1)

$$\frac{n}{\lambda} - \frac{n}{\lambda} \log \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{-\lambda} \right) - \sum_{i=1}^{n} \log x_i + \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{-\lambda} \right)^{-1} \sum_{i=1}^{n} x_i^{-\lambda} \log x_i + \frac{1}{\lambda} \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{-\lambda} \right)^{-1} \left[\log \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{-\lambda} \right) \right] \sum_{i=1}^{n} x_i^{-\lambda} \quad \text{— (4)}$$

$$= 0$$

(4) only involves λ.

So it can be solved numerically for its MLE $\hat{\lambda}$.

Then sub into (3) to obtain the MLE of σ.

Q3

$$L(\lambda, \sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{-\lambda} x_i^{\lambda-1} e^{-\left(\frac{x_i}{\sigma}\right)^{\lambda}} \right]$$

$$= \lambda^n \sigma^{-n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{\lambda-1} e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}}$$

$$\log L(\lambda, \sigma) = n \log \lambda - n\lambda \log \sigma + (\lambda-1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - n \log \sigma + \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda} \log \left(\frac{x_i}{\sigma}\right) = 0 \quad \text{—— (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n\lambda}{\sigma} + \lambda \sum_{i=1}^{n} \frac{x_i^{\lambda}}{\sigma^{\lambda+1}} = 0 \quad \text{—— (2)}$$

$$(2) \Rightarrow \quad \sigma = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right)^{\frac{1}{\lambda}} \quad \text{———— (3)}$$

Sub (3) into (1) to get

$$\frac{n}{\lambda} - \frac{n}{\lambda} \log \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right) + \sum_{i=1}^{n} \log x_i - \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right)^{-1} \sum_{i=1}^{n} x_i^{\lambda} \log x_i$$

$$+ \frac{1}{\lambda} \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right)^{-1} \left[\log \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right) \right] \sum_{i=1}^{n} x_i^{\lambda} \log x_i = 0$$

$$\left(\sum_{i=1}^{n} x_i^{\lambda} \right) \quad \text{— (4)}$$

(4) involves only λ.

Solve it numerically for the MLE $\hat{\lambda}$ of λ.

Sub $\hat{\lambda}$ into (3) to get the MLE $\hat{\sigma}$ of σ.

EXAMPLE CLASS

20 NOVEMBER

16:00-17:00PM

MATH3/4/68181

Q1

$$L(\sigma) = \prod_{i=1}^{n} \left[\frac{x_i}{\sigma} e^{-\frac{x_i}{\sigma}} \right]$$

$$= \sigma^{-n} e^{-\frac{1}{\sigma}\sum_{i=1}^{n} x_i} e^{-\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}}$$

$$\log L(\sigma) = -\frac{1}{\sigma}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}} - n \log \sigma$$

$$\frac{d \log L}{d\sigma} = \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}} - \frac{n}{\sigma}$$

$$= 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}} + n\sigma$$

This must be solved numerically for the MLE $\hat{\sigma}$ of σ.

Q2

$$L(\sigma,\lambda) = \prod_{i=1}^{n} \left[\lambda \sigma^{+\lambda} x_i^{-\lambda-1} e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}} \right]$$

$$= \lambda^n \sigma^{n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{-\lambda-1} e^{-\sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda}}$$

$$\log L(\sigma,\lambda) = n\log\lambda + n\lambda\log\sigma - (\lambda+1)\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\left(\frac{\sigma}{x_i}\right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} - \sum_{i=1}^{n}\left(\frac{\sigma}{x_i}\right)^{\lambda}\frac{\lambda}{\sigma} = 0 \quad \text{—— (1)}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n\log\sigma - \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\left(\frac{\sigma}{x_i}\right)^{\lambda}\log\left(\frac{\sigma}{x_i}\right) = 0 \quad \text{—— (2)}$$

$$(1) \Rightarrow \quad n = \sigma^{\lambda}\sum_{i=1}^{n} x_i^{-\lambda}$$

$$\Rightarrow \quad \sigma = \left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \right]^{\frac{1}{\lambda}} \quad \text{—— (3)}$$

Sub (3) into (2) to get

$$\frac{n}{\lambda} + \frac{n}{\lambda}\log\left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \right] - \sum_{i=1}^{n}\log x_i$$

$$-\left(\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}}\right)\frac{1}{\lambda}\log\left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}}\right]\sum_{i=1}^{n} x_i^{-\lambda}$$

$$+\left(\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}}\right)\sum_{i=1}^{n} x_i^{-\lambda}\log x_i = 0 \quad \text{—— (4)}$$

(4) involves only λ.

It can be solved numerically for the MLE $\hat{\lambda}$ of λ.

Sub $\hat{\lambda}$ into (3) to get

$$\hat{\sigma} = \left[\frac{n}{\sum_{i=1}^{n} x_i^{\hat{\lambda}}} \right]^{\frac{1}{\hat{\lambda}}}.$$

Q3

$$L(\sigma,\lambda) = \prod_{i=1}^{n} \left[\lambda \sigma^{-\lambda} x_i^{\lambda-1} e^{-\left(\frac{x_i}{\sigma}\right)^{\lambda}} \right]$$

$$= \lambda^n \sigma^{-n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{\lambda-1} e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}}$$

$$\log L(\sigma,\lambda] = n \log \lambda - n\lambda \log \sigma + (\lambda-1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n\lambda}{\sigma} + \frac{\lambda}{\sigma^{\lambda+1}} \sum_{i=1}^{n} x_i^{\lambda} = 0 \quad \text{—— (1)}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - n \log \sigma + \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda} \log \left(\frac{x_i}{\sigma}\right)$$

$$= 0 \quad \text{—— (2)}$$

$$(1) \Rightarrow \quad \sigma = \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right]^{\frac{1}{\lambda}} \quad \text{—————— (3)}$$

Sub (3) into (2) to get

$$\frac{n}{\lambda} - \frac{n}{\lambda} \log \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right] + \sum_{i=1}^{n} \log x_i$$

$$- \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right]^{-1} \sum_{i=1}^{n} x_i^{\lambda} \log x_i$$

$$+ \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right]^{-1} \left(\frac{1}{\lambda} \log \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right] \right) \sum_{i=1}^{n} x_i^{\lambda} = 0$$

—— (4)

(4) involves only λ.
It can be solved numerically for the MLE $\hat{\lambda}$ of λ. Sub $\hat{\lambda}$ into (3) to get

$$\hat{\sigma} = \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\hat{\lambda}} \right]^{1/\hat{\lambda}}.$$

Q4

$$L(\lambda) = \left[\prod_{i=1}^{n} (1 - \lambda x_i)\right]^{\frac{1}{\lambda} - 1}$$

$$\log L(\lambda) = \left(\frac{1}{\lambda} - 1\right) \sum_{i=1}^{n} \log (1 - \lambda x_i)$$

$$\frac{d \log L}{d\lambda} = -\frac{1}{\lambda^2} \sum_{i=1}^{n} \log (1 - \lambda x_i)$$

$$-\left(\frac{1}{\lambda} - 1\right) \sum_{i=1}^{n} \frac{x_i}{1 - \lambda x_i} = 0$$

$$\Rightarrow \quad -\sum_{i=1}^{n} \log (1 - \lambda x_i) = \lambda (1 - \lambda) \sum_{i=1}^{n} \frac{x_i}{1 - \lambda x_i}$$

This must be solved numerically for the MLE $\hat{\lambda}$ of λ.

LECTURE

26 NOVEMBER

9:00-10:00AM

MATH3/4/68181

Quiz 5

No need to derive explicit expressions for VaR or ES

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

↑ apply some transformation to get the answer.

Models for Stock Returns

1) Model based on varying volatility

(Model 1)

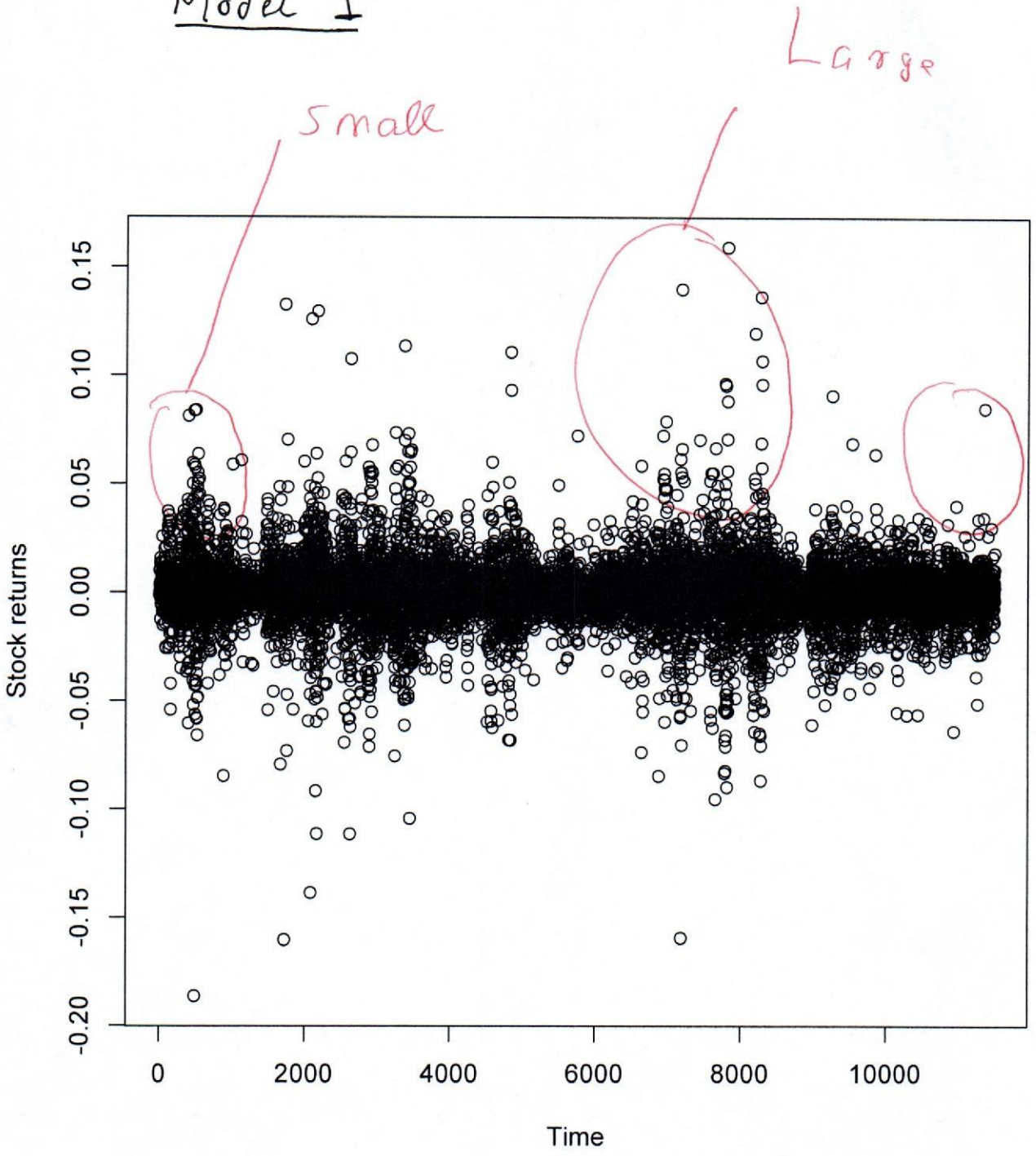
2) Model based on additive random walk

(Model 2)

3) Model based on multiplicative random walk

(Model 3)

Model I


Small
Large
Stock returns
0.15
0.10
0.05
0.00
-0.05
-0.10
-0.15
-0.20
0
2000
4000
6000
8000
10000
Time

Model I

X = stock returns

V = variability (volatility)

V is a random variable itself

V = unobservable

$X \mid V$ = observable

Hence, the PDF of X is

$$f_X(x) = \int_0^\infty \underbrace{f_{X|V}(x|v)}_{\text{conditional PDF of } X|V} \underbrace{f_V(v)}_{\text{PDF of } V} dv$$

The CDF of X is

$$F_X(x) = \int_0^\infty F_{X|V}(x|v) f_V(v) dv$$

The n^{th} moment of X is

$$E[X^n] = \int_0^\infty E[X^n|V] f_V(v)\, dv$$

$$= E[E[X^n|V]]$$

The mean of X is

$$E[X] = E[E[X|V]]$$

The variance of X is

$$Var[X] = E[X^2] - (E[X])^2$$

$$= E[E[X^2|V]] - (E[E[X|V]])^2$$

Ex 1

Suppose

$$X \mid V \sim N(0, \sigma^2)$$

$V = \sigma$ σ is a RV

σ has the PDF

$$\frac{2}{\sigma^3} e^{-\frac{1}{\sigma^2}}, \quad \sigma > 0.$$

What is the distribution of X?

$$f_X(x) = \int_0^\infty \underbrace{\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}}_{\text{Cond PDF of } X|V} \underbrace{\frac{2}{\sigma^3} \cdot e^{-\frac{1}{\sigma^2}}}_{\text{PDF of } V} d\sigma$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sigma^4} e^{-\left(\frac{x^2}{2}+1\right)\frac{1}{\sigma^2}} d\sigma$$

Set $y = \left(\frac{x^2}{2}+1\right)\frac{1}{\sigma^2}$

$$\Rightarrow \sigma^2 = \frac{\frac{x^2}{2}+1}{y} \Rightarrow \sigma = \frac{\sqrt{\frac{x^2}{2}+1}}{\sqrt{y}}$$

$$\Rightarrow \frac{d\sigma}{dy} = -\frac{\sqrt{\frac{x^2}{2}+1}}{2y^{3/2}}$$

$$= -\sqrt{\frac{2}{\pi}} \int_\infty^0 \frac{y^2}{\left(\frac{x^2}{2}+1\right)^2} \cdot e^{-y} \frac{\sqrt{\frac{x^2}{2}+1}}{2y^{3/2}} dy$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{x^2}{2}+1\right)^{-\frac{3}{2}} \boxed{\int_0^\infty y^{\frac{1}{2}} e^{-y} dy} = \Gamma\left(\frac{3}{2}\right)$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$f_X(x) = \frac{1}{2\sqrt{2}} \left(\frac{x^2}{2} + 1 \right)^{-\frac{3}{2}}.$$

$$E[X] = E[E[X|V]]$$

$$= E[0]$$

$$= 0$$

$$E[X^2] = E[E[X^2|V]]$$

$$= E[\sigma^2]$$

$$= \int_0^\infty \sigma^2 \cdot \frac{2}{\sigma^3} \cdot e^{-\frac{1}{\sigma^2}} d\sigma$$

$$= 2 \int_0^\infty \frac{1}{\sigma} e^{-\frac{1}{\sigma^2}} d\sigma$$

Set $y = \frac{1}{\sigma^2} \Rightarrow \sigma = \frac{1}{\sqrt{y}} \Rightarrow \frac{d\sigma}{dy} = -\frac{1}{2y^{3/2}}$

$$= -2 \int_\infty^0 \sqrt{y}\, e^{-y} \left(\frac{1}{2y^{3/2}} \right) dy$$

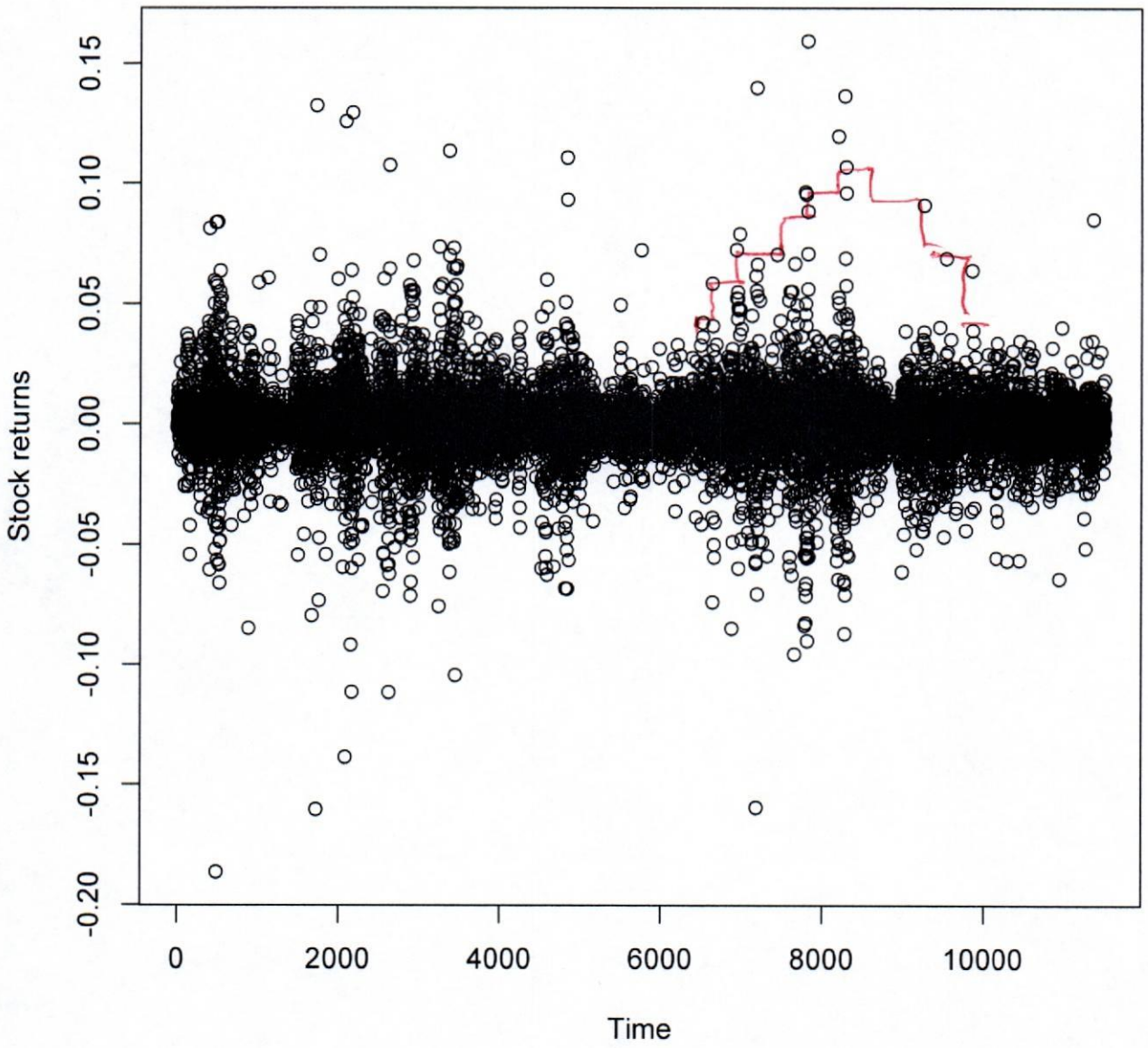
$$= \frac{2}{2} \int_0^\infty \frac{1}{y} e^{-y} dy$$

$$= \Gamma(0) = \infty.$$

$$\Rightarrow \quad Var[X] = \infty.$$

Model 2

Additive random walk



Model 2

Let X_t = stock return at time t

Let X_0 = stock " " " 0.

Assume X_0 is known.

$$X_t - X_0 = (X_t - X_{t-1}) = Y_t$$
$$+ (X_{t-1} - X_{t-2}) = Y_{t-1}$$
$$+ (X_{t-2} - X_{t-3}) = Y_{t-2}$$
$$+ \vdots$$
$$+ (X_2 - X_1) = Y_2$$
$$+ (X_1 - X_0) = Y_1$$

$$= \sum_{i=1}^{t} Y_i$$

Assume Y_i are independent RVs.

$$E[X_t - X_0] = \sum_{i=1}^{t} E[Y_i]$$

$$\Rightarrow E[X_t] = X_0 + \sum_{i=1}^{t} E[Y_i]$$

$$Var[X_t - X_0] = \sum_{i=1}^{t} Var[Y_i]$$

$$\Rightarrow Var[X_t] = \sum_{i=1}^{t} Var[Y_i]$$

If Y_i are IID then

$$E[X_t] = X_0 + t\,E[Y]$$

and

$$Var[X_t] = t\,Var[Y].$$

Ex 1 Suppose $Y_i \sim N(\mu_i, \sigma_i^2)$ are indep RVs.

$$E[X_t] = X_0 + \sum_{i=1}^{t} \mu_i$$

$$Var[X_t] = \sum_{i=1}^{t} \sigma_i^2$$

If Y_i are IID $N(\mu, \sigma^2)$

$$E[X_t] = X_0 + t\mu$$

$$Var[X_t] = t \cdot \sigma^2$$

$$X_t - X_0 = \sum_{i=1}^{t} Y_i \sim N\left(\sum_{i=1}^{t} \mu_i, \sum_{i=1}^{t} \sigma_i^2\right)$$

If Y_i are IID $N(\mu, \sigma^2)$

$$X_t - X_0 \sim N(t\mu, t\sigma^2)$$

Ex 2

Suppose Y_i are IID $\Gamma(a, b)$

Gamma distribution with parameters a & b

$$E[X_t] = X_0 + t\,E[Y]$$

$$= X_0 + t\frac{a}{b}$$

$$Var[X_t] = t \cdot Var[Y]$$

$$= t \cdot \frac{a}{b^2}$$

$$X_t - X_0 = \sum_{i=1}^{t} Y_i \sim \Gamma(ta, b)$$

[by Math 20802]

Model 3

X_t = ~~random~~ Stock return at time t

X_0 = " " " " 0

Assume X_0 is known.

$$\frac{X_t}{X_0} = \underbrace{\frac{X_t}{X_{t-1}}}_{Z_t} \cdot \underbrace{\frac{X_{t-1}}{X_{t-2}}}_{Z_{t-1}} \cdots \underbrace{\frac{X_2}{X_1}}_{Z_2} \cdot \underbrace{\frac{X_1}{X_0}}_{Z_1}$$

$$= \prod_{i=1}^{t} Z_i$$

Assume Z_i are independent RVs.

$$E\left[\frac{X_t}{X_0}\right] = \prod_{i=1}^{t} E[Z_i]$$

$$E\left[\left(\frac{X_t}{X_0}\right)^2\right] = \prod_{i=1}^{t} E[Z_i^2]$$

$$Var\left[\frac{X_t}{X_0}\right] = \prod_{i=1}^{t} E[Z_i^2] - \prod_{i=1}^{t} \left(E[Z_i]\right)^2$$

LECTURE

27 NOVEMBER

9:00-10:00AM

MATH3/4/68181

- Quiz 5

$$ES_p(x) = \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

Find a substitution that will help you to get the answer

No need to derive $F^{-1}(t)$.

- syllabus for level 3 will be completed on Tues 4 Dec
- syllabus for levels 4, 6 will be completed on Fri 7 Dec.
- Exam date will be out on 3 Dec.

Model 3

X_t = ~~rando~~ Stock return at time t

X_0 = " " " " 0

Assume X_0 is known.

$$\frac{X_t}{X_0} = \underbrace{\frac{X_t}{X_{t-1}}}_{Z_t} \cdot \underbrace{\frac{X_{t-1}}{X_{t-2}}}_{Z_{t-1}} \cdots \underbrace{\frac{X_2}{X_1}}_{Z_2} \cdot \underbrace{\frac{X_1}{X_0}}_{Z_1}$$

$$= \prod_{i=1}^{t} Z_i$$

Assume Z_i are independent RVs.

$$E\left[\frac{X_t}{X_0}\right] = \prod_{i=1}^{t} E[Z_i]$$

$$E\left[\left(\frac{X_t}{X_0}\right)^2\right] = \prod_{i=1}^{t} E[Z_i^2]$$

$$\mathrm{Var}\left[\frac{X_t}{X_0}\right] = \prod_{i=1}^{t} E[Z_i^2] - \prod_{i=1}^{t} \left(E[Z_i]\right)^2$$

Ex 1 Suppose $Z_i \sim LN(\mu_i, \sigma_i^2)$, $i = 1, 2, \ldots, t$ independently.

What is the distribution of

$$\frac{X_t}{X_0} = \prod_{i=1}^{t} Z_i \; ?$$

$$\Rightarrow \log \frac{X_t}{X_0} = \log \left[\prod_{i=1}^{t} Z_i \right]$$

$$= \sum_{i=1}^{t} \log Z_i$$

$$= \sum_{i=1}^{t} \log \left[LN(\mu_i, \sigma_i^2) \right]$$

$$= \sum_{i=1}^{t} N(\mu_i, \sigma_i^2)$$

If $Y \sim LN(a, b) \Rightarrow \log Y \sim N(a, b)$

$$= N\left(\sum_{i=1}^{t} \mu_i , \sum_{i=1}^{t} \sigma_i^2 \right)$$

$$\Rightarrow \frac{X_t}{X_0} \sim LN\left(\sum_{i=1}^{t} \mu_i , \sum_{i=1}^{t} \sigma_i^2 \right)$$

If $Y \sim LN(a, b^2)$ then

$$E[Y] = e^{a + \frac{b^2}{2}}$$

$$Var[X] = \left[e^{b^2} - 1 \right] e^{2a + b^2}$$

$$E\left[\frac{X_t}{X_0} \right] = \exp\left[\sum_{i=1}^{t} \mu_i + \frac{1}{2} \sum_{i=1}^{t} \sigma_i^2 \right]$$

$$Var\left[\frac{X_t}{X_0} \right] = \left[\exp\left(\sum_{i=1}^{t} \sigma_i^2 \right) - 1 \right]$$

$$\cdot \exp\left[2 \sum_{i=1}^{t} \mu_i + \sum_{i=1}^{t} \sigma_i^2 \right].$$

In the IID case,

$$\frac{X_t}{X_0} \sim LN(t\mu, t\sigma^2)$$

$$E\left[\frac{X_t}{X_0} \right] = e^{t\mu + \frac{t\sigma^2}{2}}$$

$$Var\left[\frac{X_t}{X_0} \right] = \left[e^{t\sigma^2} - 1 \right] \left[e^{2t\mu + t\sigma^2} \right]$$

Model 1 - More Examples

Ex 2

Suppose $X|\theta \sim \text{Uni}[-\theta, \theta]$

$\theta \sim \text{Exp}(a)$

What is the distribution of X?

$$f_X(x) = \int_0^\infty f_{X|\theta}(x|\theta)\, f(\theta)\, d\theta$$

$$= \int_0^\infty \frac{1}{2\theta} \cdot a e^{-a\theta}\, d\theta$$

$$= \frac{a}{2} \int_0^\infty \frac{1}{\theta} e^{-a\theta}\, d\theta$$

$$= \infty \quad [\text{does not exist}]$$

Ex 3

Suppose $X|\theta \sim \text{Uni}[-\theta, \theta]$

$\theta \sim \text{Uni}[a, b]$

$$f_X(x) = \int_a^b \frac{1}{2\theta} \cdot \frac{1}{b-a} \, d\theta$$

$$= \frac{1}{2(b-a)} \int_a^b \frac{1}{\theta} \, d\theta$$

$$= \frac{1}{2(b-a)} \left[\log \theta\right]_a^b$$

$$= \frac{1}{2(b-a)} \log \frac{b}{a}$$

Model 2 - More Examples

Ex 3

Suppose Y_i are IID Inverse Gaussian with PDF

$$\sqrt{\frac{\lambda}{2\pi y^3}}\, e^{-\frac{\lambda(y-\mu)^2}{2\mu^2 y}}, \quad y > 0$$

What is the distribution of

$$X_t - X_0 = \sum_{i=1}^{t} Y_i \quad ?$$

MGF Approach

$$M_{X_t - X_0}(s) = E\left[e^{s(X_t - X_0)}\right]$$

$$= E\left[e^{s\sum_{i=1}^{t} Y_i}\right]$$

$$= E\left[e^{sY_1}\right] E\left[e^{sY_2}\right] \cdots E\left[e^{sY_t}\right]$$

using independence

$$= \left(E\left[e^{sY}\right]\right)^t$$

using identical assumption

$$= \left[M_Y(s) \right]^t$$

$$= \left[e^{\frac{\lambda}{\mu}\left[1 - \sqrt{1 - \frac{2\mu^2 s}{\lambda}}\right]} \right]^t$$

MGF of an Inverse Gaussian RV

$$= e^{\frac{\lambda t}{\mu}\left[1 - \sqrt{1 - \frac{2\mu^2 s}{\lambda}}\right]}$$

$$= e^{\frac{\lambda t^2}{\mu t}\left[1 - \sqrt{1 - \frac{2(\mu t)^2 s}{\lambda t^2}}\right]}$$

$$= e^{\frac{\lambda^*}{\mu^*}\left[1 - \sqrt{1 - \frac{2(\mu^*)^2 s}{\lambda^*}}\right]}$$

where $\lambda^* = \lambda t^2$

$\mu^* = \mu t$

Hence, $X_t - X_0$ has the Inverse Gaussian distribution with parameters λ^* and μ^*

LECTURE

30 NOVEMBER

10:00-11:00AM

MATH4/68181

Bivariate Extreme Values

Often you are interested in extreme values of more than 1 variable.

Some examples are as follows.

Suppose $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ is a random sample from a joint CDF $F(x, y)$.

How to define an extreme value?
There is no one way to define an extreme value. The most definition is

$$(M_{n,1}, M_{n,2})$$

$$= (\max(X_1, X_2, \cdots, X_n), \max(Y_1, Y_2, \cdots, Y_n))$$

This definition may not be realistic because $(M_{n,1}, M_{n,2})$ may not be an actual observation.

Forest fires

Caused by extreme values of temperature and wind speed

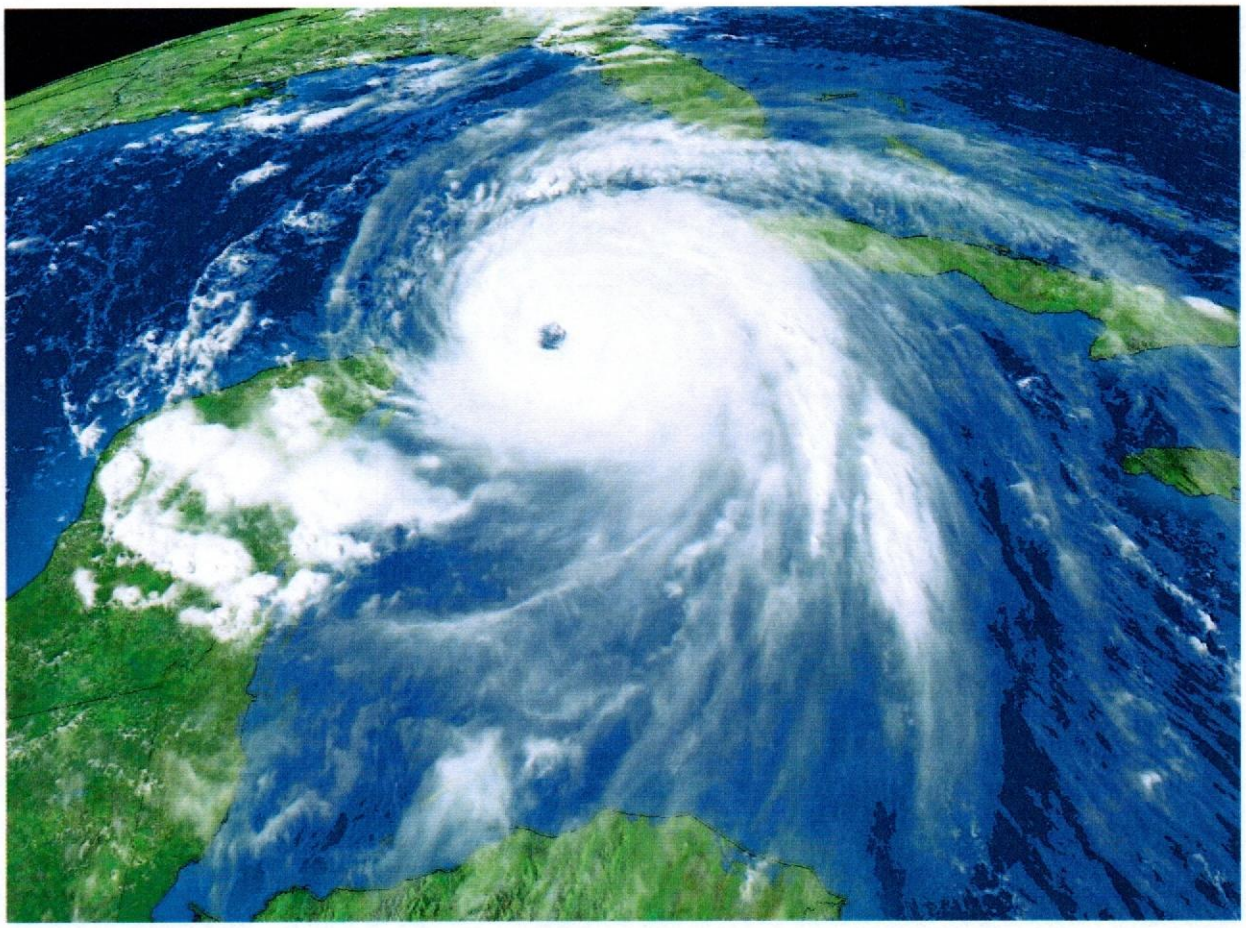
Tornado

Caused by extreme values of humidity and wind speed

Droughts

Caused by extreme values of rainfall and temperature

Hurricanes

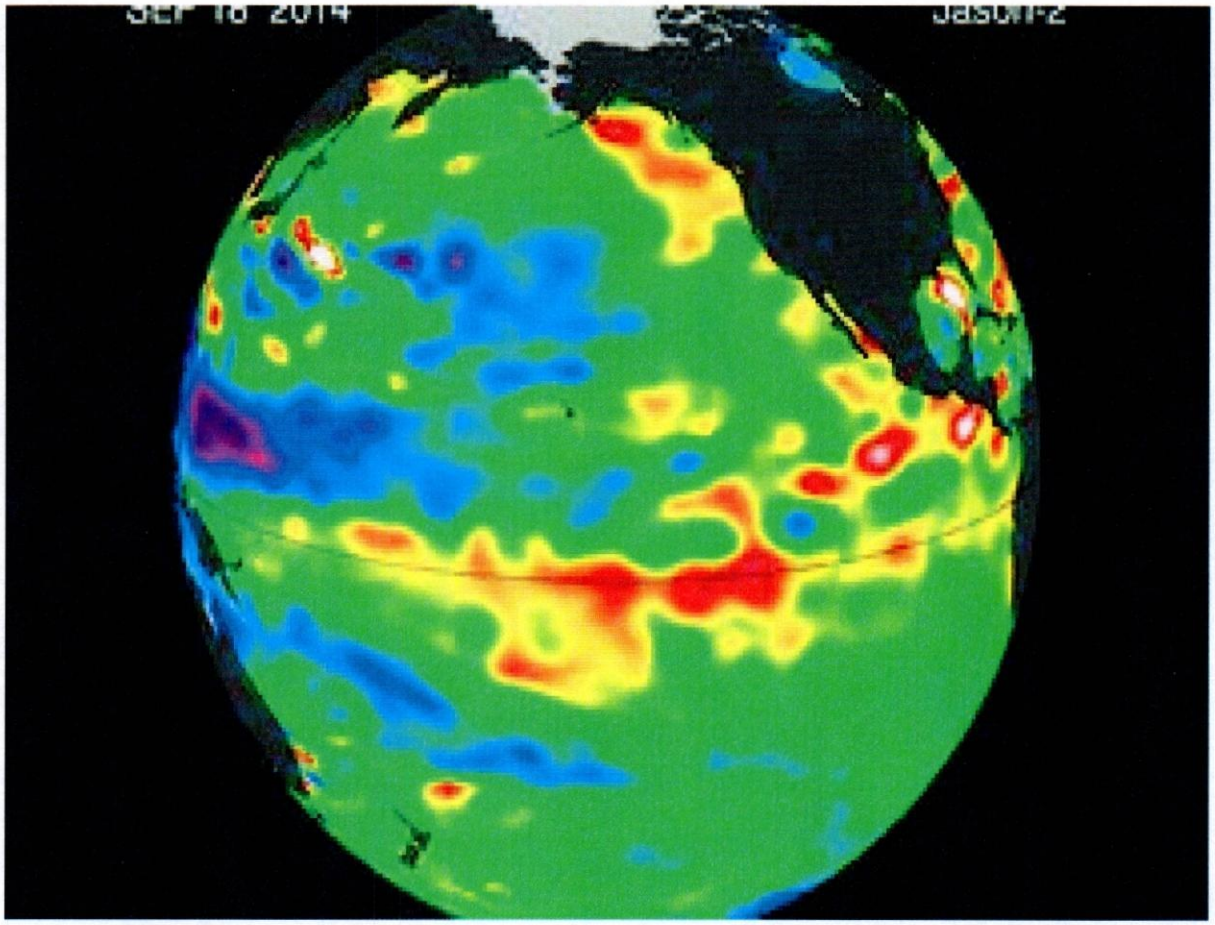


Caused by extreme values of sea temperature, rainfall and wind speed

Floods

Caused by extreme values of rainfall and wind speed

El-Nino



Caused by extreme values of sea temperature and air pressure

Ex

Data: $(1,3), (2,0), (-1,5), (6,1), (3,1)$

$(M_{n,1}, M_{n,2}) = (3,5)$

But $(3,5)$ is <u>not</u> an observation.

To model the data, we need to derive the distribution of

$$\left(\frac{M_{n,1} - b_n}{a_n}, \frac{M_{n,2} - d_n}{c_n} \right).$$

The joint CDF of this is

$$P\left(\frac{M_{n,1} - b_n}{a_n} \leq x, \frac{M_{n,2} - d_n}{c_n} \leq y \right)$$

$$= P\left(M_{n,1} \leq a_n x + b_n, M_{n,2} \leq c_n y + d_n \right)$$

$$= P\left(\max(X_1, \ldots, X_n) \leq a_n x + b_n, \right.$$
$$\left. \max(Y_1, \ldots, Y_n) \leq c_n y + d_n \right)$$

$$= P\left(X_1 \leq a_n x + b_n, \ldots, X_n \leq a_n x + b_n, \right.$$
$$\left. Y_1 \leq c_n y + d_n, \ldots, Y_n \leq c_n y + d_n \right)$$

$$= P\left(X_1 \leq a_n x + b_n, Y_1 \leq c_n y + d_n, \right.$$
$$\left. \ldots, X_n \leq a_n x + b_n, Y_n \leq c_n y + d_n \right)$$

$$\overset{\text{indep}}{=} P(X_1 \leq a_n x + b_n, Y_1 \leq c_n y + d_n)$$
$$\cdots P(X_n \leq a_n x + b_n, Y_n \leq c_n y + d_n)$$

$$= F(a_n x + b_n, c_n y + d_n)$$

$$\cdots F(a_n x + b_n, c_n y + d_n)$$

$$= \left[F(a_n x + b_n, c_n y + d_n) \right]^n .$$

We have shown that

$$P\left(\frac{M_{n,1} - b_n}{a_n} \leq x, \frac{M_{n,2} - d_n}{c_n} \leq y \right)$$

$$= \left[F(a_n x + b_n, c_n y + d_n) \right]^n .$$

We are interested in what happens as $n \longrightarrow \infty$. If there exist $a_n > 0$, $b_n \in \mathbb{R}$, $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$\left[F(a_n x + b_n, c_n y + d_n) \right]^n$$

$$\longrightarrow \quad G(x, y)$$

as $n \longrightarrow \infty$ for a non-degenerate $G(x, y)$ then possible forms for $G(x, y)$ are uncountably infinite.

For a given $F(x, y)$ [the CDF of the data] can you find $G(x, y)$ if it exists?

Yes, you can.

(i) Find $F_X(x) = F(x, \infty)$

and $F_Y(y) = F(\infty, y)$

(ii) Find the max domain of attraction of F_X and F_Y.

(iii) If F_X and F_Y belong to the Gumbel domain set

$$a_n = \gamma\left(F_X^{-1}\left(1 - \tfrac{1}{n}\right)\right), \quad b_n = F_X^{-1}\left(1 - \tfrac{1}{n}\right)$$

$$c_n = \gamma\left(F_X^{-1}\left(1 - \tfrac{1}{n}\right)\right), \quad d_n = F_Y^{-1}\left(1 - \tfrac{1}{n}\right)$$

If F_X and F_Y belong to the Frechet domain set

$$a_n = F_X^{-1}\left(1 - \tfrac{1}{n}\right), \quad b_n = 0$$

$$c_n = F_Y^{-1}\left(1 - \tfrac{1}{n}\right), \quad d_n = 0$$

If F_X and F_Y belong to the Weibull domain set

$$a_n = \omega(F_X) - F_X^{-1}\left(1 - \tfrac{1}{n}\right), \quad b_n = \omega(F_X)$$

$$c_n = \omega(F_Y) - F_Y^{-1}\left(1 - \tfrac{1}{n}\right), \quad d_n = \omega(F_Y).$$

(iv) Determine

$$G(x, y) = \lim_{n \to \infty} \left[F(a_n x + b_n, c_n y + d_n)\right]^n$$

Ex

$$F(x,y) = \left[1 + e^{-x} + e^{-y} + (1-a)e^{-x-y}\right]^{-1},$$

$$x > 0$$
$$y > 0$$

Find the corresponding $G(x,y)$.

(i) $F_X(x) = F(x,\infty) = \left[1+e^{-x}\right]^{-1}$

$F_Y(y) = F(\infty,y) = \left[1+e^{-y}\right]^{-1}$

(ii) $w(F_X) = \infty$

$w(F_Y) = \infty$

$$\lim_{t\to\infty} \frac{1 - F_X(t + x\gamma(t))}{1 - F_X(t)}$$

$$= \lim_{t\to\infty} \frac{1 - \left[1 - e^{-t-x\gamma(t)}\right]^{-1}}{1 - \left[1 - e^{-t}\right]^{-1}}$$

($e^{-t-x\gamma(t)} \to 0$, $e^{-t} \to 0$)

$$= \lim_{t\to\infty} \frac{1 - \left[1 + e^{-t-x\gamma(t)}\right]}{1 - \left[1 + e^{-t}\right]}$$

$$= \lim_{t\to\infty} \frac{e^{-t-x\gamma(t)}}{e^{-t}} = \lim_{t\to\infty} e^{-x\gamma(t)} = e^{-x}$$

if $\gamma(t) \equiv 1$.

Hence, F_X and F_Y belong to the Gumbel domain.

EXAMPLE CLASS

26 NOVEMBER

12:00-13:00PM

MATH3/4/68181

Q1

X = stock returns

$X|\lambda \sim \text{Exp}(\lambda)$

$\lambda \sim \text{Exp}(a)$

$$f_X(x) = \int_0^\infty f_{X|\lambda}(x|\lambda)\, f(\lambda)\, d\lambda$$

$$= \int_0^\infty \lambda e^{-\lambda x}\, a e^{-a\lambda}\, d\lambda$$

$$= a \int_0^\infty \lambda e^{-(a+x)\lambda}\, d\lambda$$

Set $y = (a+x)\lambda$

$\Rightarrow \lambda = \frac{y}{a+x}$

$d\lambda = \frac{dy}{a+x}$

$$= a \int_0^\infty \frac{y}{a+x} e^{-y} \frac{dy}{a+x}$$

$$= \frac{a}{(a+x)^2} \boxed{\int_0^\infty y e^{-y}\, dy} = \Gamma(2) = 1$$

$$= \frac{a}{(a+x)^2}.$$

Q2

X = stock returns

$X \mid \lambda \sim Exp(\lambda)$

$\lambda \sim Uni[a, b]$

$$f_X(x) = \int_a^b \lambda e^{-\lambda x} \frac{1}{b-a} \, d\lambda$$

$$= \frac{1}{b-a} \int_a^b \lambda e^{-\lambda x} \, d\lambda$$

Set $y = \lambda x \Rightarrow \lambda = \frac{y}{x}$

$\Rightarrow d\lambda = \frac{dy}{x}$

$$= \frac{1}{b-a} \int_{ax}^{bx} \frac{y}{x} e^{-y} \frac{dy}{x}$$

$$= \frac{1}{(b-a)x^2} \int_{ax}^{bx} y e^{-y} \, dy$$

$$= \frac{1}{(b-a)x^2} \left\{ \left[y(-e^{-y}) \right]_{ax}^{bx} + \int_{ax}^{bx} e^{-y} \, dy \right\}$$

$$= \frac{1}{(b-a)x^2} \left\{ -bx e^{-bx} + ax e^{-ax} + \left[-e^{-y} \right]_{ax}^{bx} \right\}$$

$$= \frac{1}{(b-a)x^2} \left\{ -bx e^{-bx} + ax e^{-ax} - e^{-bx} + e^{-ax} \right\}.$$

Q3

$X \mid \lambda \sim Exp(\lambda)$

λ has PDF $a\lambda^{a-1}$, $0 < \lambda < 1$

$$f_X(x) = \int_0^1 \lambda e^{-\lambda x} \cdot a\lambda^{a-1} \, d\lambda$$

$$= a \int_0^1 \lambda^a e^{-\lambda x} \, d\lambda$$

Set $y = \lambda x \Rightarrow \lambda = \frac{y}{x}$

$\Rightarrow d\lambda = \frac{dy}{x}$

$$= a \int_0^x \left(\frac{y}{x}\right)^a e^{-y} \frac{dy}{x}$$

$$= \frac{a}{x^{a+1}} \int_0^x y^a e^{-y} \, dy$$

$$= \frac{a}{x^{a+1}} \gamma(a+1, x)$$

Upper Incomplete gamma function

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} \, dt$$

Q4

$X|\lambda \sim Exp(\lambda)$

λ has PDF $\frac{a k^a}{\lambda^{a+1}}, \quad \lambda > k$

$$f_X(x) = \int_k^\infty \lambda e^{-\lambda x} \frac{a k^a}{\lambda^{a+1}} d\lambda$$

$$= a k^a \int_k^\infty \frac{1}{\lambda^a} e^{-\lambda x} d\lambda$$

Set $y = \lambda x \Rightarrow \lambda = \frac{y}{x} \Rightarrow d\lambda = \frac{dy}{x}$

$$= a k^a \int_{kx}^\infty \frac{x^a}{y^a} e^{-y} \frac{dy}{x}$$

$$= a k^a x^{a-1} \int_{kx}^\infty y^{-a} e^{-y} dy$$

$$= a k^a x^{a-1} \Gamma(1-a, kx)$$

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$$

Lower incomplete gamma function

EXAMPLE CLASS

27 NOVEMBER

16:00-17:00PM

MATH3/4/68181

Q1

X = stock returns

$X \mid \lambda \sim \text{Exp}(\lambda)$

$\lambda \sim \text{Exp}(a)$

$$f_X(x) = \int_0^\infty f_{X|\lambda}(x|\lambda)\, f(\lambda)\, d\lambda$$

$$= \int_0^\infty \lambda e^{-\lambda x}\, a e^{-\lambda a}\, d\lambda$$

$$= a \int_0^\infty \lambda e^{-\lambda(x+a)}\, d\lambda$$

Set $y = \lambda(x+a)$

$\Rightarrow \lambda = \frac{y}{x+a}$

$\Rightarrow d\lambda = \frac{dy}{x+a}$

$$= a \int_0^\infty \frac{y}{x+a} e^{-y} \frac{dy}{x+a}$$

$$= \frac{a}{(x+a)^2} \int_0^\infty y e^{-y}\, dy$$

$$= \frac{a}{(x+a)^2} \Gamma(2) = \frac{a}{(x+a)^2}.$$

Q2

X = stock returns

$X|\lambda \sim Exp(\lambda)$

$\lambda \sim Uni[a,b]$

$$f_X(x) = \int_a^b \lambda e^{-\lambda x} \frac{1}{b-a} \, d\lambda$$

$$= \frac{1}{b-a} \int_a^b \lambda e^{-\lambda x} \, d\lambda$$

$$= \frac{1}{b-a} \left\{ \left[\lambda \frac{e^{-\lambda x}}{(-x)} \right]_a^b + \frac{1}{x} \int_a^b e^{-\lambda x} \, d\lambda \right\}$$

$$= \frac{1}{b-a} \left\{ \frac{b e^{-bx} - a e^{-ax}}{(-x)} + \frac{1}{x} \left[\frac{e^{-\lambda x}}{(-x)} \right]_a^b \right\}$$

$$= \frac{1}{b-a} \left\{ \frac{b e^{-bx} - a e^{-ax}}{(-x)} - \frac{e^{-bx} - e^{-ax}}{x^2} \right\}$$

Q3

$X|\lambda \sim Exp(\lambda)$

λ has PDF $a\lambda^{a-1}$, $0 < \lambda < 1$

$$f_X(x) = \int_0^1 \lambda e^{-\lambda x} \, a\lambda^{a-1} \, d\lambda$$

$$= a \int_0^1 \lambda^a e^{-\lambda x} \, d\lambda$$

Set $y = \lambda x \Rightarrow \lambda = \frac{y}{x} \Rightarrow d\lambda = \frac{dy}{x}$

$$= a \int_0^x \left(\frac{y}{x}\right)^a e^{-y} \frac{dy}{x}$$

$$= \frac{a}{x^{a+1}} \int_0^x y^a e^{-y} \, dy$$

$$= \frac{a}{x^{a+1}} \gamma(a+1, x)$$

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} \, dt$$

(Upper) Incomplete gamma function

Q4

$$X|\lambda \sim Exp(\lambda)$$

λ has PDF $\frac{ak^a}{\lambda^{a+1}}$, $\lambda > k$

$$f_X(x) = \int_k^\infty \lambda e^{-\lambda x} \frac{ak^a}{\lambda^{a+1}} d\lambda$$

$$= ak^a \int_k^\infty \frac{1}{\lambda^a} e^{-\lambda x} d\lambda$$

Set $y = \lambda x \Rightarrow \lambda = \frac{y}{x} \Rightarrow d\lambda = \frac{dy}{x}$

$$= ak^a \int_{kx}^\infty \frac{x^a}{y^a} e^{-y} \frac{dy}{x}$$

$$= ak^a x^{a-1} \int_{kx}^\infty y^{-a} e^{-y} dy$$

$$= ak^a x^{a-1} \Gamma(1-a, kx)$$

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$$

Lower incomplete gamma function

LECTURE

3 DECEMBER

9:00-10:00AM

MATH3/4/68181

Announcements

1. syllabus for level 3 will be completed tomorrow 4 Dec
2. syllabus for levels 4 & 6 will be completed on Friday 7 Dec
3. Next week will be revision class
4. Quiz 5 due tomorrow

$$ES_p(X) = \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

Find a substitution that will help you to get the answer

The proof should not be longer than 1 A4 page.

5. UEQ

Income modeling

X = True income

Z = Reported income

eg 1

X = £ 20,000 + 1 p

Z = £ 20,000

Under reporting

$Z = XY$ where $0 < Y < 1$

eg 2

X = £ 19,000 + 10 p

Z = £ 20,000

Over reporting

$Z = X/Y$ where $0 < Y < 1$.

Z is observable

X is not observable.

The most popular model for income is the Pareto distribution. Its CDF is (Italian economist) given by

$$F_X(x) = 1 - \left(\frac{k}{x}\right)^a, \quad x > k$$

The corresponding PDF is

$$f_X(x) = \frac{a k^a}{x^{a+1}}, \quad x > k.$$

1) If X is Pareto distributed what is the distribution of Z?

2) If Z is Pareto distributed what is the distribution of X?

Assume Y is a power function RV with CDF

$$F_Y(y) = y^c, \quad 0 < y < 1$$

The corresponding PDF is

$$f_Y(y) = c\,y^{c-1}, \quad 0 < y < 1.$$

Theorem 1 (under reported income) Suppose $Z = XY$. Then X is Pareto distributed if and only if Z is also Pareto distributed.

Proof: Assume X is Pareto distributed with CDF

$$F_X(x) = 1 - \left(\frac{k}{x}\right)^a, \quad x > k.$$

The CDF of Z is

$$F_Z(z) = P(XY \le z)$$

$$= P\left(X \le \frac{z}{Y}\right)$$

Total Prob Rule ↘

$$= \int_0^1 F_X\left(\frac{z}{y}\right) f_Y(y)\,dy$$

$$= \int_0^1 \left[1 - \left(\frac{ky}{z}\right)^a\right] c\,y^{c-1}\,dy$$

$$= \int_0^1 c y^{c-1} dy - \frac{ck^a}{z^a}\int_0^1 y^{a+c-1} dy$$

$$= 1 - \frac{c\,k^a}{(a+c)\,z^a} = 1 - \frac{\left[\frac{c^{1/a}}{(a+c)^{1/a}}k\right]^a}{z^a} = 1 - \left(\frac{k^*}{z}\right)^a$$

where $K^* = \frac{c^{1/a}}{(a+c)^{1/a}} K$.

Hence, Z is also Pareto distributed.

Assume Z is Pareto distributed with CDF

$$F_Z(z) = 1 - \left(\frac{K}{z}\right)^a, \quad z > K.$$

The CDF of X is

$$F_X(x) = P(X \leq x)$$

$$= P\left(\frac{Z}{Y} \leq x\right)$$

$$= P(Z \leq Yx)$$

Total Prob Rule ↘

$$= \int_0^1 F_Z(yx) f_Y(y)\, dy$$

$$= \int_0^1 \left[1 - \left(\frac{K}{yx}\right)^a\right] c y^{c-1}\, dy$$

$$= \int_0^1 c y^{c-1} dy - \frac{c K^a}{x^a} \int_0^1 y^{c-a-1}\, dy$$

$$= 1 - \frac{c K^a}{(c-a) x^a}$$

$$= 1 - \left(\frac{K^{**}}{x}\right)^a \quad \text{where } K^* = \frac{c^{\frac{1}{a}} K}{(c-a)^{\frac{1}{a}}}.$$

Hence, X is also Pareto distributed.
Hence, Z is Pareto distributed if and only if X is also Pareto distributed.

Theorem 2 (Over reported income).
Suppose $Z = X/Y$. Then Z is Pareto distributed if and only if X is also Pareto distributed.

Proof: Assume Z is Pareto distributed with CDF

$$F_Z(z) = 1 - \left(\frac{K}{z}\right)^a, \quad z > K.$$

The CDF of X is

$$\begin{aligned} F_X(x) &= P(X \le x) \\ &= P(ZY \le x) \\ &= P\left(Z \le \frac{x}{Y}\right) \end{aligned}$$

Total Prob Rule ↘

$$= \int_0^1 F_Z\left(\frac{x}{y}\right) f_Y(y)\, dy$$

$$= \int_0^1 \left[1 - \left(\frac{Ky}{x}\right)^a\right] c y^{c-1}\, dy$$

$$= \int_0^1 c y^{c-1}\, dy - \frac{cK^a}{x^a} \int_0^1 y^{a+c-1}\, dy$$

$$= 1 - \frac{c K^a}{(a+c) x^a}$$

$$= 1 - \left(\frac{K^{***}}{x}\right)^a, \quad K^{***} = \frac{c^{\frac{1}{a}} K}{(a+c)^{\frac{1}{a}}}$$

Hence, X is also Pareto distributed.

Assume X is Pareto distributed with CDF

$$F_X(x) = 1 - \left(\frac{K}{x}\right)^a, \quad x > K.$$

The CDF of Z is

$$\begin{aligned} F_Z(z) &= P(Z \le z) \\ &= P\left(\frac{X}{Y} \le z\right) \\ &= P(X \le YZ) \\ &= \int_0^1 F_X(yz) f_Y(y)\, dy \\ &= \int_0^1 \left[1 - \left(\frac{K}{yz}\right)^a\right] c y^{c-1}\, dy \\ &= 1 - \left(\frac{K^{****}}{z}\right)^a \end{aligned}$$

where $K^{****} = \dfrac{c^{\frac{1}{a}} K}{(c-a)^{\frac{1}{a}}}$.

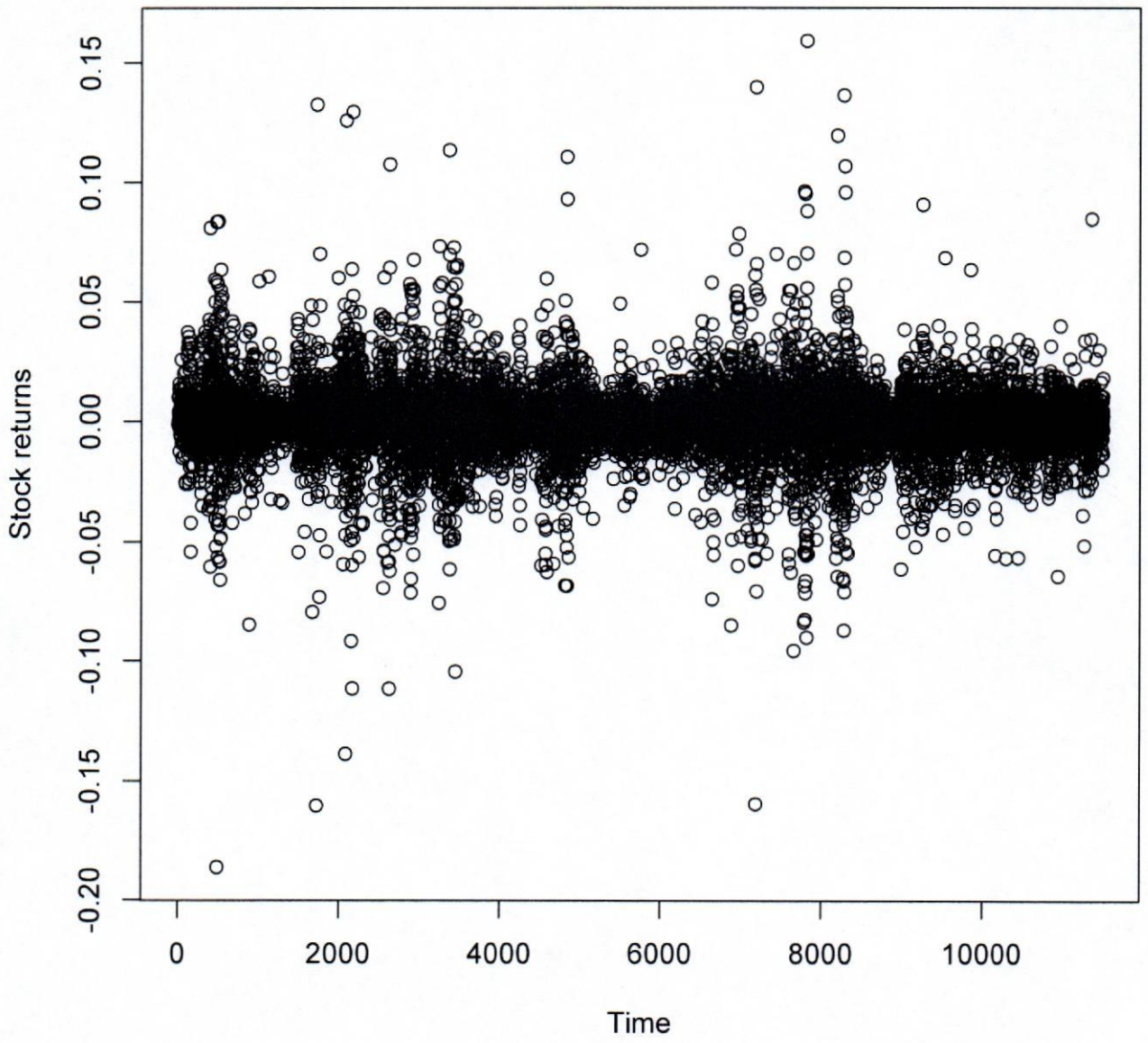
Hence, Z is also Pareto distributed.

LECTURE

4 DECEMBER

9:00-10:00AM

MATH3/4/68181


Stock returns
-0.20
-0.15
-0.10
-0.05
0.00
0.05
0.10
0.15
0
2000
4000
6000
8000
10000
Time

VEQ

< 50% complete

Please ask your frieids to complete asap.

Many thanks.

GARCH type models

$$X_t = \sigma_t Z_t$$

where

X_t = financial data at time t

σ_t = volatility at time t

Z_t = innovation at time t

Usually, $Z_t \sim N(0, 1)$ IID.

$$E[X_t] = E[\sigma_t Z_t]$$

Total Law of Expectation $\rightarrow$

$$= E[E[\sigma_t Z_t \mid \sigma_t]]$$

$$= E[\sigma_t E[Z_t]]$$

$$= E[\sigma_t \cdot 0]$$

$$= 0$$

$$E[X_t^2] = E[\sigma_t^2 Z_t^2]$$

Total Law of Expectation $\rightarrow = E\left[E[\sigma_t^2 Z_t^2 \mid \sigma_t]\right]$

$$= E[\sigma_t^2 E[Z_t^2]]$$

$$= E\left[\sigma_t^2 \left\{Var(Z_t) + [E(Z_t)]^2\right\}\right]$$

$$= E[\sigma_t^2].$$

$$\Rightarrow \quad Var[X_t] = E[X_t^2] - \left(E[X_t]\right)^2$$

$$= E[\sigma_t^2] - 0$$

$$= E[\sigma_t^2].$$

In summary,

$$E[X_t] = 0,$$

$$Var[X_t] = E[\sigma_t^2].$$

a) ARCH(q) model

$$X_t = \sigma_t Z_t,$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2$$

at time t

[volatility] is a function of the financial data at the previous q lags]

$$E[X_t] = 0$$

$$Var[X_t] = E[\sigma_t^2]$$

If the data are stationary

$$E[\sigma_t^2] = E[\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2] \quad \forall t$$

$$\Rightarrow E[\sigma_t^2] = \alpha_0 + \alpha_1 E[X_{t-1}^2] + \cdots + \alpha_q E[X_{t-q}^2]$$

$$\Rightarrow E[\sigma_t^2] = \alpha_0 + \alpha_1 E[\sigma_{t-1}^2] + \cdots + \alpha_q E[\sigma_{t-q}^2]$$

$$\Rightarrow E[\sigma_t^2] = \alpha_0 + \alpha_1 E[\sigma_t^2] + \cdots + \alpha_q E[\sigma_t^2]$$

using stationarity

$$\Rightarrow E[\sigma_t^2] = \frac{\alpha_0}{1 - \alpha_1 - \cdots - \alpha_q}.$$

b) GARCH(p, q) model

$$X_t = \sigma_t Z_t$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2$$

[volatility at time t is a function of the data values at the previous q lags as well as the volatility values at the previous p values]

$$E[X_t] = 0$$

$$Var[X_t] = E[\sigma_t^2].$$

If the data are stationary

$$E[\sigma_t^2] = \alpha_0 + \alpha_1 E[X_{t-1}^2] + \cdots + \alpha_q E[X_{t-q}^2] + \beta_1 E[\sigma_{t-1}^2] + \cdots + \beta_p E[\sigma_{t-p}^2]$$

$$= \alpha_0 + \alpha_1 E[\sigma_{t-1}^2] + \cdots + \alpha_q E[\sigma_{t-q}^2] + \beta_1 E[\sigma_{t-1}^2] + \cdots + \beta_p E[\sigma_{t-p}^2]$$

Using stationarity

$$= \alpha_0 + (\alpha_1 + \cdots + \alpha_q + \beta_1 + \cdots + \beta_p) E[\sigma_t^2]$$

$$\Rightarrow E[\sigma_t^2] = \frac{\alpha_0}{1 - (\alpha_1 + \cdots + \alpha_q + \beta_1 + \cdots + \beta_p)}$$

c) NGARCH model

$$X_t = \sigma_t Z_t$$

$$\sigma_t^2 = \omega + \alpha (X_{t-1} - \theta \sigma_{t-1})^2 + \beta \sigma_{t-1}^2$$

[volatility at time t is a function of the previous data and volatility values]

$$E[X_t] = 0$$

$$Var[X_t] = \sigma_t^2$$

If the data are stationary

$$E[\sigma_t^2] = \omega + \alpha E[(X_{t-1} - \theta \sigma_{t-1})^2] + \beta E[\sigma_{t-1}^2]$$

$$= \omega + \alpha E[X_{t-1}^2] - 2\alpha\theta E[X_{t-1}\sigma_{t-1}] + \alpha\theta^2 E[\sigma_{t-1}^2] + \beta E[\sigma_{t-1}^2]$$

$$= \omega + \alpha E[X_{t-1}^2] - 2\alpha\theta E[E[X_{t-1}\sigma_{t-1}|\sigma_{t-1}]] + (\alpha\theta^2 + \beta) E[\sigma_{t-1}^2]$$

$$= \omega + \alpha E[X_{t-1}^2] - 2\alpha\theta E[\sigma_{t-1} E[X_{t-1}]] + (\alpha\theta^2 + \beta) E[\sigma_{t-1}^2]$$

$$= \omega + \alpha E[\sigma_{t-1}^2] - 0 + (\alpha\theta^2 + \beta) E[\sigma_{t-1}^2]$$

using stationarity

$$= \omega + (\alpha + \alpha\theta^2 + \beta) E[\sigma_t^2]$$

$$\Rightarrow \quad E[\sigma_t^2] = \frac{\omega}{1 - (\alpha + \alpha\theta^2 + \beta)} .$$

d) QGARCH model

$$X_t = \sigma_t Z_t$$

$$\sigma_t^2 = k + \alpha X_{t-1}^2 + \beta \sigma_{t-1}^2 + \phi X_{t-1}$$

[This model takes account of the sign of X_{t-1}]

$$E[X_t] = 0$$

$$\text{Var}[X_t] = E[\sigma_t^2]$$

If the data are stationary,

$$E[\sigma_t^2] = k + \alpha E[X_{t-1}^2] + \beta E[\sigma_{t-1}^2] + \phi E[X_{t-1}]$$

$$\Rightarrow E[\sigma_t^2] = k + \alpha E[\sigma_{t-1}^2] + \beta E[\sigma_{t-1}^2] + 0$$

using stationarity $\Rightarrow$

$$E[\sigma_t^2] = k + (\alpha + \beta) E[\sigma_t^2]$$

$$\Rightarrow E[\sigma_t^2] = \frac{k}{1-(\alpha+\beta)}.$$

e) GJR-QGARCH model

(initials of the 3 authors)

$$X_t = \sigma_t Z_t$$

$$\sigma_t^2 = K + \delta \sigma_{t-1}^2 + \alpha X_{t-1}^2 + \phi X_{t-1}^2 I_{t-1}$$

where

$$I_{t-1} = \begin{cases} 0 & \text{if } X_{t-1} \geq 0 \\ 1 & \text{if } X_{t-1} < 0 \end{cases}$$

[volatility has an additional term if $X_{t-1} < 0$]

LECTURE

7 DECEMBER

10:00-11:00AM

MATH4/68181

For a given $F(x, y)$ [the CDF of the data] can you find $G(x, y)$ if it exists?

Yes, you can.

(i) Find $F_X(x) = F(x, \infty)$

and $F_Y(y) = F(\infty, y)$

(ii) Find the max domain of attraction of F_X and F_Y.

(iii) If F_X and F_Y belong to the Gumbel domain set

$$a_n = \gamma\left(F_X^{-1}\left(1 - \tfrac{1}{n}\right)\right), \quad b_n = F_X^{-1}\left(1 - \tfrac{1}{n}\right)$$

$$c_n = \gamma\left(F_X^{-1}\left(1 - \tfrac{1}{n}\right)\right), \quad d_n = F_Y^{-1}\left(1 - \tfrac{1}{n}\right)$$

If F_X and F_Y belong to the Frechet domain set

$$a_n = F_X^{-1}\left(1 - \tfrac{1}{n}\right), \quad b_n = 0$$

$$c_n = F_Y^{-1}\left(1 - \tfrac{1}{n}\right), \quad d_n = 0$$

If F_X and F_Y belong to the Weibull domain set

$$a_n = \omega(F_X) - F_X^{-1}\left(1 - \tfrac{1}{n}\right), \quad b_n = \omega(F_X)$$

$$c_n = \omega(F_Y) - F_Y^{-1}\left(1 - \tfrac{1}{n}\right), \quad d_n = \omega(F_Y).$$

(iv) Determine

$$G(x, y) = \lim_{n \to \infty} \left[F(a_n x + b_n, c_n y + d_n)\right]^n$$

Ex

$$F(x,y) = \left[1 + e^{-x} + e^{-y} + (1-a)e^{-x-y}\right]^{-1},$$

$$x > 0$$
$$y > 0$$

Find the corresponding $G(x,y)$.

(i) $F_X(x) = F(x,\infty) = \left[1+e^{-x}\right]^{-1}$

$F_Y(y) = F(\infty, y) = \left[1+e^{-y}\right]^{-1}$

(ii) $w(F_X) = \infty$

$w(F_Y) = \infty$

$$\lim_{t\to\infty} \frac{1 - F_X(t + x\gamma(t))}{1 - F_X(t)}$$

$$= \lim_{t\to\infty} \frac{1 - \left[1 - e^{-t-x\gamma(t)}\right]^{-1}}{1 - \left[1 - e^{-t}\right]^{-1}}$$

($e^{-t-x\gamma(t)} \to 0$, $e^{-t} \to 0$)

$$= \lim_{t\to\infty} \frac{1 - \left[1 + e^{-t-x\gamma(t)}\right]}{1 - \left[1 + e^{-t}\right]}$$

$$= \lim_{t\to\infty} \frac{e^{-t-x\gamma(t)}}{e^{-t}} = \lim_{t\to\infty} e^{-x\gamma(t)} = e^{-x}$$

if $\gamma(t) \equiv 1$.

Hence, F_X and F_Y belong to the Gumbel domain.

(iii) $F_X(x) = \left[1 + e^{-x}\right]^{-1} = z$

$$\Rightarrow \quad 1 + e^{-x} = \frac{1}{z}$$

$$\Rightarrow \quad e^{-x} = \frac{1-z}{z}$$

$$\Rightarrow \quad x = -\log\left(\frac{1-z}{z}\right)$$

$$\Rightarrow \quad F_X^{-1}(x) = -\log\left(\frac{1-x}{x}\right)$$

$$\Rightarrow \quad F_Y^{-1}(y) = -\log\left(\frac{1-y}{y}\right)$$

$$a_n = \gamma\left(F_X^{-1}\left(1-\frac{1}{n}\right)\right) = 1$$

$$b_n = F_X^{-1}\left(1-\frac{1}{n}\right) = -\log\left(\frac{1-\left(1-\frac{1}{n}\right)}{1-\frac{1}{n}}\right)$$

$$= \log(n-1)$$

$$c_n = 1$$

$$d_n = \log(n-1)$$

(iv) $$G(x,y) = \lim_{n\to\infty}\left[F(a_n x + b_n, c_n y + d_n)\right]^n$$

$$= \lim_{n\to\infty}\left[1 + e^{-a_n x - b_n} + e^{-c_n y + d_n} + (1-a)\, e^{-a_n x - b_n - c_n y - d_n}\right]^{-n}$$

$$= \lim_{n\to\infty}\left[1 + e^{-x-\log(n-1)} + e^{-y-\log(n-1)} + (1-a)\, e^{-x-\log(n-1)-y-\log(n-1)}\right]^{-n}$$

$$= \lim_{n \to \infty} \left[1 + \frac{e^{-x}}{n-1} + \frac{e^{-y}}{n-1} + (1-a)\frac{e^{-x-y}}{(n-1)^2} \right]^{-n}$$

$$= \lim_{n \to \infty} \left\{ 1 + \frac{1}{n} \left[\frac{n e^{-x}}{n-1} + \frac{n e^{-y}}{n-1} + \frac{(1-a) n e^{-x-y}}{(n-1)^2} \right] \right\}^{-n}$$

$\frac{n e^{-x}}{n-1} \to 1$, $\frac{n e^{-y}}{n-1} \to 1$, $\frac{(1-a) n e^{-x-y}}{(n-1)^2} \to 0$

$$= \lim_{n \to \infty} \left\{ 1 + \frac{1}{n} \left[e^{-x} + e^{-y} \right] \right\}^{-n}$$

$$\left(1 + \frac{z}{n}\right)^{-n} \longrightarrow e^{-z} \quad \text{as} \quad n \to \infty$$

$$= e^{-\left[e^{-x} + e^{-y} \right]}$$

Representations for $G(x, y)$

1) If F_X and F_Y belong to the max domain of Gumbel then

$$G(x, y) = e^{-\int_0^1 \min\left[\frac{f_1(s)}{e^x}, \frac{f_2(s)}{e^y}\right] ds}$$

where f_1 and f_2 are integrable functions such that

$$\int_0^1 f_1(s)\, ds = 1$$

and

$$\int_0^1 f_2(s)\, ds = 1.$$

2) If F_X and F_Y belong to the max domain of Gumbel then

$$G(x,y) = e^{-(e^{-x}+e^{-y})\,k(y-x)}$$

where $k(\cdot)$ satisfies

$$\lim_{t \to \infty} k(t) = 1$$

$$\lim_{t \to -\infty} k(t) = 1$$

$$\frac{d}{dt}\left\{(1+e^{-t})\,k(t)\right\} \leq 0 \quad \forall t$$

$$\frac{d}{dt}\left\{(1+e^{t})\,k(t)\right\} \geq 0 \quad \forall t$$

$$(1+e^{-t})\frac{d^2}{dt^2}k(t) + (1-e^{-t})\frac{d}{dt}k(t) \geq 0 \quad \forall t$$

3) If F_X and F_Y belong to the max domain of Frechet domain

$$G(x,y) = e^{-\left(\frac{1}{x}+\frac{1}{y}\right) A\left(\frac{x}{x+y}\right)}$$

where $A(\cdot)$ satisfies

$A(0) = 1$

$A(1) = 1$

$\max(w, 1-w) \leq A(w) \leq 1 \quad \forall\ 0 \leq w \leq 1$

$A(\cdot)$ is a convex function

4) Suppose $F_X(x) = 1 - e^{-x}$ and

$F_Y(y) = 1 - e^{-y}$.

Then

$$\bar{G}(x, y) = e^{-(x+y) A\left(\frac{y}{x+y}\right)}$$

where $A(\cdot)$ satisfies

$A(0) = 1$

$A(1) = 1$

$\max(w, 1-w) \leq A(w) \leq 1 \quad \forall \; 0 \leq w \leq 1$

$A(\cdot)$ is convex.

$$\boxed{G(x, y) = 1 - e^{-x} - e^{-y} + \bar{G}(x, y)}$$

↑

Relation between G and $\bar{G}$.

What is the representation for $G(x, y)$ if F_X and F_Y belong to the Weibull margin ?

Open problem.

MATH3/4/68181: Extreme values and financial risk
Semester 1
Problem sheet 12

1) Consider a bivariate distribution specified by the joint survival function

$$\overline{F}(x,y) = \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right]$$

for $x > 0$ and $y > 0$.

(a) show that the distribution is a bivariate extreme value distribution;

(b) derive the joint cdf;

(c) derive the conditional cdf if Y given $X = x$;

(d) derive the conditional cdf if X given $Y = y$;

(e) derive the joint pdf;

(f) derive the conditional pdf of Y given $X = x$;

(g) derive the conditional pdf of X given $Y = y$.

2) Consider a bivariate distribution specified by the joint survival function

$$\overline{F}(x,y) = \exp\left[\frac{\alpha xy}{x+y} - x - y\right]$$

for $x > 0$ and $y > 0$.

(a) show that the distribution is a bivariate extreme value distribution;

(b) derive the joint cdf;

(c) derive the conditional cdf if Y given $X = x$;

(d) derive the conditional cdf if X given $Y = y$;

(e) derive the joint pdf;

(f) derive the conditional pdf of Y given $X = x$;

(g) derive the conditional pdf of X given $Y = y$.

3) Consider a bivariate distribution specified by the joint survival function

$$\overline{F}(x,y) = \exp\left[-(x^a + y^a)^{1/a}\right]$$

for $x > 0$ and $y > 0$.

Q1

$$\bar{G}(x,y) = e^{-\frac{\theta y^2}{x+y} + \theta y - x - y}$$

$$\bar{G}(0,y) = e^{-\theta y + \theta y - y} = e^{-y}$$

$$\bar{G}(x,0) = e^{-x}$$

Marginal CDFs ~~Marginals~~ are $1-e^{-x}$ and $1-e^{-y}$.

This refers to representation 4.

$$\bar{G}(x,y) = e^{-(x+y)\left[\frac{\theta y^2}{(x+y)^2} - \frac{\theta y}{x+y} + 1\right]}$$

$$= e^{-(x+y)A\left(\frac{y}{x+y}\right)}$$

if $A(w) = \theta w^2 - \theta w + 1$

- $A(0) = \theta \cdot 0^2 - \theta \cdot 0 + 1 = 1$ ✓
- $A(1) = \theta \cdot 1^2 - \theta \cdot 1 + 1 = 1$ ✓
- $A(w) \leq 1$

$\Leftrightarrow \theta w^2 - \theta w + 1 \leq 1$

$\Leftrightarrow \theta w(w-1) \leq 0$

$\Leftrightarrow w - 1 \leq 0$

$\Leftrightarrow w \leq 1$ ✓

$A(w) \geq w$

$\Leftrightarrow \theta w^2 - \theta w + 1 \geq w$

$\Leftrightarrow \theta w(w-1) + 1 - w \geq 0$

$\Leftrightarrow (1-\theta w)(1-w) \geq 0$ ✓ if $\underline{\theta < 1}$

$$A(w) \geq 1 - w$$

$$\Leftrightarrow \quad \theta w^2 - \theta w + 1 \geq 1 - w$$

$$\Leftrightarrow \quad \theta w^2 - \theta w + w \geq 0$$

$$\Leftrightarrow \quad \theta w^2 + (1-\theta) w \geq 0 \quad \checkmark \quad \text{if} \quad \theta < 1$$

- $A(w) = \theta w^2 - \theta w + 1$

  $A'(w) = 2\theta w - \theta$

  $A''(w) = 2\theta > 0$

  $\Rightarrow$ $A(\cdot)$ is convex (because $0 < \theta < 1$)

Hence, $\bar{G}(x, y)$ is a bivariate extreme value distribution.

EXAMPLE CLASS

3 DECEMBER

12:00-13:00PM

MATH3/4/68181

Q1

$$F(x) = e^{-(1+\xi x)^{-1/\xi}}$$

(a) $$f(x) = \frac{dF(x)}{dx}$$

$$= (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}}$$

(b) $\boxed{\xi = 0}$ $$F(x) = e^{-e^{-x}}$$

$$f(x) = e^{-x} e^{-e^{-x}}$$

$$E(X^n) = \int_{-\infty}^{\infty} x^n e^{-x} e^{-e^{-x}} dx$$

$$y = e^{-x}$$
$$x = -\log y$$
$$\frac{dx}{dy} = -\frac{1}{y}$$

$$= \int_{\infty}^{0} (-\log y)^n y e^{-y} \left(-\frac{1}{y}\right) dy$$

$$= \int_{0}^{\infty} (-\log y)^n e^{-y} dy$$

$$\left.\frac{d^n y^a}{da^n}\right|_{a=0} = (\log y)^n$$

$$= (-1)^n \left[\int_0^{\infty} \frac{d^n y^a}{da^n} e^{-y} dy\right]\Big|_{a=0}$$

$$= (-1)^n \frac{d^n}{da^n} \Gamma(a+1) \Big|_{a=0}$$

$\boxed{\xi > 0}$: $\quad 1 + \xi x > 0$

$$x > -\frac{1}{\xi}$$

$$x \in \left(-\frac{1}{\xi}, \infty\right)$$

$$E(X^n) = \int_{-\frac{1}{\xi}}^{\infty} x^n (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

Set $y = (1+\xi x)^{-\frac{1}{\xi}}$

$\Rightarrow x = \dfrac{y^{-\xi} - 1}{\xi}$

$\Rightarrow \dfrac{dx}{dy} = -y^{-\xi-1}$

$$= \int_{\infty}^{0} \left(\frac{y^{-\xi}-1}{\xi}\right)^n \cancel{y^{1+\xi}} e^{-y} (-1) \cancel{y^{-\xi-1}} dy$$

$$= \frac{1}{\xi^n} \int_0^{\infty} (y^{-\xi} - 1)^n e^{-y} dy$$

Bin Thm

$$= \frac{1}{\xi^n} \int_0^{\infty} \sum_{k=0}^{n} \binom{n}{k} (y^{-\xi})^k (-1)^{n-k} e^{-y} dy$$

$$= \frac{1}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^{\infty} y^{-\xi k} e^{-y} dy$$

$$= \frac{1}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Gamma(1-\xi k)$$

$\xi < 0$

$1 + \xi x > 0$

$x < -\frac{1}{\xi}$

$x \in \left(-\infty, -\frac{1}{\xi}\right)$

$$E(X^n) = \int_{-\infty}^{-\frac{1}{\xi}} x^n (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

Set $y = (1+\xi x)^{-\frac{1}{\xi}}$

$\Rightarrow x = \frac{y^{-\xi} - 1}{\xi}$

$\Rightarrow \frac{dx}{dy} = -y^{-\xi-1}$

$$= \int_0^{\infty} \left(\frac{y^{-\xi}-1}{\xi}\right)^n \cancel{y^{(1+\xi)}} e^{-y} (-1) \cancel{y^{-\xi-1}} dy$$

$$= -\frac{1}{\xi^n} \int_0^{\infty} (y^{-\xi} - 1)^n e^{-y} dy$$

$$= -\frac{1}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^{\infty} y^{-\xi k} e^{-y} dy$$

$$= -\frac{1}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Gamma(1-\xi k).$$

Q2

$$F(x) = 1 - (1+\xi x)^{-\frac{1}{\xi}}$$

(a) $$f(x) = \frac{dF(x)}{dx} = (1+\xi x)^{-\frac{1}{\xi}-1}$$

(b) $\boxed{\xi = 0}$ $$F(x) = 1 - \exp(-x)$$
$$f(x) = e^{-x}$$
$$E(X^n) = \int_0^\infty x^n e^{-x}\, dx$$
$$= \Gamma(n+1) = n!$$

$\boxed{\xi > 0}$ $$E(X^n) = \int_0^\infty x^n (1+\xi x)^{-\frac{1}{\xi}-1} dx$$

Set $y = 1+\xi x$

$\Rightarrow x = \frac{y-1}{\xi}$

$\Rightarrow \frac{dx}{dy} = \frac{1}{\xi}$

$$= \int_1^\infty \left(\frac{y-1}{\xi}\right)^n y^{-\frac{1}{\xi}-1} \frac{dy}{\xi}$$
$$= \frac{1}{\xi^{n+1}} \int_1^\infty (y-1)^n y^{-\frac{1}{\xi}-1} dy$$
$$= \frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_1^\infty y^{k-\frac{1}{\xi}-1} dy$$
$$= \frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{\frac{1}{\xi}-k}$$

$\xi < 0$

$$E(X^n) = \int_0^{-1/\xi} x^n (1+\xi x)^{-\frac{1}{\xi}-1} dx$$

$$= \int_1^0 \left(\frac{y-1}{\xi}\right)^n y^{-\frac{1}{\xi}-1} \frac{dy}{\xi}$$

$$= -\frac{1}{\xi^{n+1}} \int_0^1 (y-1)^n y^{-\frac{1}{\xi}-1} dy$$

$$= -\frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^1 y^{k-\frac{1}{\xi}-1} dy$$

$$= -\frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{k-\frac{1}{\xi}}.$$

EXAMPLE CLASS

4 DECEMBER

16:00-17:00PM

MATH3/4/68181

Q1

$$F(x) = e^{-(1+\xi x)^{-\frac{1}{\xi}}}$$ GEV cdf

(a) $f(x) = \frac{d}{dx} F(x)$

$$= (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}}$$

(b) $\boxed{\xi = 0}$: From lecture notes,

$$F(x) = e^{-e^{-x}}$$

$$f(x) = e^{-x} e^{-e^{-x}}$$

$$E(X^n) = \int_{-\infty}^{\infty} x^n e^{-x} e^{-e^{-x}} dx$$

$$y = e^{-x}$$
$$\Rightarrow x = -\log y$$
$$\Rightarrow \frac{dx}{dy} = -\frac{1}{y}$$

$$= \int_{\infty}^{0} (-\log y)^n y e^{-y} \left(-\frac{1}{y}\right) dy$$

$$= \int_{0}^{\infty} (-\log y)^n e^{-y} dy$$

$$\left.\frac{d^n y^a}{da^n}\right|_{a=0} = (\log y)^n$$

$$= \left[\int_0^\infty (-1)^n \frac{d^n y^a}{da^n} e^{-y} \, dy \right] \Bigg|_{a=0}$$

$$= (-1)^n \frac{d^n}{da^n} \left[\int_0^\infty y^a e^{-y} \, dy \right] \Bigg|_{a=0}$$

$$= (-1)^n \frac{d^n}{da^n} \Gamma(a+1) \Bigg|_{a=0}$$

$\boxed{\xi > 0}$ $\quad 1 + \xi x > 0 \Rightarrow x > -\frac{1}{\xi}$

$$\Rightarrow x \in \left(-\frac{1}{\xi}, \infty\right)$$

$$E(X^n) = \int_{-\frac{1}{\xi}}^{\infty} x^n (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

Set $y = (1+\xi x)^{-\frac{1}{\xi}}$

$$\Rightarrow \frac{y^{-\xi}-1}{\xi} = x$$

$$\Rightarrow \frac{dx}{dy} = -y^{-\xi-1}$$

$$= \int_\infty^0 \left(\frac{y^{-\xi}-1}{\xi}\right)^n y^{1+\xi} e^{-y} (-1) y^{-\xi-1} \, dy$$

$$= \xi^{-n} \int_0^\infty (y^{-\xi}-1)^n e^{-y} \, dy$$

$$= \xi^{-n} \int_0^\infty \sum_{k=0}^{n} \binom{n}{k} y^{-k\xi} (-1)^{n-k} e^{-y} \, dy$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^\infty y^{-k\xi} e^{-y} \, dy$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \prod (1 - k\xi).$$

$\boxed{\xi < 0}$ $\quad 1 + \xi x > 0 \Rightarrow x < -\frac{1}{\xi}$

$$\Rightarrow x \in \left(-\infty, -\frac{1}{\xi}\right)$$

$$E(X^n) = \int_{-\infty}^{-\frac{1}{\xi}} x^n (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

$$= \int_\infty^0 \left(\frac{y^{-\xi}-1}{\xi}\right)^n \cancel{y^{1+\xi}} e^{-y} (-1) \cancel{y^{-\xi-1}} \, dy$$

$$= \xi^{-n} \int_0^\infty (y^{-\xi} - 1)^n e^{-y} \, dy$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^\infty y^{-k\xi} e^{-y} \, dy$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \prod (1 - k\xi).$$

Q2

$$F(x) = 1 - (1+\xi x)^{-\frac{1}{\xi}}$$

(a) $$f(x) = \frac{dF(x)}{dx} = (1+\xi x)^{-\frac{1}{\xi}-1}$$

(b) $\boxed{\xi = 0}$: From lecture notes,

$$F(x) = 1 - e^{-x}$$

$$f(x) = e^{-x}$$

$$E(X^n) = \int_0^\infty x^n e^{-x}\, dx$$

$$= \Gamma(n+1) = n!$$

$\boxed{\xi > 0}$: $$E(X^n) = \int_0^\infty x^n (1+\xi x)^{-\frac{1}{\xi}-1} dx$$

Set $y = 1+\xi x$

$$x = \frac{y-1}{\xi}$$

$$\frac{dx}{dy} = \frac{1}{\xi}$$

$$= \int_1^\infty \left(\frac{y-1}{\xi}\right)^n y^{-\frac{1}{\xi}-1} \frac{1}{\xi}\, dy$$

$$= \frac{1}{\xi^{n+1}} \int_1^\infty (y-1)^n y^{-\frac{1}{\xi}-1}\, dy$$

$$= \frac{1}{\xi^{n+1}} \int_1^{\infty} \sum_{k=0}^{n} \binom{n}{k} y^k (-1)^{n-k} y^{-\frac{1}{\xi}-1} dy$$

$$= \frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_1^{\infty} y^{k-\frac{1}{\xi}-1} dy$$

$$= \frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{\frac{1}{\xi} - k} .$$

$\boxed{\xi < 0}$: $E(X^n) = \int_0^{-1/\xi} x^n (1+\xi x)^{-\frac{1}{\xi}-1} dx$

$$= \int_1^0 \left(\frac{y-1}{\xi}\right)^n y^{-\frac{1}{\xi}-1} \frac{1}{\xi} dy$$

$$= -\frac{1}{\xi^{n+1}} \int_0^1 (y-1)^n y^{-\frac{1}{\xi}-1} dy$$

$$= -\frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^1 y^{k-\frac{1}{\xi}-1} dy$$

$$= -\frac{1}{\xi^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{k - \frac{1}{\xi}} .$$

REVISION CLASS

10 DECEMBER

9:00-10:00AM

MATH3/4/68181

Revision Class

Monday

10 Dec

9-10 am

Questions please!

Q5, 2017/18 Exam

$$F(x_1, \dots, x_k) = \left[\max\left(\frac{x_1}{a_1}, \dots, \frac{x_k}{a_k} \right) \right]^{-a}$$

(a) $F_T(t)$

$= P(T \leq t)$

$= 1 - P(T > t)$

$= 1 - P(\min(X_1, \dots, X_k) > t)$

$= 1 - P(X_1 > t, \dots, X_t > t)$

$= 1 - F(t, \dots, t)$

$= 1 - \left[\max\left(\frac{t}{a_1}, \dots, \frac{t}{a_k} \right) \right]^{-a}$

$= 1 - \left[t \cdot \max\left(\frac{1}{a_1}, \dots, \frac{1}{a_k} \right) \right]^{-a}$

$= 1 - \left[t \cdot \frac{1}{\min(a_1, \dots, a_k)} \right]^{-a}$

$= 1 - \left[\min(a_1, \dots, a_k) \right]^{a} t^{-a}$, $t \geq \min(a_1, \dots, a_k)$

(b) $f_T(t) = \frac{d}{dt} F_T(t)$

$= a \left[\min(a_1, \dots, a_k) \right]^{a} t^{-a-1}$, $t \geq \min(a_1, \dots, a_k)$

(c) $F_T(t) = p$

$\Rightarrow 1 - \left[\min(a_1, \dots, a_k) \right]^{a} t^{-a} = p$

$\Rightarrow t^{-a} = \frac{1-p}{\left[\min(a_1, \dots, a_k) \right]^{a}}$

$\Rightarrow VaR_p(T) = \frac{(1-p)^{-1/a}}{\left[\min(a_1, \dots, a_k) \right]^{-1}}$

(d) $ES_p(T) = \frac{1}{p}\int_0^p VaR_u(T)\,du$

$= \frac{\min(a_1,\dots,a_k)}{p}\int_0^p (1-u)^{-\frac{1}{a}}\,du$

$= \frac{\min(a_1,\dots,a_k)}{p}\left[\frac{(1-u)^{1-\frac{1}{a}}}{\frac{1}{a}-1}\right]_0^p$

$= \frac{\min(a_1,\dots,a_k)}{p}\cdot\frac{(1-p)^{1-\frac{1}{a}}-1}{\frac{1}{a}-1}.$

(e)

$L(a, a_1,\dots,a_k)$

$= \prod_{i=1}^{n}\{a[\min(a_1,\dots,a_k)]^a t_i^{-a-1}\}$

$= a^n[\min(a_1,\dots,a_k)]^{na}\left(\prod_{i=1}^{n} t_i\right)^{-a-1}$

$\prod_{i=1}^{n} I\{t_i \geq \min(a_1,\dots,a_k)\}$

$= a^n[\min(a_1,\dots,a_k)]^{na}\left(\prod_{i=1}^{n} t_i\right)^{-a-1}$

$I\{\min t_i \geq \min(a_1,\dots,a_k)\}$

$$\log L = n \log a + na \log [\min(a_1, \cdots, a_k)]$$

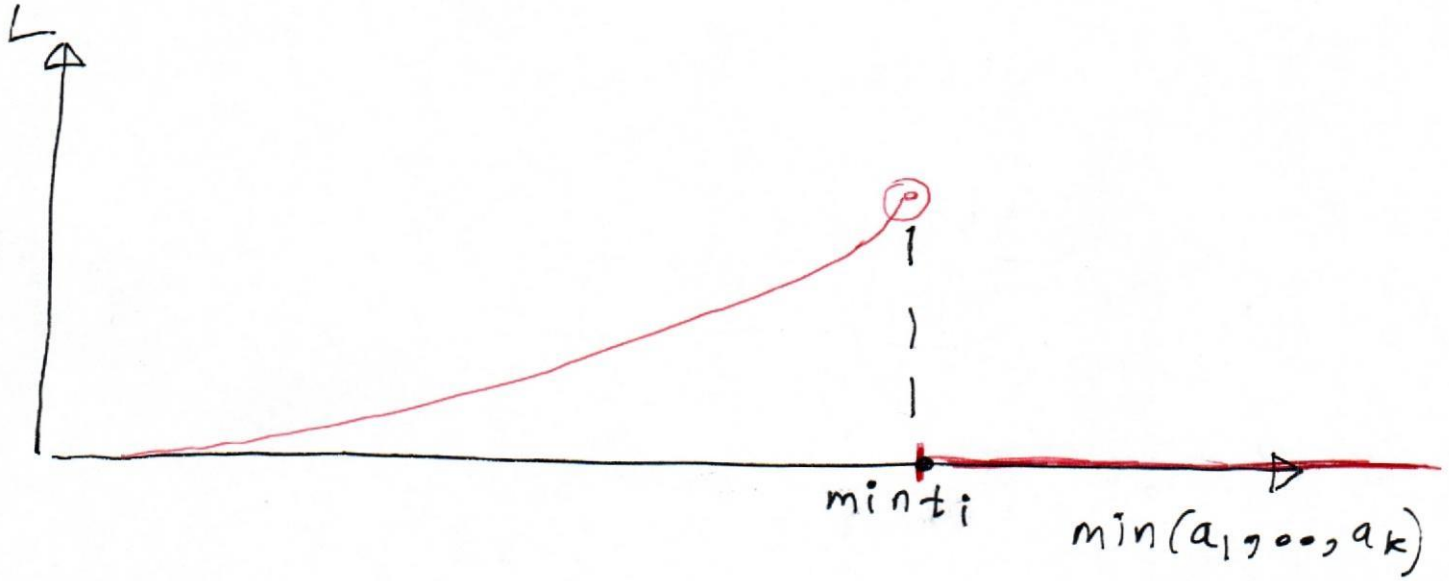
$$- (a+1) \sum_{i=1}^{n} \log t_i$$

$$+ \log I\{\min t_i \geq \min(a_1, \cdots, a_k)\}$$

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} + n \log [\min(a_1, \cdots, a_k)]$$

$$- \sum_{i=1}^{n} \log t_i = 0$$

$$\Rightarrow \hat{a} = n \left\{ \sum_{i=1}^{n} \log t_i - n \log [\min(a_1, \cdots, a_k)] \right\}^{-1}$$



$\Rightarrow$ The MLE of $\min(a_1, \cdots, a_k)$ is $\min(t_1, \cdots, t_n)$.

Quiz 5

$$ES_p(X) = \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

Set $u = F^{-1}(t)$

$t = F(u)$

$\frac{dt}{du} = f(u)$

$$= \frac{1}{p}\int_{F^{-1}(0)}^{F^{-1}(p)} u f(u)\,du$$

$$= \frac{1}{p}\int_{-\infty}^{F^{-1}(p)} u \cdot K\left(1+\frac{u^2}{a}\right)^{-\frac{a+1}{2}}\,du$$

$$= \frac{aK}{2p}\int_{-\infty}^{F^{-1}(p)} \frac{2u}{a}\left(1+\frac{u^2}{a}\right)^{-\frac{a+1}{2}}\,du$$

$$= \frac{aK}{2p}\int_{-\infty}^{F^{-1}(p)} \left(1+\frac{u^2}{a}\right)^{-\frac{a+1}{2}} d\left(1+\frac{u^2}{a}\right)$$

$$= \frac{aK}{2p}\left[\frac{\left(1+\frac{u^2}{a}\right)^{1-\frac{a+1}{2}}}{1-\frac{a+1}{2}}\right]_{-\infty}^{F^{-1}(p)}$$

$$= \frac{aK}{2p}\cdot\frac{2}{1-a}\left\{\left(1+\frac{[F^{-1}(p)]^2}{a}\right)^{\frac{1-a}{2}} - 0\right\}$$

$$= \frac{aK}{p(1-a)}\left(1+\frac{1}{a}[F^{-1}(p)]^2\right)^{\frac{1-a}{2}}.$$

Alternative proof

Assume

$$ES_p(X) = \frac{aK}{(1-a)p}\left\{1+\frac{1}{a}\left[VaR_p(X)\right]^2\right\}^{\frac{1-a}{2}}$$

$$\Rightarrow \frac{1}{p}\int_0^p F^{-1}(u)\,du = \frac{aK}{(1-a)p}\left\{1+\frac{1}{a}\left[F^{-1}(p)\right]^2\right\}^{\frac{1-a}{2}}$$

$$\Rightarrow \int_0^p F^{-1}(u)\,du = \frac{aK}{1-a}\left\{1+\frac{1}{a}\left[F^{-1}(p)\right]^2\right\}^{\frac{1-a}{2}}$$

$$\Rightarrow \frac{d}{dp}\int_0^p F^{-1}(u)\,du = \frac{aK}{1-a}\frac{d}{dp}\left\{1+\frac{1}{a}\left[F^{-1}(p)\right]^2\right\}^{\frac{1-a}{2}}$$

$$\Rightarrow F^{-1}(p) = \frac{aK}{1-a}\cdot\frac{1-a}{2}\left\{1+\frac{1}{a}\left[F^{-1}(p)\right]^2\right\}^{\frac{1-a}{2}-1}$$

$$\times \frac{2}{a}F^{-1}(p)\frac{d}{dp}F^{-1}(p)$$

$$\Rightarrow 1 = K\left\{1+\frac{1}{a}\left[F^{-1}(p)\right]^2\right\}^{-\frac{a+1}{2}}\cdot\frac{d}{dp}F^{-1}(p)$$

$$\Rightarrow f\left(F^{-1}(p)\right) = K\left\{1+\frac{1}{a}\left[F^{-1}(p)\right]^2\right\}^{-\frac{a+1}{2}}$$

This is true.

Hence the assumption must be true.

REVISION CLASS

11 DECEMBER

9:00-10:00AM

MATH3/4/68181

REVISION

CLASS

TUESDAY 11 DEC

9:00- 10:00 AM

Q2, Exam 2017/18

$$f(x) = C\, g(x) \left[1 - G(x)\right]^{\lambda b - 1} \times \left\{1 - \left[1 - G(x)\right]^{\lambda}\right\}^{a-1}$$

Show that F and G belong to the same domain of attraction.

(i) Assume G belongs to the Gumbel max domain. There exists $\gamma(t) > 0$ such that

$$\lim_{t \to w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x} \quad \cdots (*).$$

$$\lim_{t \to w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$\overset{LH}{=} \lim_{t \to w(F)} \frac{-f(t + x\gamma(t))\,(1 + x\gamma'(t))}{-f(t)}$$

$$= \lim_{t \to w(F)} \frac{f(t + x\gamma(t))\,(1 + x\gamma'(t))}{f(t)}$$

$$= \lim_{t \to w(G)} \frac{g(t + x\gamma(t))}{g(t)} \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)} \right]^{\lambda b - 1}$$

$$\times \left\{ \frac{1 - [1 - G(t + x\gamma(t))]^{\lambda}}{1 - [1 - G(t)]^{\lambda}} \right\}^{a-1}$$

$$\times \left(1 + x\gamma'(t) \right)$$

$$= \lim_{t \to w(G)} \frac{g(t + x\gamma(t))}{g(t)} \cdot \left(1 + x\gamma'(t) \right)$$

$$\times \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)} \right]^{\lambda b - 1}$$

$$\overset{\text{LH in reverse}}{=} \lim_{t \to w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} \cdot \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)} \right]^{\lambda b - 1}$$

$$= \lim_{t \to w(G)} \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)} \right]^{\lambda b}$$

$$\overset{(*)}{=} \left[e^{-x} \right]^{\lambda b} = e^{-\lambda b x}$$

$\Rightarrow$ F also belongs to the Gumbel max domain.

Level 3

2 hrs

5 Q

must answer any 4 Q

Levels 4 & 6

3 hrs

Section A

- 3 Q
- must answer any 2 Q

Section B

- 5 Q
- must answer any 4 Q

Q3 (e), 2017/18 Exam

$$F(x) = \left[1 + e^{-ax}\right]^{-b}, \quad -\infty < x < \infty$$

$$w(F) = +\infty$$

$$I: \lim_{t \to \infty} \frac{1 - \left[1 + e^{-a(t + x\gamma(t))}\right]^{-b}}{1 - \left[1 + e^{-at}\right]^{-b}}$$

$$= \lim_{t \to \infty} \frac{1 - \left[1 - b\, e^{-a(t + x\gamma(t))}\right]}{1 - \left[1 - b e^{-at}\right]}$$

$$(1+z)^{-b} \sim 1 - bz$$

$$= \lim_{t \to \infty} \frac{e^{-a(t + x\gamma(t))}}{e^{-at}}$$

$$= \lim_{t \to \infty} e^{-ax\gamma(t)}$$

$$= e^{-x} \quad \text{if} \quad \gamma(t) \equiv \frac{1}{a}$$

Hence, F belongs to the Gumbel max domain.

Q4(b), Exam 2017/18

$$X_i \sim N(\mu_i, \sigma_i^2) \quad \text{independently}$$

(i) $$T = X_1 + X_2 + \cdots + X_k$$

$$\sim N\left(\sum_{i=1}^{k} \mu_i \,,\, \sum_{i=1}^{k} \sigma_i^2\right)$$

(ii) $$F_T(t) = p$$

$$\Rightarrow \Phi\left(\frac{t - \sum_{i=1}^{k} \mu_i}{\sqrt{\sum_{i=1}^{k} \sigma_i^2}}\right) = p$$

$$\Rightarrow \quad VaR_p(T) = \left(\sum_{i=1}^{k} \mu_i\right) + \sqrt{\sum_{i=1}^{k} \sigma_i^2}\ \Phi^{-1}(p)$$

(iii) $$ES_p(T) = \frac{1}{p} \int_0^p VaR_u(T)\, du$$

$$= \left(\sum_{i=1}^{k} \mu_i\right) + \sqrt{\sum_{i=1}^{k} \sigma_i^2}\ \frac{1}{p} \int_0^p \Phi^{-1}(u)\, du.$$

Q 2 (b), 2017 / 18 Exam

$$p(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$k = \max(0, n+K-N), \ldots, \min(n, K).$$

$$w(F) = \min(n, K)$$

$$\lim_{k \to w(F)} \frac{P(X=k)}{1-F(k-1)} = \frac{P(X=\min(n,K))}{1-F(\min(n,K)-1)}$$

$$= \frac{P(X=\min(n,K))}{1-P(X \leq \min(n,K)-1)}$$

$$= \frac{P(X=\min(n,K))}{P(X \geq \min(n,K))}$$

$$= \frac{P(X=\min(n,K))}{P(X=\min(n,K))} = 1 \neq 0$$

Hence, ETT does not hold.

REVISION CLASS

14 DECEMBER

10:00-11:00AM

MATH4/68181

REVISION

CLASS

FRI 14 DEC

10- 11 AM

A2 (a), Exam 2017/18

$$C(u_1, u_2) = \min\left(u_1^{1-\alpha} u_2,\ u_1 u_2^{1-\beta}\right), \quad \begin{matrix} 0<\alpha<1 \\ 0<\beta<1 \end{matrix}$$

(i) $C(0, u_2) = \min(0, 0) = 0$ ✓

(ii) $C(u_1, 0) = \min(0, 0) = 0$ ✓

(iii) $C(1, u_2) = \min(u_2, u_2^{1-\beta}) = u_2$ ✓

(iv) $C(u_1, 1) = \min(u_1^{1-\alpha}, u_1) = u_1$ ✓

(v) $$\frac{\partial C}{\partial u_1} = \frac{\partial}{\partial u_1} \begin{cases} u_1^{1-\alpha} u_2 & \text{if } u_1^{1-\alpha} u_2 < u_1 u_2^{1-\beta} \\ u_1 u_2^{1-\beta} & \text{if } u_1 u_2^{1-\beta} < u_1^{1-\alpha} u_2 \end{cases}$$

$$= \begin{cases} (1-\alpha) u_1^{-\alpha} u_2 \geq 0 & \text{if } u_1^{1-\alpha} u_2 < u_1 u_2^{1-\beta} \\ u_2^{1-\beta} \geq 0 & \text{if } u_1 u_2^{1-\beta} < u_1^{1-\alpha} u_2 \end{cases}$$

≥ 0 ✓

(vi) $$\frac{\partial C}{\partial u_2} = \frac{\partial}{\partial u_2} \begin{cases} u_1^{1-\alpha} u_2 & \text{if } u_1^{1-\alpha} u_2 < u_1 u_2^{1-\beta} \\ u_1 u_2^{1-\beta} & \text{if } u_1 u_2^{1-\beta} < u_1^{1-\alpha} u_2 \end{cases}$$

$$= \begin{cases} u_1^{1-\alpha} \geq 0 & \text{if } u_1^{1-\alpha} u_2 < u_1 u_2^{1-\beta} \\ (1-\beta) u_1 u_2^{-\beta} \geq 0 & \text{if } u_1 u_2^{1-\beta} < u_1^{1-\alpha} u_2 \end{cases}$$

≥ 0 ✓

Hence, C is a copula.

A2(d) Exam 2017/18

$$C(u_1, u_2) = \left\{ \left[(u_1^{-\theta} - 1)^{\delta} + (u_2^{-\theta} - 1)^{\delta} \right]^{\frac{1}{\delta}} + 1 \right\}^{-\frac{1}{\theta}}$$

$$\theta > 0$$

$$\delta \geq 1$$

(i) $C(0, u_2) = \left\{ \left[(\infty - 1)^{\delta} + (u_2^{-\theta} - 1)^{\delta} \right]^{\frac{1}{\delta}} + 1 \right\}^{-\frac{1}{\theta}}$

$= 0$ ✓

(ii) $C(u_1, 0) = \left\{ \left[(u_1^{-\theta} - 1)^{\delta} + (\infty - 1)^{\delta} \right]^{\frac{1}{\delta}} + 1 \right\}^{-\frac{1}{\theta}}$

$= 0$ ✓

(iii) $C(1, u_2) = \left\{ \left[(1-1)^{\delta} + (u_2^{-\theta} - 1)^{\delta} \right]^{\frac{1}{\delta}} + 1 \right\}^{-\frac{1}{\theta}}$

$= \left\{ u_2^{-\theta} - 1 + 1 \right\}^{-\frac{1}{\theta}}$

$= u_2$ ✓

(iv) $C(u_1, 1) = \left\{ \left[(u_1^{-\theta} - 1)^{\delta} + (1-1)^{\delta} \right]^{\frac{1}{\delta}} + 1 \right\}^{-\frac{1}{\theta}}$

$= \left\{ u_1^{-\theta} - 1 + 1 \right\}^{-\frac{1}{\theta}}$

$= u_1$ ✓

(v) $\frac{\partial C}{\partial u_1} = \left(-\frac{1}{\theta}\right) \left\{ \left[(u_1^{-\theta} - 1)^{\delta} + (u_2^{-\theta} - 1)^{\delta} \right]^{\frac{1}{\delta}} + 1 \right\}^{-\frac{1}{\theta} - 1}$

$\cdot \frac{1}{\delta} \left[(u_1^{-\theta} - 1)^{\delta} + (u_2^{-\theta} - 1)^{\delta} \right]^{\frac{1}{\delta} - 1}$

$\cdot \delta (u_1^{-\theta} - 1)^{\delta - 1}$

$\cdot (-\theta) u_1^{-\theta - 1}$

$$= \left\{ \left[(u_1^{-\theta} - 1)^{\delta} + (u_2^{-\theta} - 1)^{\delta} \right]^{\frac{1}{\delta}} + 1 \right\}^{-\frac{1}{\theta} - 1} \geq 0$$

- $\left[(u_1^{-\theta} - 1)^{\delta} + (u_2^{-\theta} - 1)^{\delta} \right]^{\frac{1}{\delta} - 1} \geq 0$
- $(u_1^{-\theta} - 1)^{\delta - 1} \quad u_1^{-\theta - 1}$
  $\geq 0 \quad \geq 0$

≥ 0 ✓

(vi) $\dfrac{\partial C}{\partial u_2} = \left\{ \left[(u_1^{-\theta} - 1)^{\delta} + (u_2^{-\theta} - 1)^{\delta} \right]^{\frac{1}{\delta}} + 1 \right\}^{-\frac{1}{\theta} - 1} \geq 0$

- $\left[(u_1^{-\theta} - 1)^{\delta} + (u_2^{-\theta} - 1)^{\delta} \right]^{\frac{1}{\delta} - 1} \geq 0$
- $(u_2^{-\theta} - 1)^{\delta - 1} \quad u_2^{-\theta - 1}$
  $\geq 0 \quad \geq 0$

≥ 0 ✓

Hence, C is a copula.

AI, Exam 2017/18

$$f_{X,Y}(x,y) = \frac{4xy + 2x + 2y + 1}{4}$$

$$0 < x < 1$$
$$0 < y < 1$$

(a) $$F_{X,Y}(x,y) = \int_0^x \int_0^y f_{X,Y}(u,v)\, dv\, du$$

$$= \frac{1}{4} \int_0^x \int_0^y [4uv + 2u + 2v + 1]\, dv\, du$$

$$= \frac{1}{4} \int_0^x [2uv^2 + 2uv + v^2 + v]_0^y\, du$$

$$= \frac{1}{4} \int_0^x [2uy^2 + 2uy + y^2 + y]\, du$$

$$= \frac{1}{4} [u^2y^2 + u^2y + uy^2 + uy]_0^x$$

$$= \frac{1}{4} [x^2y^2 + x^2y + xy^2 + xy]$$

(b) $$F_X(x) = F_{X,Y}(x,1) = \frac{x^2 + x}{2}$$

$$F_Y(y) = F_{X,Y}(1,y) = \frac{y^2 + y}{2}$$

(c) $$w(F_X) = 1,$$

$$\lim_{t \to 0} \frac{1 - F_X(1 - tx)}{1 - F_X(1 - t)}$$

$$= \lim_{t \to 0} \frac{1 - \frac{(1-tx)^2 + 1 - tx}{2}}{1 - \frac{(1-t)^2 + 1 - t}{2}}$$

$$= \lim_{t \to 0} \frac{\frac{3tx}{2} - \frac{(tx)^2}{2}}{\frac{3t}{2} - \frac{t^2}{2}}$$

$$= \lim_{t \to 0} \frac{\frac{3x}{2} - \frac{tx^2}{2} \to 0}{\frac{3}{2} - \frac{t}{2} \to 0}$$

$$= x$$

Hence F_X belongs to the Weibull max domain.

(d) $\omega(F_Y) = 1$

$$\lim_{t \to 0} \frac{1 - F_Y(1 - ty)}{1 - F_Y(1 - t)}$$

$$= \lim_{t \to 0} \frac{\frac{3y}{2} - \frac{ty^2}{2} \to 0}{\frac{3}{2} - \frac{t}{2} \to 0}$$

$$= y$$

Hence F_Y belongs to the Weibull max domain.

(e) $a_n = \omega(F_X) - F_X^{-1}\left(1 - \frac{1}{n}\right)$

$b_n = \omega(F_X)$

$$F_X(x) = \frac{x^2 + x}{2} = z$$

$$\Rightarrow x^2 + x - 2z = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1 + 8z}}{2}$$

$$\Rightarrow x = \frac{-1 + \sqrt{1 + 8z}}{2}$$

$$F_X^{-1}(x) = \frac{-1 + \sqrt{1 + 8x}}{2}$$

$$F_X^{-1}\left(1 - \frac{1}{n}\right) = \frac{-1 + \sqrt{1 + 8\left(1 - \frac{1}{n}\right)}}{2}$$

$$= \frac{-1 + \sqrt{9 - \frac{8}{n}}}{2}$$

$$\Rightarrow \quad a_n = \frac{3}{2} - \frac{1}{2}\sqrt{9 - \frac{8}{n}}$$

$$b_n = 1$$

$$c_n = \frac{3}{2} - \frac{1}{2}\sqrt{9 - \frac{8}{n}}$$

$$d_n = 1$$

(g) $$\lim_{n \to \infty} \left[F_{X,Y}(a_n x + b_n, c_n y + d_n)\right]^n$$

$$= \lim_{n \to \infty} \frac{(a_n x + b_n)^n (c_n y + d_n)^n}{4^n}$$

- $$\left[(a_n x + b_n)(c_n y + d_n) + a_n x + b_n + c_n y + d_n + 1\right]^n$$

FIND THE LIMIT OF THIS

REVISION CLASS

10 DECEMBER

12:00-13:00PM

MATH3/4/68181

REVISION

CLASS

MON 10 DEC

12 - 1 PM

Quiz 4

$$X = \min(Y_1, Y_2, \ldots, Y_N)$$

$$Y_i = \max(Z_{i,1}, Z_{i,2}, \ldots, Z_{i,\alpha})$$

The CDF of Y_i is

$$F_{Y_i}(y) = P[Y_i \le y]$$

$$= P[\max(Z_{i,1}, \ldots, Z_{i,\alpha}) \le y]$$

$$= P[Z_{i,1} \le y, \ldots, Z_{i,\alpha} \le y]$$

$$\overset{\text{indep}}{=} P[Z_{i,1} \le y] \cdots P[Z_{i,\alpha} \le y]$$

$$= (1 - e^{-\beta y}) \cdots (1 - e^{-\beta y})$$

$$= (1 - e^{-\beta y})^{\alpha}$$

The CDF of X is

$$F_X(x) = P(X \le x)$$

$$= 1 - P(X > x)$$

$$\overset{\text{total prob rule}}{=} 1 - \sum_{n=1}^{\infty} P(X > x / N = n) P(N = n)$$

$$= 1 - \sum_{n=1}^{\infty} P(\min(Y_1, \ldots, Y_N) > x / N = n) P(N = n)$$

$$= 1 - \sum_{n=1}^{\infty} P(Y_1 > x, \ldots, Y_n > x) P(N = n)$$

$$\overset{\text{indep}}{=} 1 - \sum_{n=1}^{\infty} P(Y_1 > x) \cdots P(Y_n > x)\, P(N = n)$$

$$= 1 - \sum_{n=1}^{\infty} [P(Y > x)]^n P(N = n)$$

$$= 1 - \sum_{n=1}^{\infty} \left[1 - (1 - e^{-\beta x})^{\alpha}\right]^n \frac{1}{e^{\lambda} - 1} \frac{\lambda^n}{n!}$$

$$= 1 - \frac{1}{e^{\lambda} - 1} \sum_{n=1}^{\infty} \frac{\{[1 - (1 - e^{-\beta x})^{\alpha}]\lambda\}^n}{n!}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{\theta^n}{n!} = e^{\theta} - 1}$$

$$= 1 - \frac{1}{e^{\lambda} - 1} \left\{ e^{[1 - (1 - e^{-\beta x})^{\alpha}]\lambda} - 1 \right\}$$

To find $\text{VaR}_p(X)$, set

$$F_X(x) = p$$

$$\Leftrightarrow 1 - \frac{e^{[1 - (1 - e^{-\beta x})^{\alpha}] \cdot \lambda} - 1}{e^{\lambda} - 1} = p$$

$$\Leftrightarrow e^{[1 - (1 - e^{-\beta x})^{\alpha}] \cdot \lambda} - 1 = (1 - p)(e^{\lambda} - 1)$$

$$\Leftrightarrow e^{[1 - (1 - e^{-\beta x})^{\alpha}] \cdot \lambda} = 1 + (1 - p)(e^{\lambda} - 1)$$

$$\Leftrightarrow [1 - (1 - e^{-\beta x})^{\alpha}] \lambda = \log[1 + (1 - p)(e^{\lambda} - 1)]$$

$$\Leftrightarrow \quad 1 - (1-e^{-\beta x})^{\alpha} = \frac{1}{\lambda} \log\left[1+(1-p)(e^{\lambda}-1)\right]$$

$$\Leftrightarrow \quad (1-e^{-\beta x})^{\alpha} = 1 - \frac{1}{\lambda} \log\left[1+(1-p)(e^{\lambda}-1)\right]$$

$$\Leftrightarrow \quad 1-e^{-\beta x} = \left\{1 - \frac{1}{\lambda} \log\left[1+(1-p)(e^{\lambda}-1)\right]\right\}^{\frac{1}{\alpha}}$$

$$\Leftrightarrow \quad VaR_p(X) = -\frac{1}{\beta} \log\left[1 - \left\{1 - \frac{1}{\lambda} \log\left[1+(1-p)(e^{\lambda}-1)\right]\right\}^{\frac{1}{\alpha}}\right]$$

Q2, Exam 2016 / 2017 (c)

$$f(x) = C\, g(x) [G(x)]^{a-1} [1 - G(x)]^{b-1}$$

Show that F and G belong to the same domain of attraction.

i) Assume G belongs to Gumbel max domain and show that F also belongs to the Gumbel max domain

ii) Repeat for Frechet max domain

iii) " " Weibull " " .

i) Assume G belongs to the Gumbel max domain. So there exists $\gamma(t) > 0$ such that

$$\lim_{t \to w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x} \quad \cdots (*)$$

$$\lim_{t \to w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$\overset{LH}{=} \lim_{t \to w(F)} \frac{-f(t + x\gamma(t))\,(1 + x\gamma'(t))}{-f(t)}$$

$$= \lim_{t \to w(G)} \frac{f(t + x\gamma(t))\ (1 + x\gamma'(t))}{f(t)}$$

$$= \lim_{t \to w(G)} \frac{g(t + x\gamma(t))}{g(t)} \left[\frac{G(t + x\gamma(t))}{G(t)}\right]^{a-1} \cdot \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b-1} (1 + x\gamma'(t))$$

$$= \lim_{t \to w(G)} \frac{g(t + x\gamma(t))}{g(t)} \cdot (1 + x\gamma'(t)) \cdot \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b-1}$$

$$\overset{\text{LH in reverse}}{=} \lim_{t \to w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} \cdot \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b-1}$$

$$= \lim_{t \to w(G)} \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b}$$

$$\overset{(*)}{=} \left[e^{-x}\right]^{b} = e^{-bx}$$

Hence F also belongs to the Gumbel max domain.

REVISION CLASS

14 DECEMBER

10:00-11:00AM

MATH4/68181

REVISION

CLASS

TUESDAY 11 DEC

4 - 5 PM

Q2 (c) Exam 2017/18

$$f(x) = Ce^{\beta x - \eta e^{bx}}, \quad x > 0$$

$$\omega(F) = +\infty$$

I: $$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$\overset{LH}{=} \lim_{t \to \infty} \frac{-f(t+x\gamma(t))\,(1 + x\gamma'(t))}{-f(t)}$$

$$= \lim_{t \to \infty} \frac{\cancel{C}\, e^{\beta(t+x\gamma(t)) - \eta e^{b(t+x\gamma(t))}} (1 + x\gamma'(t))}{\cancel{C}\, e^{\beta t - \eta e^{bt}}}$$

$$= \lim_{t \to \infty} e^{\beta x \gamma(t)}\, e^{\eta e^{bt} - \eta e^{b(t+x\gamma(t))}} (1 + x\gamma'(t))$$

$$= \lim_{t \to \infty} e^{\beta x \gamma(t)}\, e^{\eta e^{bt}\left[1 - e^{bx\gamma(t)}\right]} (1 + x\gamma'(t))$$

Assume $\gamma(t) \to 0$

$e^z \sim 1 + z$

$$= \lim_{t \to \infty} e^{\beta x \gamma(t)}\, e^{\eta e^{bt}\left[1 - 1 - bx\gamma(t)\right]} (1 + x\gamma'(t))$$

$$= \lim_{t \to \infty} e^{\beta x \gamma(t)}\, e^{-\eta b x e^{bt} \gamma(t)} (1 + x\gamma'(t))$$

$\beta x \gamma(t) \to 0$; $\eta b x e^{bt}\gamma(t) = 1$; $x\gamma'(t) \to 0$

$$\eta b e^{bt} \gamma(t) = 1$$

$$\Rightarrow \gamma(t) = \frac{1}{\eta b e^{bt}} \to 0 \quad \text{as} \quad t \to \infty$$

$$\gamma'(t) = -\frac{1}{\eta b e^{bt}} \to 0 \quad \text{as} \quad t \to \infty$$

$$= \lim_{t \to \infty} e^{0} e^{-x} (1 + 0)$$

$$= e^{-x}$$

Hence, cond I is satisfied and F belongs to the Gumbel domain.

Q3(d) Exam 2015/16

$$f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$$

$$w(F) = \infty$$

I : $$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$\overset{LH}{=} \lim_{t \to \infty} \frac{-f(t + x\gamma(t))\,(1 + x\gamma'(t))}{-f(t)}$$

$$= \lim_{t \to \infty} \frac{\frac{1}{2} e^{-|t + x\gamma(t)|}\,(1 + x\gamma'(t))}{\frac{1}{2} e^{-|t|}}$$

$$= \lim_{t \to \infty} \frac{e^{-t - x\gamma(t)}\,(1 + x\gamma'(t))}{e^{-t}}$$

$$= \lim_{t \to \infty} e^{-x\gamma(t)}\,(1 + x\gamma'(t))$$

$\gamma(t) \equiv 1$

$\gamma'(t) \equiv 0$

$$= e^{-x}$$

Hence, condition I is satisfied and F belongs to the Gumbel max domain.

Q3(e) Exam 2016/17

$$F(x) = 1 - \left[1 - e^{-\frac{1}{x}}\right]^a, \quad x > 0$$

$$w(F) = \infty.$$

II : $\lim_{t \to \infty} \dfrac{1 - F(tx)}{1 - F(t)}$

$$= \lim_{t \to \infty} \frac{1 - \left\{1 - \left[1 - e^{-\frac{1}{tx}}\right]^a\right\}}{1 - \left\{1 - \left[1 - e^{-\frac{1}{t}}\right]^a\right\}}$$

$$= \lim_{t \to \infty} \left[\frac{1 - e^{-\frac{1}{tx}}}{1 - e^{-\frac{1}{t}}}\right]^a$$

$\frac{1}{tx} \to 0$, $\frac{1}{t} \to 0$

$e^{-z} \sim 1 - z$ as $z \to 0$

$$= \lim_{t \to \infty} \left[\frac{1 - \left(1 - \frac{1}{tx}\right)}{1 - \left(1 - \frac{1}{t}\right)}\right]^a$$

$$= x^{-a}$$

Hence, condition II is satisfied and F belongs to the Fréchet max domain.

Q 3 (c) Exam 2016/17

$$f(x) = \frac{1}{(\log b - \log a)\, x}, \quad 0 < a < x < b < \infty$$

$$w(F) = b$$

III : $\lim_{t \to 0} \frac{1 - F(b - tx)}{1 - F(b - t)}$

$\overset{L'H}{=} \lim_{t \to 0} \frac{-f(b - tx) \cdot (-x)}{-f(b - t) \cdot (-1)}$

$= \lim_{t \to 0} \frac{\frac{1}{(\log b - \log a)} \frac{1}{b - tx} \cdot x}{\frac{1}{(\log b - \log a)} \frac{1}{b - t} \cdot}$ → b → b

$= x$

Hence, condition III is satisfied and F belongs to the Weibull max domain.

Q3(a) Exam 2015/16

$$f(x) = C\, x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1$$

$$w(F) = 1$$

III : $$\lim_{t \to 0} \frac{1 - F(1-tx)}{1 - F(1-t)}$$

$$\overset{LH}{=} \lim_{t \to 0} \frac{-f(1-tx)\cdot(-x)}{-f(1-t)\cdot(-1)}$$

$$= \lim_{t \to 0} \frac{f(1-tx)\cdot x}{f(1-t)}$$

$$= \lim_{t \to 0} \frac{C\,(1-tx)^{\alpha-1}(tx)^{\beta-1}\, x}{C\,(1-t)^{\alpha-1}\, t^{\beta-1}}$$

($(1-tx)^{\alpha-1} \to 1$, $(1-t)^{\alpha-1} \to 1$)

$$= x^{\beta-1} \cdot x = x^{\beta}$$

$\Rightarrow$ Cond III is satisfied and F belongs to the Weibull max domain.