

**LECTURE**

**26 SEPTEMBER**

**9:00-10:00AM**

**MATH3/4/68181**

# Extreme Values

## \* Financial Risk

### Lecture times

Tues	9-10 am	3/4/6
Fri <del>Thurs</del>	9-10 am	3/4/6
Thurs	12-1	4/6

### Example

Mon	12-1	3/4/6
Tues	10-11	3/4/6

### Contact

Room : ATB 2.223

Email: \_

office hrs : 2-3 pm Tues

2-3 pm Thurs

# Assessment

20% In-class test 14 Nov

80% Exam Jan 2018

2% Bonus (last 2 weeks  
of Semester I)

## Website

http://www.maths.manchester.ac.uk/  
~sarales / extreme3.html  
Level 3

extreme4.html  
Level 4

extreme6.html  
Level 6

## Credits

10	-	Level 3
15	-	Level 4
15	-	Level 6

## Pre-requisite

- Statistical Methods
- Statistical Inference

## Books

No need



WHAT IS THE USE OF  
THIS COURSE ?

## Answer

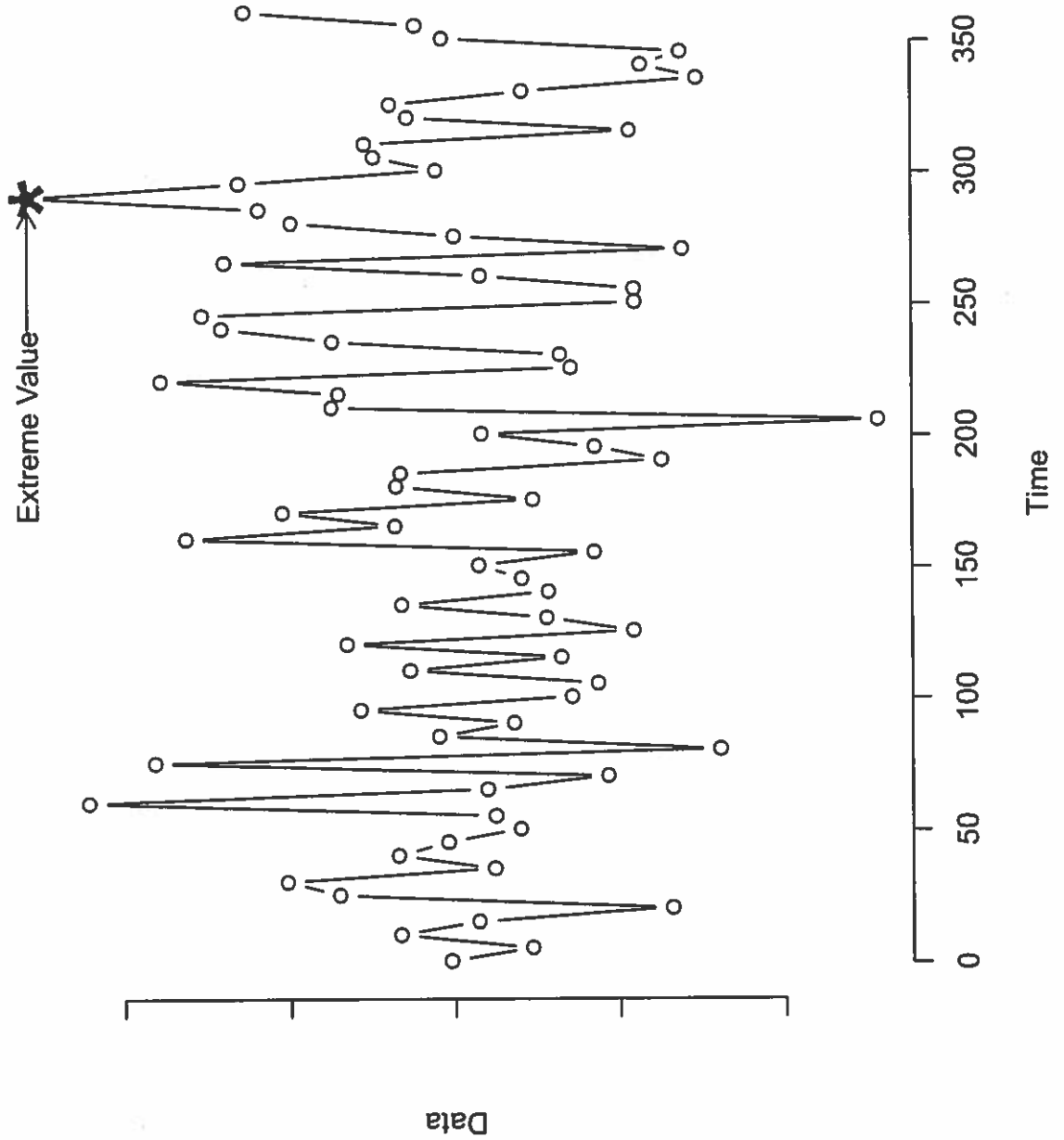
1. To minimize the chance of financial disasters

2. ~~Get~~ Help to catch the crooks.

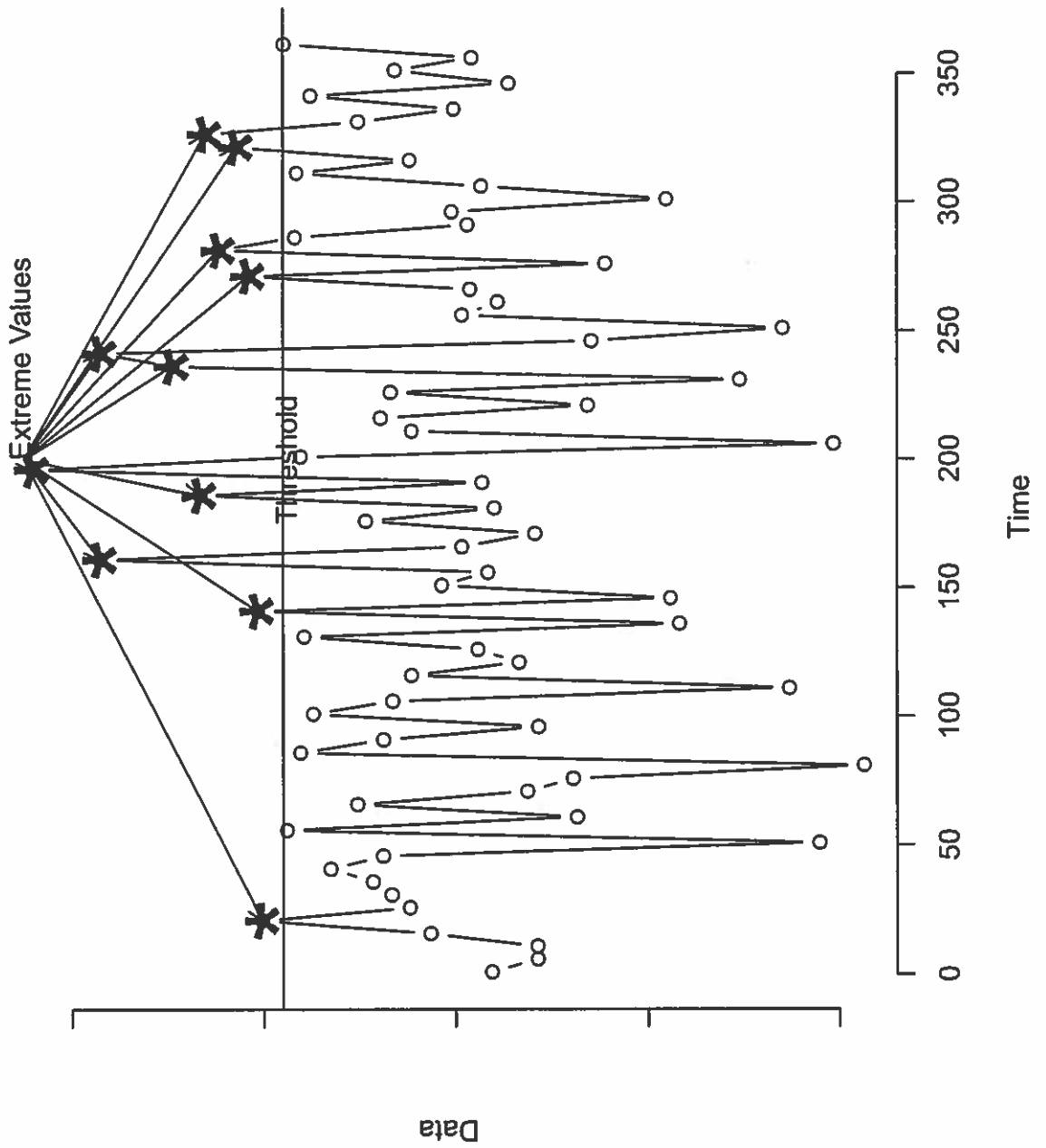
# Extreme Values & Financial Risk

WHAT IS AN EXTREME  
VALUE?

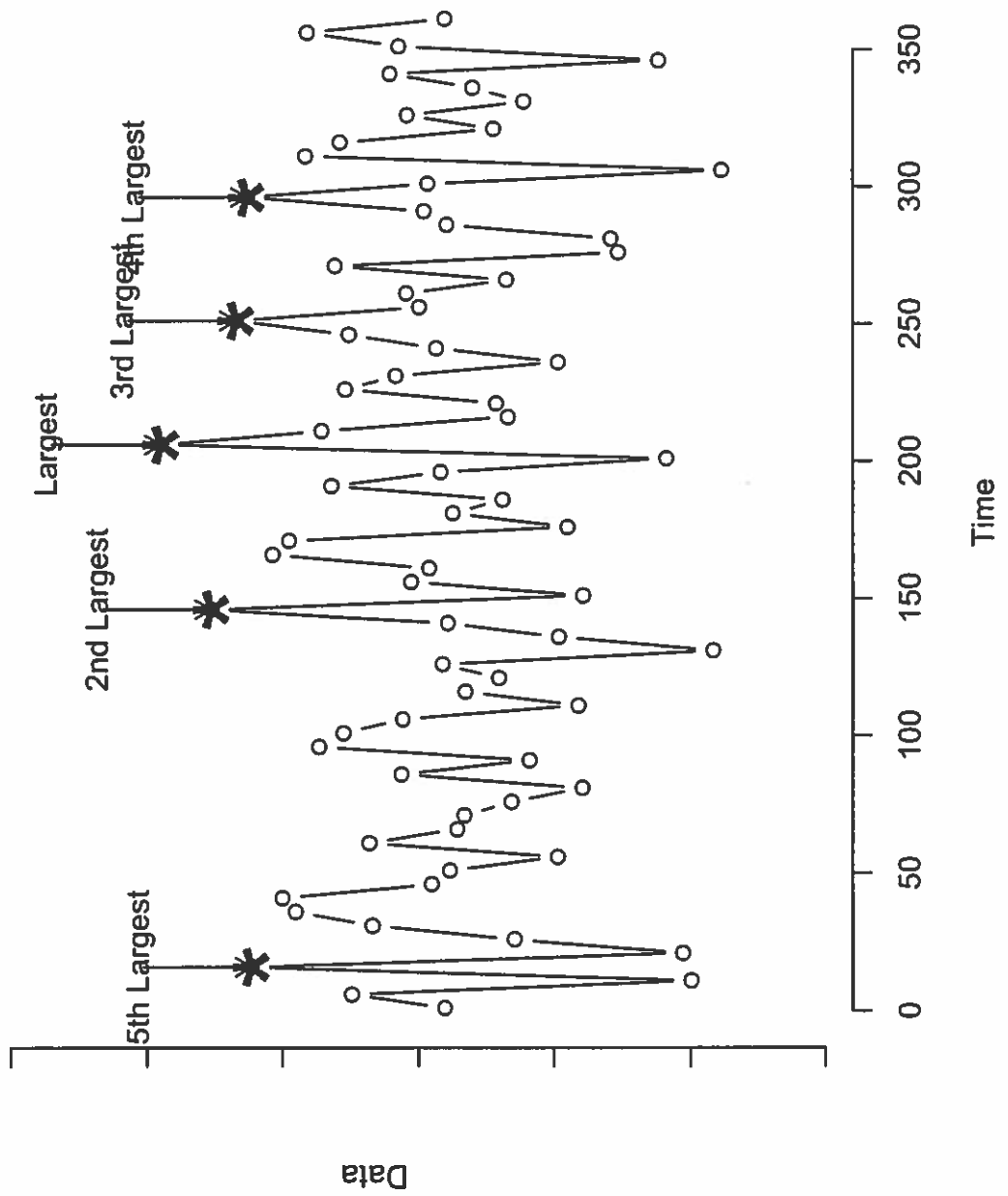
# DEFINITION 1



DEFINITION 2



DEFINITION 3



## Definition 1

Suppose  $X_1, X_2, \dots, X_n$  are IID with CDF  $F(\cdot)$ .

The extreme value

$$M_n = \max(X_1, \dots, X_n)$$

What is the distribution of  $M_n$ ?

$$P(M_n \leq x)$$

$$= P(\max(X_1, \dots, X_n) \leq x)$$

$$= P(X_1 \leq x, \dots, X_n \leq x)$$

indep

$$= P(X_1 \leq x) \cdots P(X_n \leq x)$$

$$= F(x) \cdots F(x)$$

$$= F^n(x)$$

Usually,  $n$  is very large.

We like to know the distribution of the extreme value as  $n \rightarrow \infty$ .

$$\lim_{n \rightarrow \infty} P(M_n \leq x)$$

$$= \lim_{n \rightarrow \infty} F^n(x)$$

$$= \begin{cases} 0 & \text{if } F(x) < 1 \\ 1 & \text{if } F(x) = 1 \end{cases}$$

"Not very helpful"  
for modelling real data.



Suppose  $X_1, X_2, \dots, X_n$   
are IID with mean  $\mu$   
and variance  $\sigma^2$ .

$$\lim_{n \rightarrow \infty} \frac{X_1 + X_2 + \dots + X_n}{n} = \mu$$

SLLN  
(Strong Law of Large Numbers)

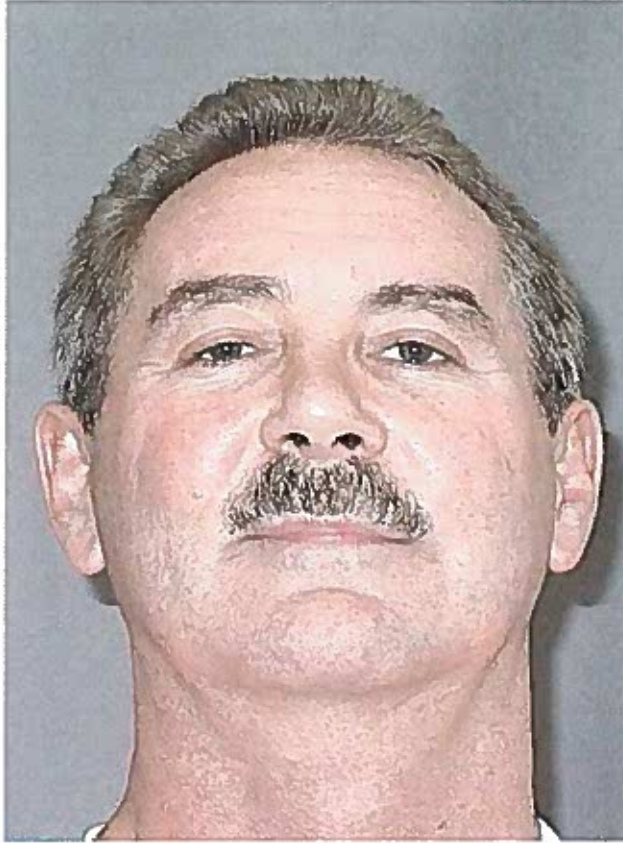
$$\lim_{n \rightarrow \infty} P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq y\right) = \Phi(y)$$

$\Phi$  is the  
CDF of  $N(0, 1)$

CLT

(Central Limit Thm)

$$P \left( \frac{M_n - b_n}{a_n} \leq x \right)$$



# **Allen Stanford**

Stanford is serving a 110 year prison sentence, having been convicted of charges of fraud. Stanford was the chairman of the now defunct Standord Financial Group of companies.



# Jordan Belfort

Belfort founded the brokerage firm Stratton Oakmont, and went on to defraud investors through sales of stock while employing more than 1,000 people. The firm was shut in 1998. Belfort was charged with money laundering and fraud.



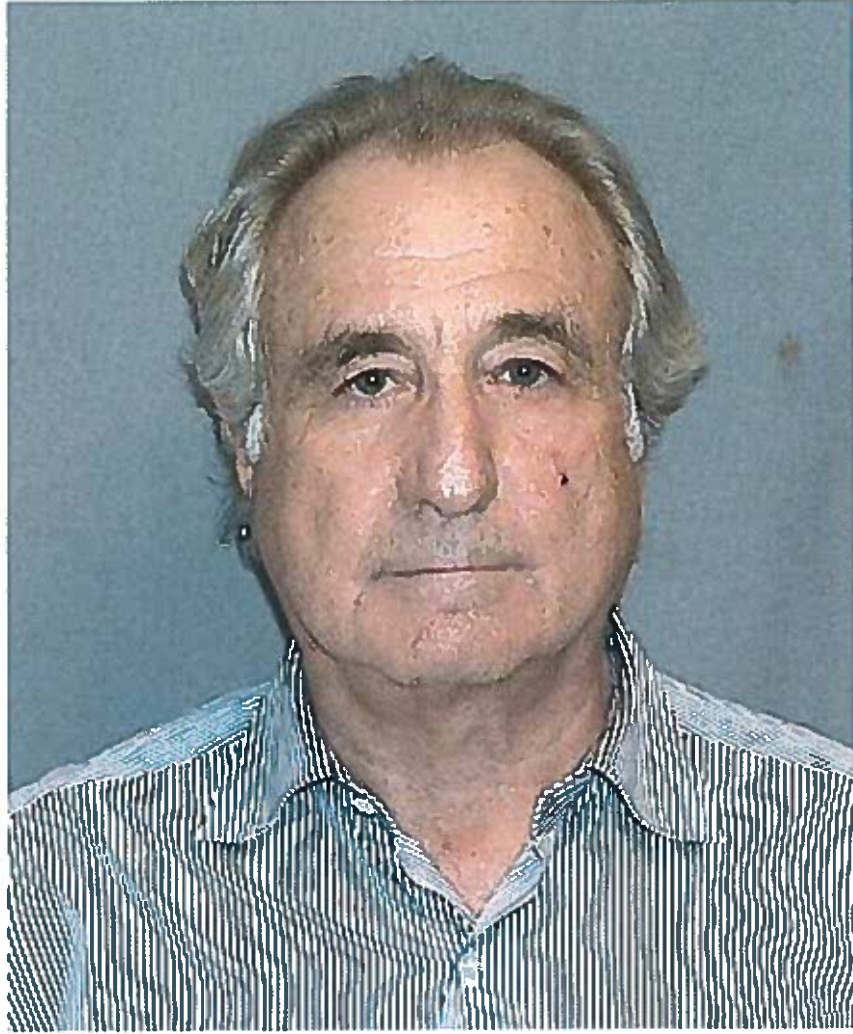
## **Bruno Iksil**

Former trader for JPMorgan Chase & Co and also known as “the London Whale”. His position in the credit default swap market caused more than \$6.2 billion in losses for the firm in 2012.



## **Albert H. Wiggin**

Career spanning from head of Chase National Bank and now JP Morgan chase. He shorted his own company with 40000 shares in 1929 and profited immensely to the tune of \$4 million and wasn't even found guilty of breaking any laws.



# **Bernard Madoff**

Former stockbroker and investment adviser. His name is connected to the most well known Ponzi scheme fraud. He earned \$62 billion through such fraud and is currently sentenced to a maximum of 150 years in prison.





## **Barry Minkow**

Founded ZZZZ Best which appeared to be an immensely successful carpet cleaning and restoration company. However, it was actually a front to attract investment for a massive Ponzi scheme. It collapsed in 1987 costing investors and lenders \$100 million.





# Martha Stewart

Martha is an American businesswoman, writer and television personality. In 2004, Stewart was convicted relating to the ImClone insider trading affair and sentenced to prison.



## **Bernard Ebbers**

A Canadian born businessman and cofounded the telecommunications company WorldCom. In 2005, he was convicted of fraud and conspiracy as a result of WorldCom's false financial reporting. He was sentenced to 25 years for his role in an \$11 billion accounting fraud. It is the largest accounting scandal in US history.



# Donald Johnson

Executive of stock exchange for Nasdaq's. Convicted in 2011 of using insider information to trade shares of United Therapeutics, Honda and other companies from 2006 to 2009. He was sentenced to 3 and half years in prison for a total of \$755,000 of fraud.



## **Doug Whitman**

Managing more than \$100 million through his Whitman Capital hedge fund. He was convicted in August 2012 of insider trading charges involving stocks of Google, Polycom and Marvell Technology Group. The total value of his fraud accumulated to \$1 million and lead to 2 years in prison.



# Raj Rajaratnam

Former head of Galleon Capital. He received the longest insider trading sentence in history of 11 years for securities fraud and five counts of conspiracy. He was ordered to pay back the \$53.8 million of fraud.



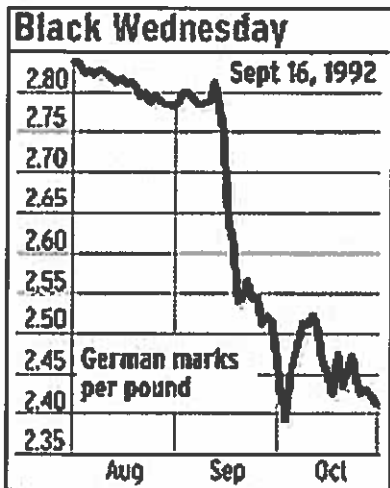
# Jeff Skilling

Jeff was the CEO of Enron Corp. He was convicted of federal felony charges relating to Enron's financial collapse and is currently serving 14 years prison sentence.

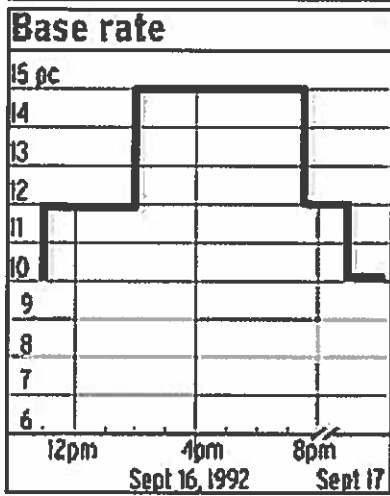


## Ivan Boesky

Boesky is a former American stock trader who is notable for his prominent role in a Wall Street insider trading scandal that occurred in the US in the mid-1980s. In 1987, Boesky was sentenced to 3 years in prison.



Exchange rate of DEM/GBP



UK Interest rate

Hour

Figure 5: Black Wednesday crash of 16 September 1992. Top image shows the exchange rate of Deutsche mark to British pounds. Bottom image shows the UK interest rate on the day.





Figure 4: Asian financial crisis (Asian dollar index) in July 1997. Not fully recovered even in 2011.

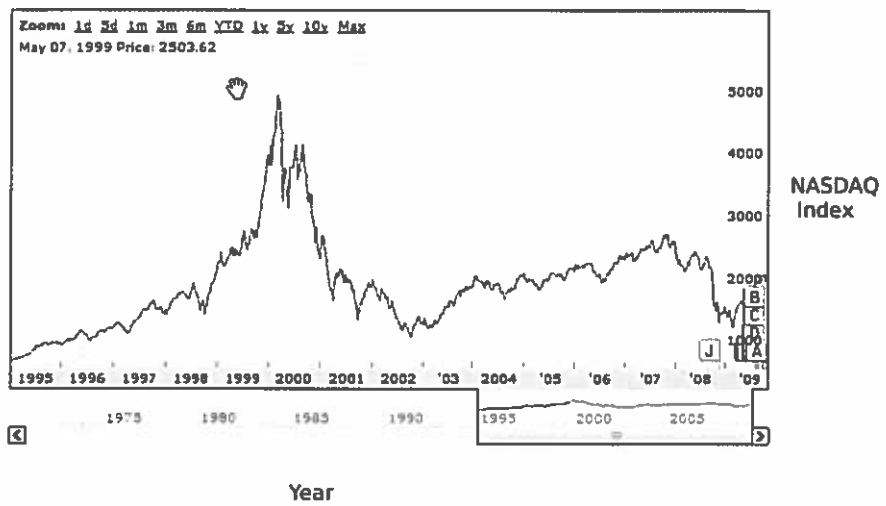


Figure 3: Dot com bubble (the NASDAQ index) during 1999 and 2000. The bubble burst on 10 March 2000. The peak on that day was \$5048.62. There is a recovery after 2002. Never recovered to attain the peak.

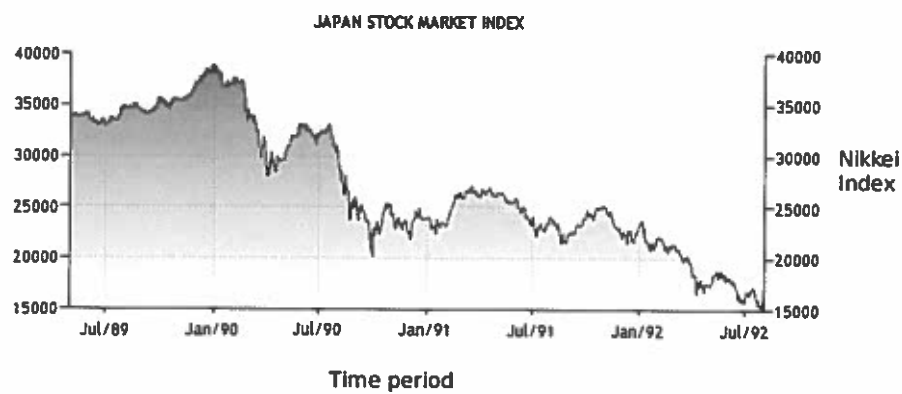


Figure 2: Japan stock price bubble near the end of 1989. A loss of \$2.7 trillion in capital. A recovery happened after mid-1990.

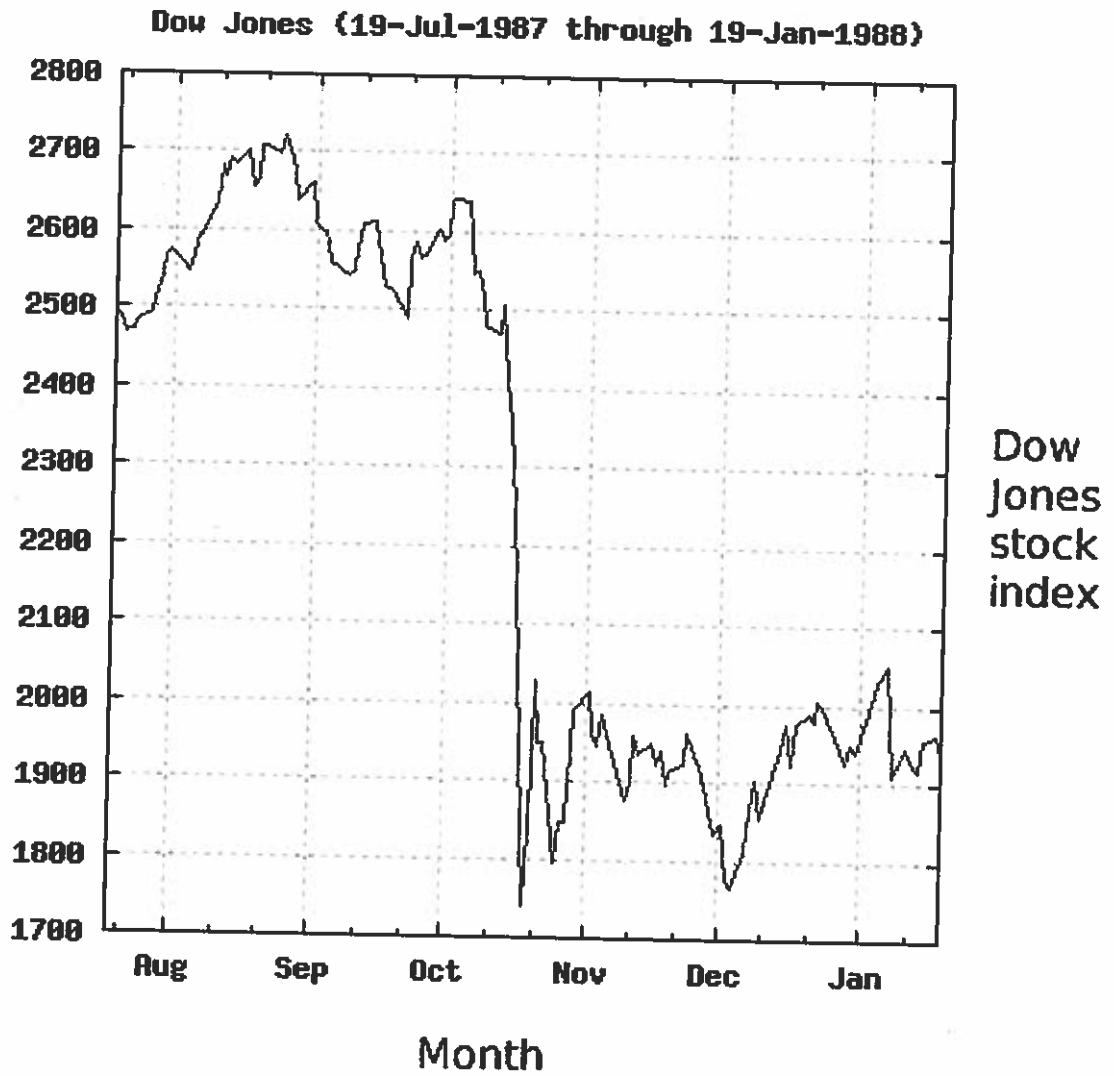


Figure 1: Black Monday crash on 19 October 1987. The Dow Jones stock index crashed down by 22.6 percent (by 508 points). Overall the stock market lost \$0.5 trillion.

**LECTURE**

**28 SEPTEMBER**

**12:00-13:00PM**

**MATH4/68181**

$$\begin{aligned}
& P\left(\frac{M_n - b_n}{a_n} \leq x\right) \\
&= P(M_n \leq b_n + a_n x) \\
&= P(\max(X_1, \dots, X_n) \leq b_n + a_n x) \\
&= P(X_1 \leq b_n + a_n x, \dots, X_n \leq b_n + a_n x) \\
&\stackrel{\text{indep}}{=} P(X_1 \leq b_n + a_n x) \dots P(X_n \leq b_n + a_n x) \\
&= F(b_n + a_n x) \dots F(b_n + a_n x) \\
&= F^n(b_n + a_n x)
\end{aligned}$$

What is

$$\lim_{n \rightarrow \infty} F^n(b_n + a_n x) \quad (*)$$

# Extremal Types Theorem (ETT)

If there exist  $a_n > 0$  and  $b_n \in \mathbb{R}$  such that

$$P\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n(b_n + a_n x)$$

as  $n \rightarrow \infty$  then  $G(x)$  must be of the same type as

I:  $\Lambda(x) = e^{-e^{-x}}, -\infty < x < +\infty$   
"Gumbel"

II:  $\Phi_\alpha(x) = \begin{cases} 0 & x < 0 \\ e^{-x^\alpha} & x \geq 0 \end{cases}$  "Frechet"

III:  $\Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha} & x < 0 \\ 1 & x \geq 0 \end{cases}$  "Weibull"

## Same type

Two cdfs  $G_1$  &  $G_2$  are of the same type if

$$G_1(x) = G_2(ax + b)$$

for all  $x \in \mathbb{R}$ ,  $a > 0$  &  $b \in \mathbb{R}$ .

Eg 1

$$G_1(x) = e^{-e^{-x}}$$

$$G_2(x) = e^{-e^{-(2x+3)}}$$

$G_1$  &  $G_2$  are of the same type.

Eg 2

$$G_1(x) = e^{-x^{-1}}$$

$$G_2(x) = e^{-(-3x)^{-1}}$$

$G_1$  &  $G_2$  are not of the same type.



Which limit will be attained if there is a limit?

Suppose  $X_1, \dots, X_n$  are IID with CDF  $F$ . Let  $M_n = \max(X_1, \dots, X_n)$ .

$$\text{Let } a(t) = F^{-1}\left(1 - \frac{1}{t}\right)$$

$$b(t) = t f(a(t)), \quad f(x) = \frac{dF(x)}{dx}$$

Then

$$\text{I: } P\left(\frac{M_n - b_n}{a_n} \leq x\right) \rightarrow \Lambda(x) \text{ if}$$

$$\frac{b(tx)}{b(t)} \rightarrow 1 \text{ as } t \rightarrow \infty$$

$$\text{II: } P\left(\frac{M_n - b_n}{a_n} \leq \infty\right) \rightarrow \Phi_\alpha(x) \text{ if}$$

$$w(F) = \infty, \quad \lim_{t \rightarrow \infty} a(t)b(t) = \alpha$$

$$\text{III: } P\left(\frac{M_n - b_n}{a_n} \leq x\right) \rightarrow \Psi_\alpha(x) \text{ if}$$

$$w(F) < \infty, \quad \lim_{t \rightarrow \infty} \{w(F) - a(t)\} b(t) = \alpha$$

What is  $w(F)$ ?

$w(F)$  = the upper end point of  $F$ .

$$w(F) = \sup \{x : F(x) < 1\}$$

Set  $F(x) = 1$  and solve for  $x$ .  
The solution is  $w(F)$ .

eg 1  $F(x) = 1 - e^{-x}$

$$F(x) = 1 \Rightarrow 1 - e^{-x} = 1$$

$$\Rightarrow e^{-x} = 0 \Rightarrow x = +\infty$$

$$\Rightarrow w(F) = +\infty$$

eg 2  $F(x) = x, \quad 0 < x < 1$

$$F(x) = 1 \Rightarrow x = 1 \Rightarrow w(F) = 1$$

eg 3  $F(x) = 1 - \frac{1}{x}, \quad x > 1$

$$F(x) = 1 \Rightarrow 1 - \frac{1}{x} = 1$$

$$\Rightarrow \frac{1}{x} = 0 \Rightarrow x = +\infty$$

$$\Rightarrow w(F) = +\infty$$

eg 1

$$F(x) = 1 - e^{-x}, \quad f(x) = e^{-x}$$

$$W(F) = +\infty$$

$$1 - e^{-x} = y \Rightarrow e^{-x} = 1 - y$$

$$\Rightarrow x = -\log(1-y)$$

$$F^{-1}(x) = -\log(1-x)$$

$$\begin{aligned} a(t) &= F^{-1}\left(1 - \frac{1}{t}\right) \\ &= -\log\left(1 - \left(1 - \frac{1}{t}\right)\right) \\ &= \log t \end{aligned}$$

$$\begin{aligned} b(t) &= t f(a(t)) = t \cdot e^{-\log t} \\ &= t \cdot \frac{1}{t} = 1 \end{aligned}$$

$$\frac{b(tx)}{b(t)} = \frac{1}{1} = 1$$

Hence, (I) is satisfied.

There exist  $a_n > 0$ ,  $b_n \in \mathbb{R}$   
such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \rightarrow e^{-e^{-x}}$$

as  $n \rightarrow \infty$ .

eg 2

$$\begin{aligned} F(x) &= 1 - \frac{1}{x}, \quad x > 1 \\ f(x) &= \frac{1}{x^2} \\ w(F) &= +\infty \end{aligned}$$

$$\begin{aligned} F(x) = y &\Rightarrow 1 - \frac{1}{x} = y \\ \Rightarrow \frac{1}{x} &= 1 - y \Rightarrow x = \frac{1}{1-y} \\ \Rightarrow F^{-1}(x) &= \frac{1}{1-x} \end{aligned}$$

$$\begin{aligned} a(t) &= F^{-1}\left(1 - \frac{1}{t}\right) \\ &= \frac{1}{1 - \left(1 - \frac{1}{t}\right)} = t \end{aligned}$$

$$b(t) = t \cdot f(a(t)) = t \cdot \frac{1}{t^2} = \frac{1}{t}$$

$$\text{I: } \frac{b(tx)}{b(t)} = \frac{\frac{1}{tx}}{\frac{1}{t}} = \frac{1}{x} \neq 1$$

$\Rightarrow$  (I) not satisfied

$$\text{II: } w(F) = \infty \quad \checkmark$$

$$\lim_{t \rightarrow \infty} a(t) \cdot b(t) = \lim_{t \rightarrow \infty} t \cdot \frac{1}{t} = 1$$

if  $\alpha = 1$

$\Rightarrow$  (II) is satisfied with  $\alpha = 1$ .

Hence, there exists  $a_n > 0$  &  $b_n \in \mathbb{R}$  such that

$$P\left(\frac{M_n - b_n}{a_n}\right) \rightarrow e^{-x^{-1}} \quad \text{as } n \rightarrow \infty.$$

## Max-stability

A cdf  $G$  is said to be max-stable if there exist  $\alpha_n > 0$  and  $\beta_n \in \mathbb{R}$  such that

$$G^n(x) = G(\alpha_n x + \beta_n).$$

$\Lambda(x)$ ,  $\Phi_\alpha(x)$  and  $\Psi_\alpha(x)$  are all max-stable cdfs.

Proof  $\Lambda(x)$  is a max-stable cdf

Take  $\alpha_n = 1$ ,  $\beta_n = -\log n$ .

Then

$$\begin{aligned} & G(1 \cdot x - \log n) \\ &= e - e^{-(x - \log n)} \\ &= e - e^{-x + \log n} \\ &= e - e^{-x} \cdot e^{\log n} \\ &= e - e^{-x} \cdot n \\ &= \left[ e - e^{-x} \right]^n = G^n(x). \end{aligned}$$

Home work :

Prove that

$$\Phi_{\alpha}(x)$$

and  $\Psi_{\alpha}(x)$

are max-stable cdfs.

**LECTURE**

**29 SEPTEMBER**

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**9:00-10:00AM**

**MATH3/4/68181**

Suppose  $X_1, X_2, \dots, X_n$   
are IID with mean  $\mu$   
and variance  $\sigma^2$ .

$$\lim_{n \rightarrow \infty} \frac{X_1 + X_2 + \dots + X_n}{n} = \mu$$

SLLN  
(Strong Law of Large Numbers)

$$\lim_{n \rightarrow \infty} P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq y\right) = \Phi(y)$$

$\Phi$  is the  
CDF of  $N(0, 1)$

CLT

(Central Limit Thm)



Usually,  $n$  is very large.

We like to know the distribution of the extreme value as  $n \rightarrow \infty$ .

$$\begin{aligned} & \lim_{n \rightarrow \infty} P(M_n \leq x) \\ &= \lim_{n \rightarrow \infty} F^n(x) \\ &= \begin{cases} 0 & \text{if } F(x) < 1 \\ 1 & \text{if } F(x) = 1 \end{cases} \end{aligned}$$

"Not very helpful"  
for modelling real data.

$$\begin{aligned}
& P\left(\frac{M_n - b_n}{a_n} \leq x\right) \\
&= P(M_n \leq a_n x + b_n) \\
&= P(\max(X_1, \dots, X_n) \leq a_n x + b_n) \\
&= P(X_1 \leq a_n x + b_n, \dots, X_n \leq a_n x + b_n)
\end{aligned}$$

$$\begin{aligned}
& \stackrel{\text{indep}}{=} P(X_1 \leq a_n x + b_n) \cdots P(X_n \leq a_n x + b_n) \\
&= F(a_n x + b_n) \cdots F(a_n x + b_n) \\
&= F^n(a_n x + b_n)
\end{aligned}$$

$$\lim_{n \rightarrow \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right)$$

$$= \lim_{n \rightarrow \infty} F^n(a_n x + b_n) \quad (*)$$

# Extremal Types Theorem (ETT)

If (\*) exists it must be of the same type as

$$I: \Lambda(x) = e^{-e^{-x}}, \quad -\infty < x < \infty$$

"Gumbel"

$$II: \Phi_{\alpha}(x) = \begin{cases} e^{-x^{\alpha}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

"Fréchet"

$$III: \Psi_{\alpha}(x) = \begin{cases} e - (-x)^{\alpha}, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

"Weibull"

The same type

Two cdfs  $G_1$  and  $G_2$  are of the same type if

$$G_1(x) = G_2(ax + b)$$

for all  $x$ , where  $a > 0, b \in \mathbb{R}$ .

Eg 1

$$G_1(x) = e^{-e^{-x}}$$

$$G_2(x) = e^{-e^{-(x+3)}}$$

Clearly,  $G_1$  &  $G_2$  are of the same type.

Eg 2

$$G_1(x) = e^x$$

$$G_2(x) = e^{-2x+3}$$

$G_1$  and  $G_2$  are not of the same type.

Suppose  $X_1, \dots, X_n$  are IID with cdf  $F$ . Let  $M_n = \max(X_1, \dots, X_n)$ .  
How to know which limit

$P\left(\frac{M_n - b_n}{a_n} \leq x\right)$  will approach to ?

## Conditions

$$I: P\left(\frac{M_n - b_n}{a_n} < x\right) \rightarrow \Lambda(x)$$

if there exists  $\gamma(t) > 0$

such that  $\lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = e^{-x}$ .

$$II: P\left(\frac{M_n - b_n}{a_n} < x\right) \rightarrow \Phi_\alpha(x)$$

if  $w(F) = \infty$ ,  $\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}$

$$III: P\left(\frac{M_n - b_n}{a_n} < x\right) \rightarrow \Psi_\alpha(x)$$

if  $w(F) < \infty$ ,  $\lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^{-\alpha}$

If any only one of (I)-(III) will be satisfied

What is  $w(F)$

$w(F)$  = "upper end point" of  $F$

$$w(F) = \sup \{ x : F(x) < 1 \}$$

Easy way: set  $F(x) = 1$   
solve for  $x$   
the solution is  $w(F)$

eg 1

$$F(x) = 1 - e^{-x}$$

$$F(x) = 1 \Rightarrow 1 - e^{-x} = 1$$

$$\Rightarrow e^{-x} = 0 \Rightarrow x = +\infty$$

$$w(F) = +\infty$$

eg 2

$$F(x) = 1 - \frac{1}{x}, \quad x > 1$$

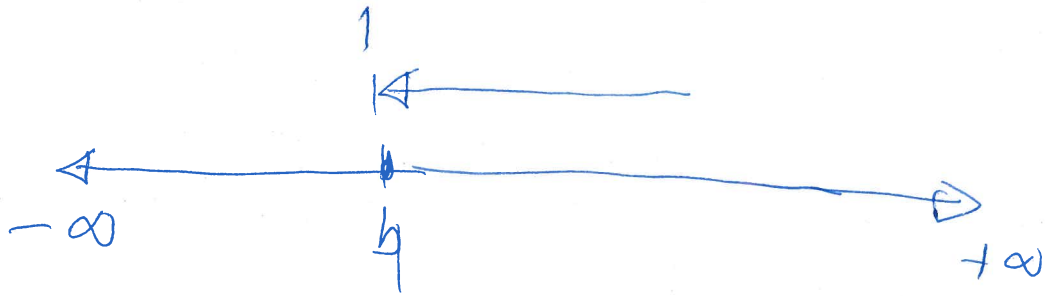
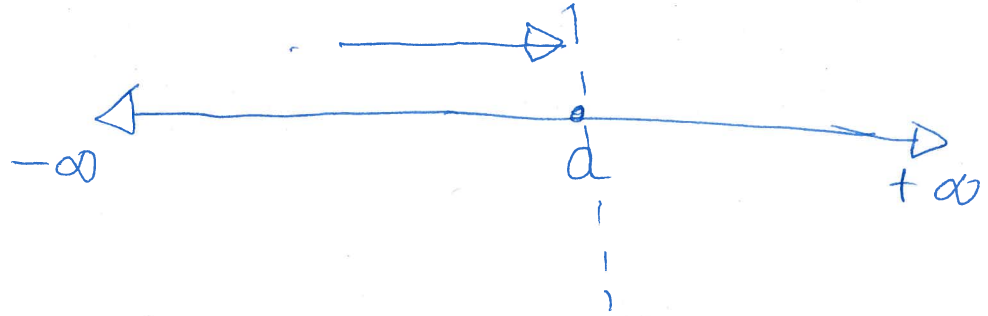
$$F(x) = 1 \Rightarrow 1 - \frac{1}{x} = 1$$

$$\Rightarrow \frac{1}{x} = 0 \Rightarrow x = +\infty$$

$$w(F) = +\infty$$

limit

arrows





eg 1

$$F(x) = 1 - e^{-x}$$

$$W(F) = +\infty$$

$$\lim_{t \uparrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{1 - [1 - e^{-t - x\gamma(t)}]}{1 - [1 - e^{-t}]}$$

$$= \lim_{t \uparrow \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t \uparrow \infty} e^{-x\gamma(t)}$$

$$= e^{-x} \quad \text{if } \gamma(t) \equiv 1$$

Condition (I) is satisfied.

Hence, there exist  $a_n > 0$  &  $b_n \in \mathbb{R}$  such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \rightarrow e^{-e^{-x}}$$

as  $n \rightarrow \infty$ .

eg 2

$$F(x) = 1 - \frac{1}{x}, \quad x > 1$$

$$w(F) = +\infty$$

$$I: \lim_{t \uparrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{1 - \left[ 1 - \frac{1}{t + x\gamma(t)} \right]}{1 - \left[ 1 - \frac{1}{t} \right]}$$

$$= \lim_{t \uparrow \infty} \frac{t}{t + x\gamma(t)}$$

$\neq e^{-x} \Rightarrow$  Condition (I) not satisfied

$$II: \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{1 - \left( 1 - \frac{1}{tx} \right)}{1 - \left( 1 - \frac{1}{t} \right)}$$

$$= \lim_{t \uparrow \infty} \frac{t}{tx} = \frac{1}{x} \Rightarrow \text{Condition (II) is satisfied with } \alpha = 1.$$

There exists  $a_n > 0, b_n \in \mathbb{R}$  such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \rightarrow \begin{cases} e^{-x^{-1}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

How to choose  $a_n$  and  $b_n$  ?

$$\text{I} : a_n = \gamma \left( F^{-1} \left( 1 - \frac{1}{n} \right) \right)$$

$$b_n = F^{-1} \left( 1 - \frac{1}{n} \right)$$

$$\text{II} : a_n = F^{-1} \left( 1 - \frac{1}{n} \right)$$

$$b_n = 0$$

$$\text{III} : a_n = w(F) - F^{-1} \left( 1 - \frac{1}{n} \right)$$

$$b_n = w(F)$$

eg 1 (contd)

$$F(x) = 1 - e^{-x}$$

$$1 - e^{-x} = y$$

$$\Rightarrow e^{-x} = 1 - y$$

$$\Rightarrow x = -\log(1 - y)$$

$$\Rightarrow F^{-1}(x) = -\log(1 - x)$$

$$\begin{cases} a_n = \gamma \left( -\log \left( 1 - \left( 1 - \frac{1}{n} \right) \right) \right) \\ b_n = -\log \left( 1 - \left( 1 - \frac{1}{n} \right) \right) \end{cases}$$

$$\Rightarrow \begin{cases} a_n = \gamma \left( -\log \left( \frac{1}{n} \right) \right) \\ b_n = -\log \left( \frac{1}{n} \right) \end{cases}$$

$$\Rightarrow \begin{cases} a_n = \gamma \left( \log n \right) = 1 \\ b_n = \log n \end{cases}$$

Hence,

$$P \left( \frac{M_n - \log n}{1} \leq x \right) \rightarrow e^{-e^{-x}}$$

as  $n \rightarrow \infty$ .

eg 2 (contd)

$$F(x) = 1 - \frac{1}{x}$$

$$1 - \frac{1}{x} = y \Rightarrow \frac{1}{x} = 1 - y$$

$$\Rightarrow x = \frac{1}{1-y}$$

$$\Rightarrow F^{-1}(x) = \frac{1}{1-x}$$

$$a_n = F^{-1}\left(1 - \frac{1}{n}\right) = \frac{1}{1 - \left(1 - \frac{1}{n}\right)} = n$$

$$b_n = 0$$

Hence,  $P\left(\frac{M_n - 0}{n} < x\right)$

$$\rightarrow \begin{cases} e^{-x^{-1}} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

# **EXAMPLE CLASS**

**2 OCTOBER**

**12:00-13:00PM**

**MATH3/4/68181**

Q1

$$\Lambda(x) = e^{-e^{-x}}$$

$$\begin{aligned} \frac{d\Lambda(x)}{dx} &= e^{-e^{-x}} (-1) e^{-x} (-1) \\ &= e^{-e^{-x}} \cdot e^{-x} \end{aligned}$$

$$\bar{\Phi}_\alpha(x) = \begin{cases} e^{-x^{-\alpha}} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

$$\frac{d\bar{\Phi}_\alpha(x)}{dx} = \begin{cases} e^{-x^{-\alpha}} (-1) (-\alpha) x^{-\alpha-1} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

$$= \begin{cases} \frac{\alpha e^{-x^{-\alpha}} x^{-\alpha-1}}{1} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

$$\Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha} & , x \leq 0 \\ 1 & , x > 0 \end{cases}$$

$$\frac{d\Psi_\alpha(x)}{dx} = \begin{cases} e^{-(-x)^\alpha} (-1) \alpha (-x)^{\alpha-1} (-1) & , x \leq 0 \\ 0 & , x > 0 \end{cases}$$

Q2

$$E(x) = \int_{-\infty}^{\infty} x \cdot \underbrace{e^{-e^{-x}} e^{-x}}_{\text{pdf}} dx$$

$$\boxed{\begin{aligned} y = e^{-x} &\Rightarrow x = -\log y \\ &\Rightarrow \frac{dx}{dy} = -\frac{1}{y} \end{aligned}}$$

$$= \int_{-\infty}^0 (-\log y) \cdot e^{-y} y \left(-\frac{1}{y}\right) dy$$

$$= \int_0^{\infty} (\log y) \cdot e^{-y} dy$$

$$= - \int_0^{\infty} \left( \frac{d}{da} y^a \Big|_{a=0} \right) \cdot e^{-y} dy$$

$$= - \frac{d}{da} \left( \int_0^{\infty} y^a e^{-y} dy \right) \Big|_{a=0}$$

$$= - \frac{d}{da} \Gamma(a+1) \Big|_{a=0}$$

$$= - \Gamma'(1)$$



$$\frac{d}{da} y^a = y^a \log y$$

$$\boxed{a=0} : \left. \frac{d}{da} y^a \right|_{a=0} = y^0 \log y = \log y$$

$$\Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} dt$$

Gamma Function

$$\begin{aligned}\frac{d^2}{da^2} y^a &= \frac{d}{da} (y^a \log y) \\ &= y^a (\log y)^2\end{aligned}$$

$$\boxed{a=0} : \frac{d^2}{da^2} y^a \Big|_{a=0} = y^0 (\log y)^2 \\ = (\log y)^2$$

Q3

$$\text{Var}[X] = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot e^{-x} \cdot e^{-x} dx$$

$$\begin{aligned} y = e^{-x} &\Rightarrow x = -\log y \\ &\Rightarrow \frac{dx}{dy} = -\frac{1}{y} \end{aligned}$$

$$= \int_{-\infty}^0 (\log y)^2 e^{-y} y \left(-\frac{1}{y}\right) dy$$

$$= \int_0^{\infty} \underline{(\log y)^2} e^{-y} dy$$

$$= \int_0^{\infty} \left( \frac{d^2}{da^2} y^a \Big|_{a=0} \right) \cdot e^{-y} dy$$

$$= \frac{d^2}{da^2} \left( \int_0^{\infty} y^a e^{-y} dy \right) \Big|_{a=0}$$

$$= \frac{d^2}{da^2} \Gamma(a+1) \Big|_{a=0}$$

$$= \Gamma''(1)$$

$$\text{Var}[X] = \Gamma''(1) - [\Gamma'(1)]^2$$

Q2

$$F(x) = [1 - e^{-x}]^\alpha$$

$$F(x) = 1 \Rightarrow [1 - e^{-x}]^\alpha = 1$$

$$\Rightarrow 1 - e^{-x} = 1$$

$$\Rightarrow e^{-x} = 0 \Rightarrow x = +\infty$$

$$\Rightarrow w(F) = +\infty$$

$$I: \lim_{t \uparrow \infty} \frac{1 - [1 - e^{-t - x\delta(t)}]^\alpha}{1 - [1 - e^{-t}]^\alpha}$$

$$(1-a)^\alpha \approx 1 - a\alpha \text{ as } a \rightarrow 0$$

$$= \lim_{t \uparrow \infty} \frac{1 - [1 - \alpha e^{-t - x\delta(t)}]}{1 - [1 - \alpha e^{-t}]}$$

$$= \lim_{t \uparrow \infty} \frac{e^{-t - x\delta(t)}}{e^{-t}}$$

$$= \lim_{t \uparrow \infty} e^{-x\delta(t)} = e^{-x}$$

if  $\delta(t) \equiv 1$

$\Rightarrow$  Cond (I) is satisfied. There exists  $a_n > 0$  &  $b_n \in \mathbb{R}$  such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \rightarrow e^{-e^{-x}}$$

Q9

$$F(x) = x, \quad 0 < x < 1$$

$$w(F) = 1$$

$$\lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)}$$

$$= \lim_{t \downarrow 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)}$$

$$= \lim_{t \downarrow 0} \frac{x - (x - tx)}{x - (x - t)} = \lim_{t \downarrow 0} \frac{tx}{t}$$

$$= x$$

and (III) is satisfied with  $\alpha = 1$ .

There exists  $a_n > 0$  &  $b_n \in \mathbb{R}$  such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \rightarrow \begin{cases} e^{-x} & x \leq 0 \\ 1 & x > 0 \end{cases}$$

**LECTURE**

**3 OCTOBER**

**9:00-10:00AM**

**MATH3/4/68181**

# More Examples on ETT

eg 1  
max  $F(x) = x e^{-\frac{1}{x}}$ ,  $x > 0$ . Find the domain of attraction.

///

$$F(x) = 1$$

$$\Rightarrow e^{-\frac{1}{x}} = 1$$

$$\Rightarrow -\frac{1}{x} = 0$$

$$\Rightarrow x = +\infty \Rightarrow w(F) = +\infty.$$

$$I: \lim_{t \uparrow \infty} \frac{1 - e^{-\frac{1}{t+x\gamma(t)}}}{1 - e^{-\frac{1}{t}}} \rightarrow 0$$

$e^{-y} \approx 1 - y$   
as  $y \rightarrow 0$

$$= \lim_{t \uparrow \infty} \frac{x - \left[ x - \frac{1}{t+x\gamma(t)} \right]}{x - \left[ x - \frac{1}{t} \right]}$$

$$= \lim_{t \uparrow \infty} \frac{t}{t+x\gamma(t)}$$

$\neq e^{-x} \Rightarrow$  cond (I) not satisfied

$$\text{II} : \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{1 - e^{-\frac{1}{tx}}}{1 - e^{-\frac{1}{t}}}$$

$$= \lim_{t \uparrow \infty} \frac{\cancel{x} - \left(\cancel{x} - \frac{1}{tx}\right)}{\cancel{x} - \left(\cancel{x} - \frac{1}{t}\right)}$$

$$= \frac{1}{x}$$

$$e^{-y} \approx 1 - y \text{ as } y \rightarrow 0$$

$\Rightarrow$  cond (II) satisfied with  $\alpha = 1$

There exists  $a_n > 0$  &  $b_n \in \mathbb{R}$  such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \rightarrow \begin{cases} e^{-x^{-1}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

as  $n \rightarrow \infty$ .



## L' Hôpital's Rule

$$\lim_{t \rightarrow \infty} \frac{f_1(x)}{f_2(x)} = \lim_{t \rightarrow \infty} \frac{f_1'(x)}{f_2'(x)}$$

eg 2 Find the max domain of attraction of  $F(x) = \Phi(x)$ , the cdf of  $N(0, 1)$ .

$$\begin{aligned}
 F(x) = 1 &\Rightarrow \Phi(x) = 1 \\
 &\Rightarrow x = \Phi^{-1}(1) = +\infty. \\
 &\Rightarrow w(F) = +\infty.
 \end{aligned}$$

$$(I) : \lim_{t \uparrow \infty} \frac{1 - \Phi(t + x\gamma(t))}{1 - \Phi(t)}$$

$$\stackrel{LH}{=} \lim_{t \uparrow \infty} \frac{-\phi(t + x\gamma(t)) (1 + x\gamma'(t))}{-\phi(t)}$$

$\phi$  is the PDF of  $N(0, 1)$

$$= \lim_{t \uparrow \infty} \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(t + x\gamma(t))^2}{2}} (1 + x\gamma'(t))}{\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}}$$

$$= \lim_{t \uparrow \infty} e^{\frac{t^2}{2} - \frac{(t + x\gamma(t))^2}{2}} (1 + x\gamma'(t))$$

$$= \lim_{t \uparrow \infty} e^{-xt\gamma(t) - \frac{x^2\gamma^2(t)}{2}} (1 + x\gamma'(t))$$

Choose  $\gamma(t) = \frac{1}{t}$ ,  $\gamma'(t) = -\frac{1}{t^2}$

$$= \lim_{t \uparrow \infty} e^{-x} e^{-\frac{x^2}{2t^2}} \left(1 - \frac{x}{t^2}\right)$$

$$= e^{-x}$$

$\Rightarrow$  cond (I) satisfied

Hence there exist  $a_n > 0$  &  $b_n \in \mathbb{R}$   
such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \rightarrow e^{-e^{-x}}$$

as  $n \rightarrow \infty$ .

# Example where ETT fails

$$F(x) = 1 - \frac{1}{\log x}, \quad x > e$$

$$F(x) = 1 \quad \Rightarrow \quad 1 - \frac{1}{\log x} = 1$$

$$\Rightarrow -\frac{1}{\log x} = 0 \quad \Rightarrow \quad x = +\infty \quad \Rightarrow \quad w(F) = +\infty$$

$$(I): \quad \lim_{t \uparrow \infty}$$

$$\frac{1 - F(t + x\delta(t))}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{1 - \left[ 1 - \frac{1}{\log(t + x\delta(t))} \right]}{1 - \left[ 1 - \frac{1}{\log t} \right]}$$

$$= \lim_{t \uparrow \infty} \frac{\log t \rightarrow +\infty}{\log(t + x\delta(t)) \rightarrow +\infty}$$

$$\stackrel{L'H}{=} \lim_{t \uparrow \infty}$$

$$\frac{\frac{1}{t}}{\frac{1 + x\delta'(t)}{t + x\delta(t)}}$$

$$= \lim_{t \uparrow \infty} \frac{t + x\delta(t)}{t(1 + x\delta'(t))} \neq e^{-x}$$

Cond (I) not satisfied

$$(II) : \lim_{t \uparrow \infty} \frac{1 - \left[ 1 - \frac{1}{\log(tx)} \right]}{1 - \left[ 1 - \frac{1}{\log t} \right]}$$

$$= \lim_{t \uparrow \infty} \frac{\log t}{\log(tx)}$$

$$= \lim_{t \uparrow \infty} \frac{\log t}{\log t + \log x}$$

$$= \lim_{t \uparrow \infty} \frac{1}{1 + \frac{\log x}{\log t}} \rightarrow 0$$

$= 1 \Rightarrow$  cond (II) not satisfied

(III) : clearly not satisfied  
since  $W(F) = +\infty$

Hence ETT fails.

Ex 11

$$f(x) = k g(x) [G(x)]^{a-1} [1-G(x)]^{b-1} e^{-cG(x)}$$

- (a) Assume  $G$  belongs to Gumbel limit  
& show  $F$  also " " " "
- (b) Assume  $G$  belongs to Frechet limit  
& show  $F$  also " " " "
- (c) Assume  $G$  belongs to Weibull limit  
& show  $F$  also " " " "

(a) Assume  $G$  belongs to Gumbel domain. That is, there exists  $\delta(t) > 0$  such that

$$\lim_{t \uparrow w(G)} \frac{1 - G(t + x\delta(t))}{1 - G(t)} = e^{-x} \quad (*)$$

We need to find

$$\lim_{t \uparrow w(F)} \frac{1 - F(t + x\delta(t))}{1 - F(t)}$$

$$\stackrel{LH}{=} \lim_{t \uparrow w(F)} \frac{-f(t + x\delta(t)) (1 + x\delta'(t))}{-f(t)} \quad \left( \frac{(1+x\delta'(t))}{b-1} \right)$$

$$= \lim_{t \uparrow w(F)} \frac{g(t + x\delta(t)) [G(t + x\delta(t))]^{a-1} [1 - G(t + x\delta(t))]^{b-1} e^{-cG(t + x\delta(t))}}{g(t) [G(t)]^{a-1} [1 - G(t)]^{b-1} e^{-cG(t)}}$$

$$w(F) = w(G) = \lim_{t \uparrow w(G)} \frac{g(t + x\delta(t)) (1 + x\delta'(t))}{g(t)}$$

- $\frac{G(t + x\delta(t))^{a-1}}{G(t)^{a-1}} \rightarrow 1$

- $\left[ \frac{1 - G(t + x\delta(t))}{1 - G(t)} \right]^{b-1} \rightarrow 1$

- $e^{-cG(t)} \rightarrow e^{-cG(t + x\delta(t))}$

$$= \lim_{t \uparrow w(G)} \frac{g(t+x\delta(t))(1+x\delta'(t))}{g(t)}$$

$$\cdot \left[ \frac{1-G(t+x\delta(t))}{1-G(t)} \right]^{b-1}$$

$$\stackrel{(*)}{=} \lim_{t \uparrow w(G)} \frac{g(t+x\delta(t))(1+x\delta'(t))}{g(t)} \cdot (e^{-x})^{b-1}$$

$$\stackrel{LH}{=} \lim_{t \uparrow w(G)} \frac{1-G(t+x\delta(t))}{1-G(t)} \cdot e^{-(b-1)x}$$

$$\stackrel{(*)}{=} e^{-x} \cdot e^{-(b-1)x} = e^{-bx}$$

same type as  $e^{-x}$ .

Hence  $F$  also belongs to the Gumbel domain.



By definition,

$$G\left(\frac{w(G)}{\quad}\right) = \underline{1}$$

$\wedge$

$$G\left(w(G) + \underset{\vee}{\alpha} \underbrace{\gamma(w(G))}_{\vee}\right) = 1$$

$\quad \quad \quad \vee \quad \quad \vee$   
 $\quad \quad \quad 0 \quad \quad 0$

$x < y \iff G(x) \leq G(y)$
-----------------------------

# **EXAMPLE CLASS**

**3 OCTOBER**

**10:00-11:00AM**

**MATH3/4/68181**

$$\underline{\Phi 1} \quad \Lambda(x) = e^{-e^{-x}}$$

$$\frac{d\Lambda(x)}{dx} = e^{-e^{-x}} \cdot (-1) \cdot e^{-x} \cdot (-1)$$

$$= e^{-x} e^{-e^{-x}}$$

$$\Phi_{\alpha}(x) = \begin{cases} e^{-x^{-\alpha}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\frac{d\Phi_{\alpha}(x)}{dx} = \begin{cases} e^{-x^{-\alpha}} (-1)(-\alpha) x^{-\alpha-1}, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\Psi_{\alpha}(x) = \begin{cases} e^{-(-x)^{\alpha}} & x \leq 0 \\ 1 & x > 0 \end{cases}$$

$$\frac{d\Psi_{\alpha}(x)}{dx} = \begin{cases} e^{-(-x)^{\alpha}} (-1) \alpha (-x)^{\alpha-1} (-1), & x \leq 0 \\ 0 & x > 0 \end{cases}$$

Q2

$$E(X) = \int_{-\infty}^{\infty} x \cdot \underbrace{e^{-e^{-x}} \cdot e^{-x}} dx$$

$$y = e^{-x} \Rightarrow x = -\log y \Rightarrow \frac{dx}{dy} = -\frac{1}{y}$$

$$= \int_{\infty}^0 (-\log y) \cdot e^{-y} y \left(-\frac{1}{y}\right) dy$$

$$= - \int_0^{\infty} \log y \cdot e^{-y} dy$$

$$= - \int_0^{\infty} \left( \frac{d}{da} y^a \Big|_{a=0} \right) \cdot e^{-y} dy$$

$$= - \frac{d}{da} \left( \int_0^{\infty} y^a e^{-y} dy \right) \Big|_{a=0}$$

$$= - \frac{d}{da} \Gamma(a+1) \Big|_{a=0} = -\Gamma'(1)$$

$$\underline{Q3} \quad \text{Var}[X] = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot e^{-e^{-x}} e^{-x} dx$$

$$y = e^{-x} \Rightarrow x = -\log y \Rightarrow \frac{dx}{dy} = -\frac{1}{y}$$

$$= \int_{\infty}^0 (\log y)^2 e^{-y} y \left(-\frac{1}{y}\right) dy$$

$$= \int_0^{\infty} (\log y)^2 e^{-y} dy$$

$$= \int_0^{\infty} \left( \frac{d^2}{da^2} y^a \Big|_{a=0} \right) e^{-y} dy$$

$$= \frac{d^2}{da^2} \left( \int_0^{\infty} y^a e^{-y} dy \right) \Big|_{a=0}$$

$$= \frac{d^2}{da^2} \Gamma(a+1) \Big|_{a=0} = \Gamma''(1)$$

$$\text{Var}[X] = \Gamma''(1) - [\Gamma'(1)]^2$$

$$\frac{d}{da} y^a = y^a \log y$$

$$\boxed{a=0} : \left. \frac{d}{da} y^a \right|_{a=0} = y^0 \log y = \log y$$

$$\Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} dt$$

Gamma function

$$\frac{d^2}{da^2} y^a = \frac{d}{da} [y^a \log y] = y^a (\log y)^2$$

$$\boxed{a=0} : \left. \frac{d^2}{da^2} y^a \right|_{a=0} = y^0 (\log y)^2 = (\log y)^2$$

Q2

$$F(x) = [1 - e^{-x}]^\alpha$$

$$F(x) = 1 \Rightarrow [1 - e^{-x}]^\alpha = 1$$

$$\Rightarrow 1 - e^{-x} = 1$$

$$\Rightarrow e^{-x} = 0$$

$$\Rightarrow x = +\infty \Rightarrow w(F) = +\infty$$

$$(I) \quad \lim_{t \uparrow \infty} \frac{1 - [1 - e^{-t - x\gamma(t)}]^\alpha}{1 - [1 - e^{-t}]^\alpha}$$

$$(1 - a)^\alpha \approx 1 - a\alpha \quad \text{as } a \rightarrow 0$$

$$= \lim_{t \uparrow \infty} \frac{1 - [1 - \alpha \cdot e^{-t - x\gamma(t)}]}{1 - [1 - \alpha \cdot e^{-t}]}$$

$$= \lim_{t \uparrow \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t \rightarrow \infty} e^{-x\gamma(t)} = e^{-x} \quad \text{if } \gamma(t) \equiv 1$$

Cond (I) is satisfied. Hence there exist  $a_n > 0$  &  $b_n \in \mathbb{R}$  such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \rightarrow e^{-e^{-x}}$$

**LECTURE**

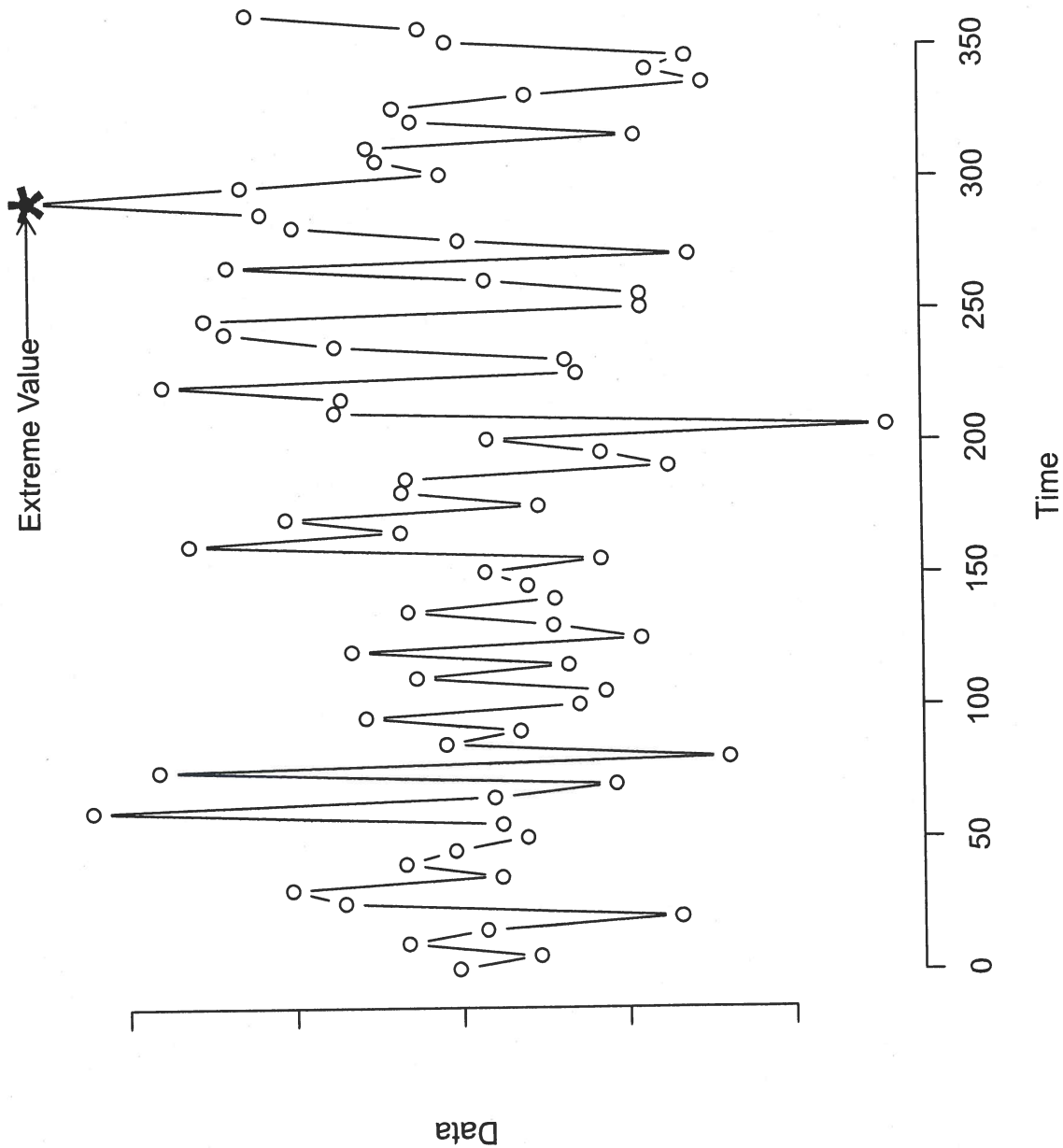
**5 OCTOBER**

**12:00-13:00PM**

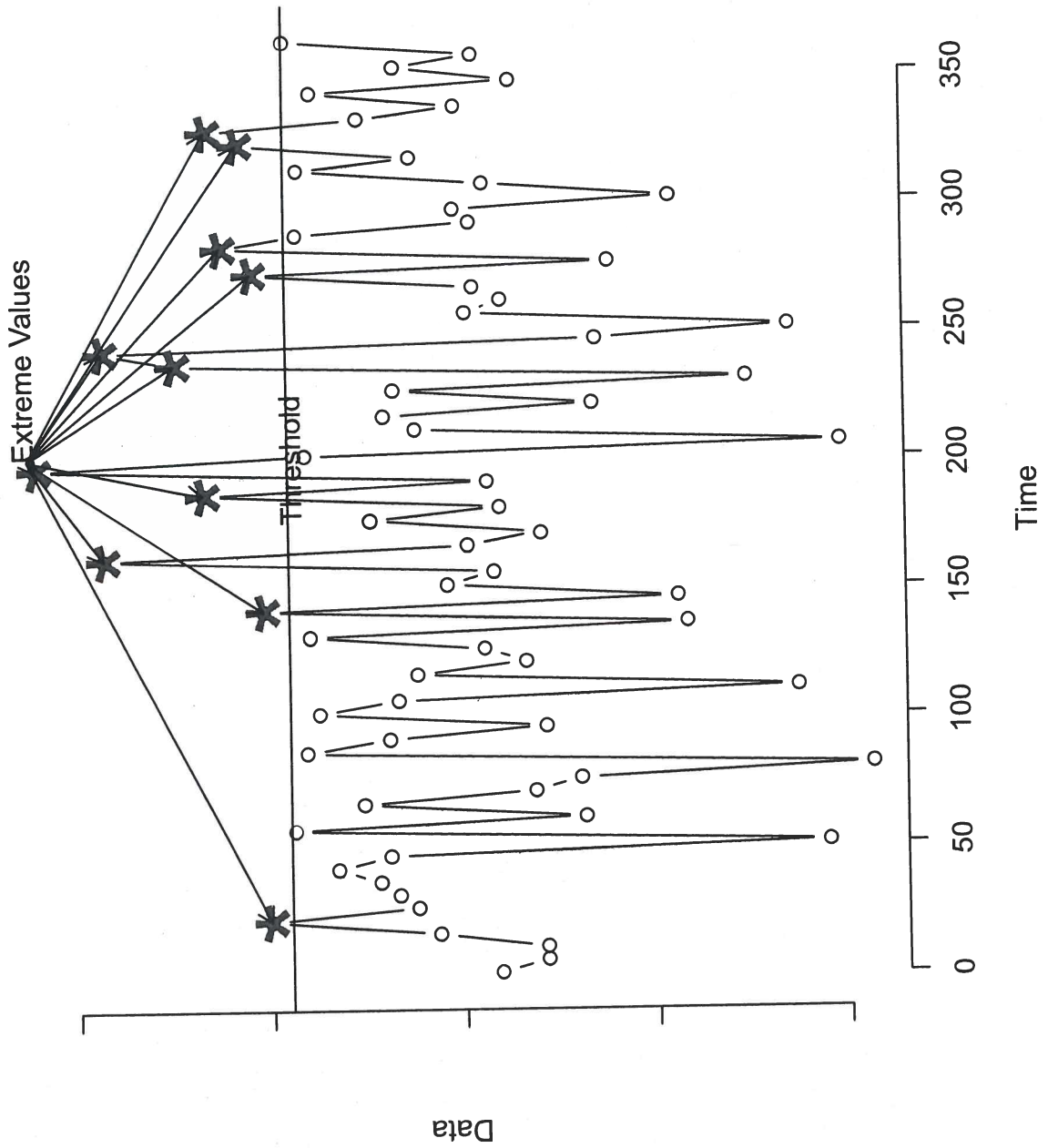
**MATH4/68181**



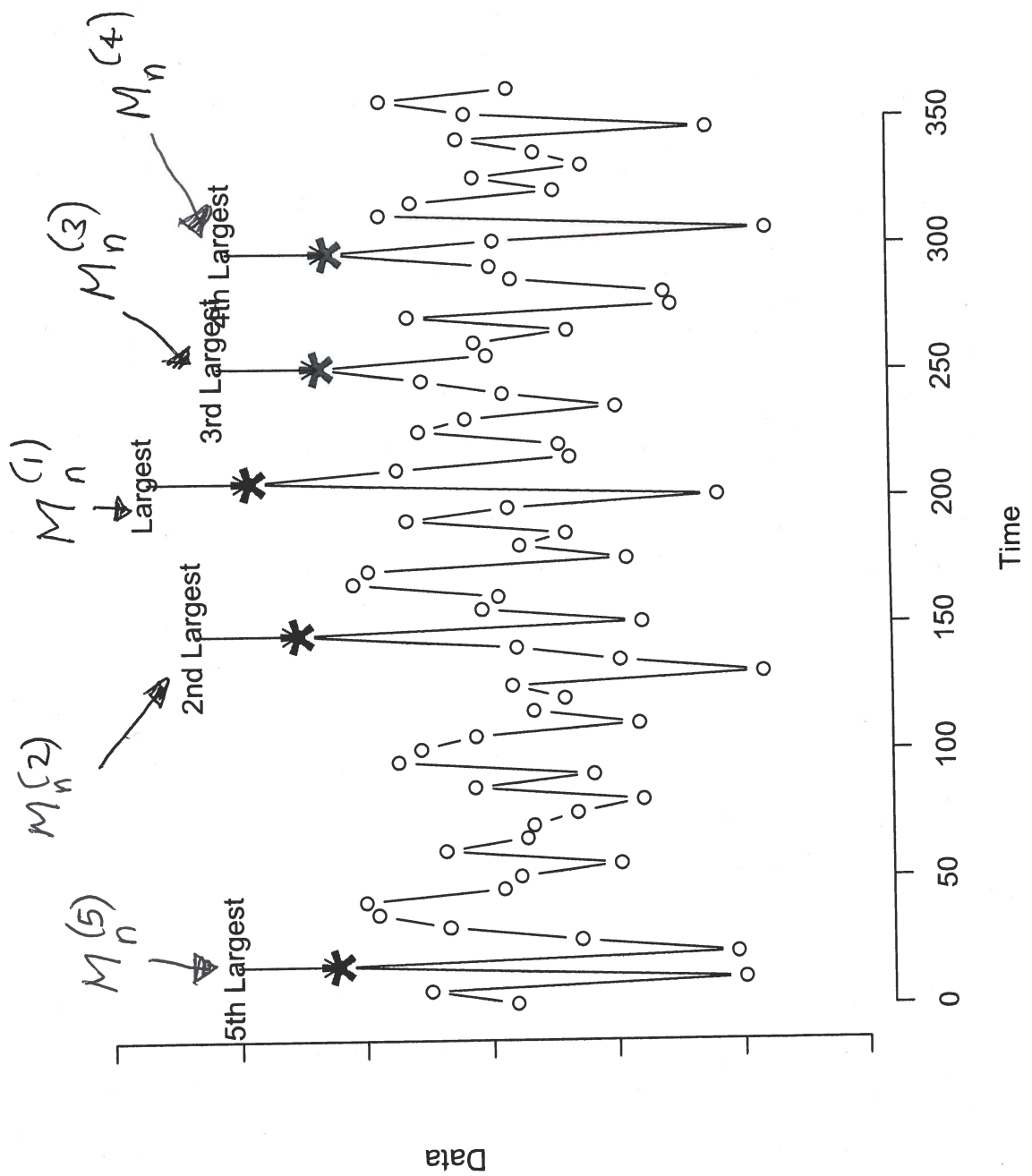
# DEFINITION 1



DEFINITION 2



# DEFINITION 3



Definition	1	}	Level	3/4/6
"	2			
"	3	}	Level	4/6

### Definition 3

Let  $M_n^{(1)}$  = largest observation

$M_n^{(2)}$  = 2<sup>nd</sup> " "

$M_n^{(3)}$  = 3<sup>rd</sup> " "

⋮

$M_n^{(r)}$  = r<sup>th</sup> " "

Under suitable conditions,

$$P \left( \frac{M_n^{(1)} - b_n}{a_n} < x_1, \dots, \frac{M_n^{(r)} - b_n}{a_n} < x_r \right)$$

$$\begin{aligned} \rightarrow \sum_{s_1=0}^1 \sum_{s_2=0}^{2-s_1} \dots \sum_{s_{r-1}=0}^{r-1-s_1-\dots-s_{r-2}} \frac{(\gamma_2 - \gamma_1)^{s_1}}{s_1!} \\ \dots \frac{(\gamma_r - \gamma_{r-1})^{s_{r-1}}}{s_{r-1}!} e^{-\gamma_r} \quad (*) \end{aligned}$$

where  $\gamma_i = \left( 1 + \binom{x_i}{5} \right)^{-\frac{1}{5}}$

$i = 1, 2, \dots, r$

## Joint PDF

From (\*), the joint PDF of  $(M_n^{(1)}, \dots, M_n^{(r)})$  can be expressed as

$$f(x_1, \dots, x_r) = \frac{1}{\sigma^r} e^{-\left(1 + \sum_{i=1}^r \frac{x_i - \mu}{\sigma}\right)^{\frac{1}{\alpha}}} \cdot e^{-\left(1 + \frac{1}{\alpha}\right) \sum_{i=1}^r \log\left(1 + \sum_{j=1}^r \frac{x_j - \mu}{\sigma}\right)}$$

where  $x_1 \geq \dots \geq x_r$  are such that  $1 + \sum_{j=1}^r \frac{x_j - \mu}{\sigma} > 0, i = 1, \dots, r$

## Data

$$(x_{1,1}, \dots, x_{1,r})$$

r largest  
obsns for year 1

$$(x_{2,1}, \dots, x_{2,r})$$

r largest  
obsns for year 2

⋮

$$(x_{n,1}, \dots, x_{n,r})$$

r largest  
obsns for year n

## Likelihood

$$L(\mu, \sigma, \xi) = \prod_{i=1}^n f(x_{i,1}, \dots, x_{i,r})$$

$$= \frac{1}{\sigma^{nr}} e^{-\sum_{i=1}^n \left(1 + \xi \frac{x_{i,r} - \mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$\cdot e^{-\left(1 + \frac{1}{\xi}\right) \sum_{i=1}^n \sum_{j=1}^r \log \left(1 + \xi \frac{x_{i,j} - \mu}{\sigma}\right)}$$

$$\log L = -nr \log \sigma$$

$$- \sum_{i=1}^n \left( 1 + \xi \frac{x_{i,r} - \mu}{\sigma} \right)^{-\frac{1}{\xi}}$$

$$- \left( 1 + \frac{1}{\xi} \right) \sum_{i=1}^n \sum_{j=0}^{r-1} \log \left( 1 + \xi \frac{x_{ij} - \mu}{\sigma} \right)$$

$$\frac{\partial \log L}{\partial \mu} = 0$$

$$\frac{\partial \log L}{\partial \sigma} = 0$$

$$\frac{\partial \log L}{\partial \xi} = 0$$

the solution  
of these will  
be the  
MLE of  
( $\mu, \sigma, \xi$ ).

<http://www.maths.manchester.ac.uk/~sarales/extremesoft.pdf>

~sarales/extremesoft.pdf



## MLE equations for the $r$ largest distribution

The MLEs of  $\mu$ ,  $\sigma$  and  $\xi$  are the simultaneous solutions of

$$\begin{aligned} \frac{\partial \log L}{\partial \mu} &= -\frac{1}{\sigma} \sum_{i=1}^n \left(1 + \xi \frac{x_{i,r} - \mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &\quad - \frac{1 + \xi}{\sigma} \sum_{i=1}^n \sum_{j=1}^r \left(1 + \xi \frac{x_{i,j} - \mu}{\sigma}\right)^{-1} \\ &= 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial \log L}{\partial \sigma} &= -\frac{nr}{\sigma} - \frac{1}{\sigma^2} \sum_{i=1}^n (x_{i,r} - \mu) \left(1 + \xi \frac{x_{i,r} - \mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &\quad + \frac{1 + \xi}{\sigma^2} \sum_{i=1}^n \sum_{j=1}^r (x_{i,j} - \mu) \left(1 + \xi \frac{x_{i,j} - \mu}{\sigma}\right)^{-1} \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \log L}{\partial \xi} &= -\frac{1}{\xi^2} \sum_{i=1}^n \log \left(1 + \xi \frac{x_{i,r} - \mu}{\sigma}\right) \left(1 + \xi \frac{x_{i,r} - \mu}{\sigma}\right)^{-\frac{1}{\xi}} \\ &\quad + \frac{1}{\xi \sigma} \sum_{i=1}^n (x_{i,r} - \mu) \left(1 + \xi \frac{x_{i,r} - \mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &\quad + \frac{1}{\xi^2} \sum_{i=1}^n \sum_{j=1}^r \log \left(1 + \xi \frac{x_{i,j} - \mu}{\sigma}\right) \\ &\quad - \frac{1 + \xi}{\xi \sigma} \sum_{i=1}^n \sum_{j=1}^r (x_{i,j} - \mu) \left(1 + \xi \frac{x_{i,j} - \mu}{\sigma}\right)^{-1} \\ &= 0. \end{aligned}$$

$w(G)$  is the point where the cdf  $G$  becomes equal to 1.

Therefore,

$$G(w(G)) = 1.$$

$$\left[ G\left(w(G) + \underbrace{x}_0 \underbrace{\gamma(w(G))}_0\right) \right] \geq G(w(G)) = 1$$

$$\Rightarrow G\left(w(G) + x \gamma(w(G))\right) = 1.$$

**LECTURE**

**6 OCTOBER**

**9:00-10:00AM**

**MATH3/4/68181**

Ex 11

$$f(x) = k g(x) [G(x)]^{a-1} [1-G(x)]^{b-1} e^{-cG(x)}$$

✓ (a) Assume  $G$  belongs to Gumbel limit  
& show  $F$  also " " " "

✓ (b) Assume  $G$  belongs to Frechet limit  
& show  $F$  also " " " "

(c) Assume  $G$  belongs to Weibull limit  
& show  $F$  also " " " "

(a) Assume  $G$  belongs to Gumbel domain. That is, there exists  $\delta(t) > 0$  such that

$$\lim_{t \uparrow w(G)} \frac{1 - G(t + x\delta(t))}{1 - G(t)} = e^{-x} \quad (*)$$

We need to find

$$\lim_{t \uparrow w(F)} \frac{1 - F(t + x\delta(t))}{1 - F(t)}$$

$$\stackrel{LH}{=} \lim_{t \uparrow w(F)} \frac{-f(t + x\delta(t)) (1 + x\delta'(t))}{-f(t)} \quad ((1+x\delta'(t)))$$

$$= \lim_{t \uparrow w(F)} \frac{g(t + x\delta(t)) [G(t + x\delta(t))]^{a-1} [1 - G(t + x\delta(t))]^{b-1} e^{-cG(t + x\delta(t))}}{g(t) [G(t)]^{a-1} [1 - G(t)]^{b-1} e^{-cG(t)}}$$

$$w(F) = w(G) \\ = \lim_{t \uparrow w(G)} \frac{g(t + x\delta(t)) (1 + x\delta'(t))}{g(t)}$$

- $\frac{G(t + x\delta(t))}{G(t)} \rightarrow 1$

- $\left[ \frac{1 - G(t + x\delta(t))}{1 - G(t)} \right]^{b-1} \rightarrow 1$

- $e^{-cG(t)} \rightarrow e^{-cG(t + x\delta(t))}$

$$= \lim_{t \uparrow w(G)} \frac{g(t + x\delta(t)) (1 + x\delta'(t))}{g(t)}$$

$$\cdot \left[ \frac{1 - G(t + x\delta(t))}{1 - G(t)} \right]^{b-1}$$

$$\stackrel{(*)}{=} \lim_{t \uparrow w(G)} \frac{g(t + x\delta(t)) (1 + x\delta'(t))}{g(t)} \cdot (e^{-x})^{b-1}$$

$$\stackrel{LH}{=} \lim_{t \uparrow w(G)} \frac{1 - G(t + x\delta(t))}{1 - G(t)} \cdot e^{-(b-1)x}$$

$$\stackrel{(*)}{=} e^{-x} \cdot e^{-(b-1)x} = e^{-bx}$$

same type as  $e^{-x}$ .

Hence  $F$  also belongs to the Gumbel domain.

By defn of  $w(G)$ ,

$$G(w(G)) = 1.$$

Also

$$G\left(w(G) + \underbrace{x}_{\substack{\vee \\ 0}} \cdot \underbrace{\gamma(w(G))}_{\substack{\vee \\ 0}}\right)$$

$$\begin{array}{c} \vee \\ G(w(G)) \end{array}$$

||

1

$$\Rightarrow G(w(G) + x \cdot \gamma(w(G))) = 1.$$

(6) Assume  $G$  belongs to Frechet domain, that is

$$\omega(G) = \infty, \quad \lim_{t \uparrow \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{-\alpha}$$

..... (\*)

We want to find

$$\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$\stackrel{LH}{=} \lim_{t \uparrow \infty} \frac{-x f(tx)}{-f(t)}$$

$$= \lim_{t \uparrow \infty} \frac{x \cdot g(tx) [G(tx)]^{a-1} [1 - G(tx)]^{b-1} e^{-cG(tx)}}{g(t) [G(t)]^{a-1} [1 - G(t)]^{b-1} e^{-cG(t)}}$$

$$= \lim_{t \uparrow \infty} \left[ \frac{x g(tx)}{g(t)} \right] \cdot \left[ \frac{G(tx)}{G(t)} \right]^{a-1} \cdot \left[ \frac{1 - G(tx)}{1 - G(t)} \right]^{b-1}$$

$$e^{c(G(t) - cG(tx))}$$

$$= \lim_{t \uparrow \infty} \left[ \frac{x g(tx)}{g(t)} \right] \left[ \frac{1 - G(tx)}{1 - G(t)} \right]^{b-1}$$

$$\stackrel{(*)}{=} \lim_{t \uparrow \infty} \frac{x g(tx)}{g(t)} \cdot (x^{-\alpha})^{b-1}$$

$$\stackrel{LH}{=} \lim_{t \uparrow \infty} \frac{1 - G(tx)}{1 - G(t)} \cdot (x^{-\alpha})^{b-1}$$

$$\stackrel{(*)}{=} (x^{-\alpha}) \cdot (x^{-\alpha})^{b-1} = x^{-b\alpha}$$



$x^{-b\alpha}$  is of the same type  
as  $x^{-\alpha}$ .

Hence  $F$  also belongs to  
the Fréchet domain.

(c) Assume  $G$  belongs to Weibull domain. That is

$$w(G) < \infty, \quad \lim_{t \downarrow 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^a \dots (*)$$

We want to find

$$\lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)}$$

$$\stackrel{L'H}{=} \lim_{t \downarrow 0} \frac{x f(w(F) - tx)}{f(w(F) - t)}$$

$$\stackrel{w(F) = w(G)}{=} \lim_{t \downarrow 0} \frac{x f(w(G) - tx)}{f(w(G) - t)}$$

$$= \lim_{t \downarrow 0} \frac{x \cdot g(w(G) - tx) [G(w(G) - tx)]^{a-1} [1 - G(w(G) - tx)]^{b-1}}{g(w(G) - t) [G(w(G) - t)]^{a-1} [1 - G(w(G) - t)]^{b-1}}$$

$$= \lim_{t \downarrow 0} \frac{e^{-c G(w(G) - tx)}}{e^{-c G(w(G) - t)}}$$

$$= \lim_{t \downarrow 0} \frac{x g(w(G) - tx)}{g(w(G) - t)} \cdot \left[ \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} \right]^{b-1}$$

$$= \lim_{t \downarrow 0} \frac{x g(w(G) - tx)}{g(w(G) - t)} \cdot (x^a)^{b-1}$$

$$\stackrel{\text{LH}}{=} \lim_{t \downarrow 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} \cdot (x^\alpha)^{b-1}$$

$$\stackrel{(*)}{=} x^\alpha \cdot (x^\alpha)^{b-1} = x^{b\alpha},$$

the same type as  $x^\alpha$ .

Hence  $F$  also belongs to the Weibull domain.

$F$  belongs to the same domain of attraction as  $G$ .

## The ETT

For a practitioner, it is difficult to deal with the 3 types. Can the 3 limits be combined ~~it~~ into one form?

The GEV distribution.  
(Generalized Extreme Value)

It has the CDF

$$G(x) = e^{-\left(1 + \xi \frac{x - \mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$-\infty < \mu < +\infty$  location parameter

$\sigma > 0$  scale ||

$-\infty < \xi < +\infty$  shape ||

$$1 + \xi \frac{x - \mu}{\sigma} > 0$$

# Special cases of GEV

a)  $\boxed{\xi = 0}$

$$\lim_{\xi \rightarrow 0} G(x) = \lim_{\xi \rightarrow 0} e^{-\left(1 + \xi \frac{x-\mu}{\sigma}\right)^{\frac{1}{\xi}}}$$

$$= \lim_{\xi \rightarrow 0} e^{-\left[1 + \frac{\frac{x-\mu}{\sigma}}{\frac{1}{\xi}}\right]^{\frac{1}{\xi}}}$$

[set  $m = \frac{1}{\xi}$ ]

$$= \lim_{m \rightarrow \infty} e^{-\left[1 + \frac{\frac{x-\mu}{\sigma}}{m}\right]^m}$$

$$\left[ \lim_{m \rightarrow \infty} \left(1 + \frac{z}{m}\right)^m = e^z \right]$$

$$= e^{-\left(e^{\frac{x-\mu}{\sigma}}\right)^{-1}}$$

$$= e^{-e^{-\frac{x-\mu}{\sigma}}}$$

the same type as  $\Lambda(x)$

b)  $\boxed{\sigma > 0}$

$$\begin{aligned} G(x) &= e^{-\left(1 + \sigma \frac{x - \mu}{\sigma}\right)^{-\frac{1}{\sigma}}} \\ &= e^{-\left(\frac{\sigma x}{\sigma} + 1 - \frac{\sigma \mu}{\sigma}\right)^{-\frac{1}{\sigma}}} \\ &= e^{-(ax + b)^{-\frac{1}{\sigma}}} \end{aligned}$$

the same type as

$$\boxed{e^{-x^{-\frac{1}{\sigma}}}}$$

$$\stackrel{||}{\Phi_{\frac{1}{\sigma}}}(x)$$

$$a > 0$$

$$b \in \mathbb{R}$$

$$c) \quad \boxed{\xi < 0}$$

$$G(x) = e - \left(1 + \xi \frac{x - \mu}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$= e - \left(\frac{\xi x}{\sigma} + 1 - \frac{\xi \mu}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$= e - \left(-\left(\frac{-\xi x}{\sigma} + \frac{\xi \mu}{\sigma} - 1\right)\right)^{-\frac{1}{\xi}}$$

$\parallel a$                        $\parallel b$

$$= e - \left(-\left(ax + b\right)\right)^{-\frac{1}{\xi}}$$

$a > 0$   
,  $b \in \mathbb{R}$

the same type as

$$e - (-x)^{-\frac{1}{\xi}} = \Psi_{-\frac{1}{\xi}}(x)$$



## PDF

$$f(x) = \frac{1}{\sigma} \left( 1 + \sqrt{\frac{x-\mu}{\sigma}} \right)^{-\frac{1}{\sqrt{\pi}} - 1}$$
$$\cdot e^{-\left( 1 + \sqrt{\frac{x-\mu}{\sigma}} \right)^{-\frac{1}{\sqrt{\pi}}}}$$

## quantile

$$\text{Set } G(x) = p$$

$$\Rightarrow e^{-\left( 1 + \sqrt{\frac{x-\mu}{\sigma}} \right)^{-\frac{1}{\sqrt{\pi}}}} = p$$

$$\Rightarrow \left( 1 + \sqrt{\frac{x-\mu}{\sigma}} \right)^{-\frac{1}{\sqrt{\pi}}} = -\log p$$

$$\Rightarrow 1 + \sqrt{\frac{x-\mu}{\sigma}} = (-\log p)^{-\sqrt{\pi}}$$

$$\Rightarrow x = \mu + \frac{\sigma}{\sqrt{\pi}} \left[ (-\log p)^{-\sqrt{\pi}} - 1 \right],$$

$p$ th quantile

# Likelihood

Data:  $x_1, x_2, \dots, x_n$

$$L(\mu, \sigma, \xi) = \prod_{i=1}^n g(x_i)$$

$$= \frac{1}{\sigma^n} \left[ \prod_{i=1}^n \left( 1 + \xi \frac{x_i - \mu}{\sigma} \right) \right]^{-\frac{1}{\xi} - 1}$$

$$= e^{-\sum_{i=1}^n \left( 1 + \xi \frac{x_i - \mu}{\sigma} \right)^{-\frac{1}{\xi}}}$$

$$\log L = -n \log \sigma$$

$$- \left( \frac{1}{\xi} + 1 \right) \sum_{i=1}^n \log \left( 1 + \xi \frac{x_i - \mu}{\sigma} \right)$$

$$- \sum_{i=1}^n \left( 1 + \xi \frac{x_i - \mu}{\sigma} \right)^{-\frac{1}{\xi}}$$

## MLEs

The MLEs of  $\mu$ ,  $\sigma$  &  $\xi$  are the solutions of

$$\frac{\partial \log L}{\partial \mu} = 0$$

$$\frac{\partial \log L}{\partial \sigma} = 0$$

$$\frac{\partial \log L}{\partial \xi} = 0$$

## MLE equations for the GEV distribution

The MLEs of  $\mu$ ,  $\sigma$  and  $\xi$  are the simultaneous solutions of

$$\begin{aligned}\frac{\partial \log L}{\partial \mu} &= \frac{1 + \xi}{\sigma} \sum_{i=1}^n \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-1} \\ &\quad - \frac{1}{\sigma} \sum_{i=1}^n \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{1 + \xi}{\sigma^2} \sum_{i=1}^n (x_i - \mu) \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-1} \\ &\quad - \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu) \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \xi} &= \frac{1}{\xi^2} \sum_{i=1}^n \log \left(1 + \xi \frac{x_i - \mu}{\sigma}\right) \\ &\quad - \frac{1 + \xi}{\xi \sigma} \sum_{i=1}^n (x_i - \mu) \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-1} \\ &\quad - \frac{1}{\xi^2} \sum_{i=1}^n \log \left(1 + \xi \frac{x_i - \mu}{\sigma}\right) \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi}} \\ &\quad + \frac{1}{\xi \sigma} \sum_{i=1}^n (x_i - \mu) \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0.\end{aligned}$$

fgev in  $\mathbb{R}$  that can  
return the MLEs,

# **EXAMPLE CLASS**

**9 OCTOBER**

**12:00-13:00PM**

**MATH3/4/68181**

Q1

$$L(\sigma) = \prod_{i=1}^n \left[ \frac{1}{\sigma} e^{-\frac{x_i}{\sigma}} e^{-e^{-\frac{x_i}{\sigma}}} \right]$$
$$= \frac{1}{\sigma^n} e^{-\frac{1}{\sigma} \sum_{i=1}^n x_i} e^{-\sum_{i=1}^n e^{-\frac{x_i}{\sigma}}}$$

$$\log L(\sigma) = -n \log \sigma - \frac{1}{\sigma} \sum_{i=1}^n x_i - \sum_{i=1}^n e^{-\frac{x_i}{\sigma}}$$

$$\frac{d \log L}{d \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^n x_i - \sum_{i=1}^n e^{-\frac{x_i}{\sigma}} \cdot \frac{(-x_i)(-1)}{\sigma^2}$$
$$= -\frac{n}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^n x_i - \frac{1}{\sigma^2} \sum_{i=1}^n e^{-\frac{x_i}{\sigma}} \cdot x_i$$

The MLE of  $\sigma$  is the root of

$$-\frac{n}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^n x_i - \frac{1}{\sigma^2} \sum_{i=1}^n e^{-\frac{x_i}{\sigma}} \cdot x_i = 0$$

solve this using a numerical routine.

Q2

$$L(\lambda, \sigma) = \prod_{i=1}^n \left[ \lambda \sigma^\lambda x_i^{-\lambda-1} e^{-\left(\frac{\sigma}{x_i}\right)^\lambda} \right]$$

$$= \lambda^n \sigma^{n\lambda} \left( \prod_{i=1}^n x_i \right)^{-\lambda-1} e^{-\sum_{i=1}^n \left(\frac{\sigma}{x_i}\right)^\lambda}$$

$$\log L(\lambda, \sigma) = n \log \lambda + n\lambda \log \sigma - (\lambda+1) \sum_{i=1}^n \log x_i - \sum_{i=1}^n \left(\frac{\sigma}{x_i}\right)^\lambda$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n \log \sigma - \sum_{i=1}^n \log x_i$$

$$- \sum_{i=1}^n \left(\frac{\sigma}{x_i}\right)^\lambda \log \left(\frac{\sigma}{x_i}\right) = 0 \quad (1)$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} - \sum_{i=1}^n \frac{\lambda \sigma^{\lambda-1}}{x_i^\lambda} = 0 \quad (2)$$

$$(2) \Rightarrow \frac{n}{\sigma} = \sum_{i=1}^n x_i^{-\lambda}$$

$$\Rightarrow \sigma = \left[ \frac{n}{\sum_{i=1}^n x_i^{-\lambda}} \right]^{\frac{1}{\lambda}} \quad (3)$$

Sub (3) into (1) gives

$$\frac{n}{\lambda} + \frac{n}{\lambda} \log \left[ \frac{n}{\sum_{i=1}^n x_i^{-\lambda}} \right] - \sum_{i=1}^n \log x_i - \left[ \frac{n}{\sum_{i=1}^n x_i^{-\lambda}} \right]^{\frac{1}{\lambda}} \sum_{i=1}^n x_i^{-\lambda} \log \left(\frac{\sigma}{x_i}\right) = 0$$

$$\Rightarrow \frac{n}{\lambda} + \frac{n}{\lambda} \log \left[ \frac{n}{\sum_{i=1}^n x_i^{-\lambda}} \right] - \sum_{i=1}^n \log x_i$$

$$- \frac{1}{\lambda} \left( \log \left[ \frac{n}{\sum_{i=1}^n x_i^{-\lambda}} \right] \right) \left( \frac{n}{\sum_{i=1}^n x_i^{-\lambda}} \right) \cdot \left( \sum_{i=1}^n x_i^{-\lambda} \right)$$

$$+ \frac{n}{\sum_{i=1}^n x_i^{-\lambda}} \sum_{i=1}^n x_i^{-\lambda} \log x_i = 0 \quad (4)$$

$\hat{\lambda}$  is the root of (4)

Sub into eq (3) to get  $\hat{\sigma}$ .

Q3

$$L(\lambda, \sigma) = \prod_{i=1}^n \left[ \frac{\lambda}{\sigma^\lambda} x_i^{\lambda-1} e^{-\left(\frac{x_i}{\sigma}\right)^\lambda} \right]$$

$$= \frac{\lambda^n}{\sigma^{n\lambda}} \left( \prod_{i=1}^n x_i \right)^{\lambda-1} e^{-\sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^\lambda}$$

$$\log L(\lambda, \sigma) = n \log \lambda - n \lambda \log \sigma + (\lambda-1) \sum_{i=1}^n \log x_i - \sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^\lambda$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - n \log \sigma + \sum_{i=1}^n \log x_i - \sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^\lambda \log \left(\frac{x_i}{\sigma}\right) = 0 \quad (1)$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n\lambda}{\sigma} + \lambda \sum_{i=1}^n \frac{x_i^\lambda}{\sigma^{\lambda+1}} = 0 \quad (2)$$

$$(2) \Rightarrow \sigma = \left[ \frac{1}{n} \sum_{i=1}^n x_i^\lambda \right]^{\frac{1}{\lambda}} \quad (3)$$

Sub (3) into (1)

$$\frac{n}{\lambda} - \frac{n}{\lambda} \log \left[ \frac{1}{n} \sum_{i=1}^n x_i^\lambda \right] + \sum_{i=1}^n \log x_i$$

$$- \left[ \frac{1}{n} \sum_{i=1}^n x_i^\lambda \right]^{-1} \sum_{i=1}^n x_i^\lambda \log x_i$$

$$+ \frac{1}{\lambda} \left[ \frac{1}{n} \sum_{i=1}^n x_i^\lambda \right]^{-1} \log \left[ \frac{1}{n} \sum_{i=1}^n x_i^\lambda \right] \sum_{i=1}^n x_i^\lambda = 0 \quad (4)$$

$\hat{\lambda}$  is the root of (4).  
Sub into (3) to get  $\hat{\sigma}$ .

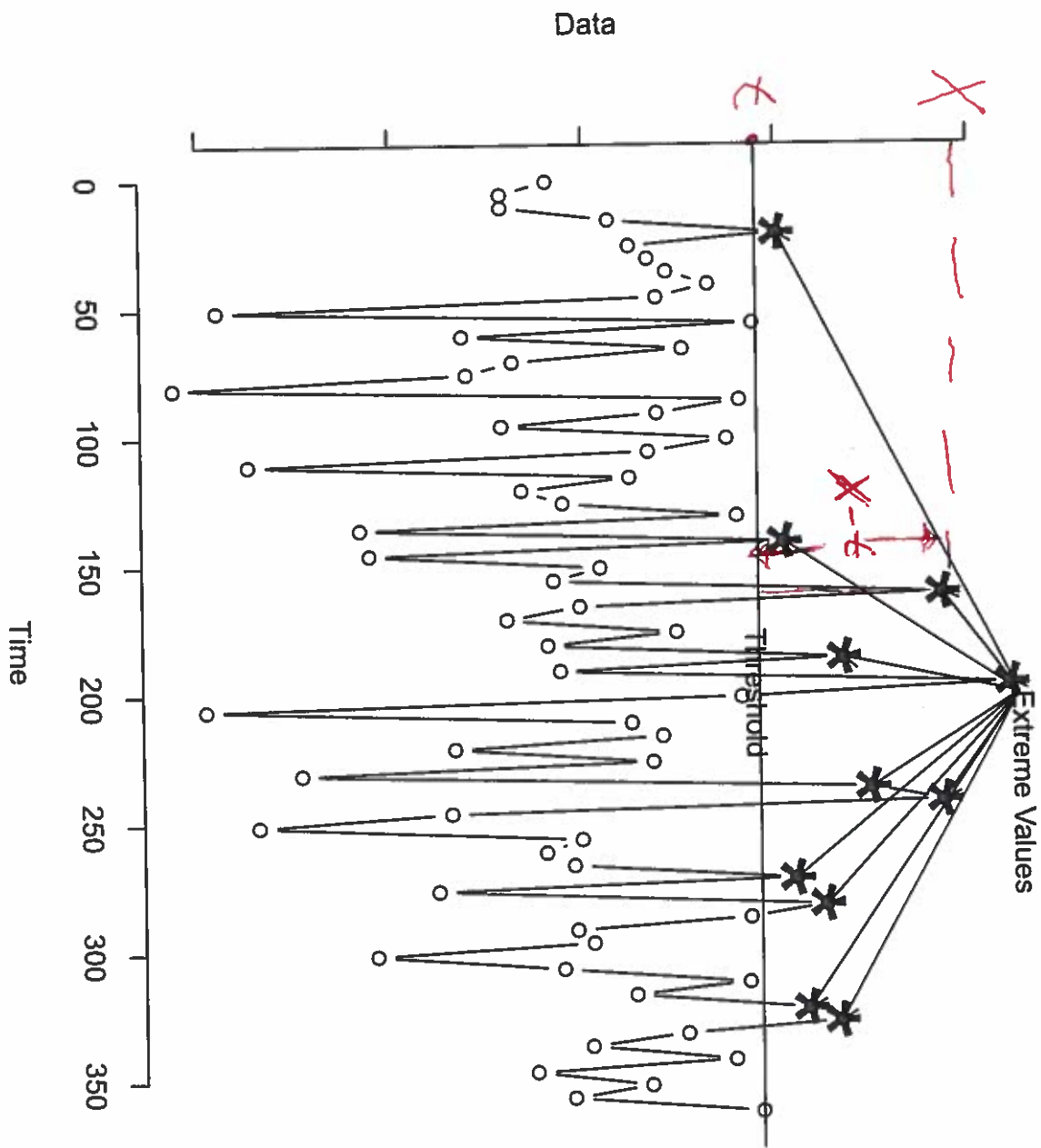


**LECTURE**

**10 OCTOBER**

**9:00-10:00AM**

**MATH3/4/68181**



DEFINITION 2

$$\underline{E_{\text{excess}}} = X - t$$

$$P(\underbrace{X - t}_{\text{Excess}} > x \mid X > t)$$

$$= \frac{P(X - t > x, X > t)}{P(X > t)}$$

$$= \frac{P(X > t + x, X > t)}{P(X > t)}$$

$$= \frac{P(X > t + x)}{P(X > t)} = \frac{1 - P(X \leq t + x)}{1 - P(X \leq t)}$$

$$= \frac{1 - F(t + x)}{1 - F(t)}$$

$F$  cdf of  $X$

↓ as  $t \rightarrow \omega(F)$

$$\left(1 + \frac{x}{\sigma}\right)^{-\frac{1}{\alpha}}$$

due to James Pickand (1975).

If  $t$  is large enough,

$$\frac{1 - F(t+x)}{1 - F(t)} \approx \left(1 + \frac{x}{\sigma}\right)^{-\frac{1}{\alpha}}$$

$$\Rightarrow 1 - F(t+x) \approx \underbrace{[1 - F(t)]}_{\text{"p"}} \left(1 + \frac{x}{\sigma}\right)^{-\frac{1}{\alpha}}$$

$$\Rightarrow F(\underbrace{t+x}_{\text{"y"}}) \approx 1 - P \left(1 + \frac{x}{\sigma}\right)^{-\frac{1}{\alpha}}$$

$$\Rightarrow F(y) \approx 1 - P \left(1 + \frac{y-t}{\sigma}\right)^{-\frac{1}{\alpha}}$$

Generalized Pareto distribution

# Special case

$$\xi = 0$$

$$F(y) = 1 - p \lim_{m \rightarrow \infty} \left( 1 + \frac{y-t}{\sigma} \frac{1}{m} \right)^{-m}$$

$$= 1 - p \cdot \lim_{m \rightarrow \infty} \left[ 1 + \frac{\frac{y-t}{\sigma}}{m} \right]^{-m}$$

$$= 1 - p \lim_{m \rightarrow \infty} \left( 1 + \frac{(y-t)/\sigma}{m} \right)^{-m}$$

$$= 1 - p \cdot \left( e^{-\frac{y-t}{\sigma}} \right)^{-1}$$

$$\left( 1 + \frac{z}{m} \right)^m \rightarrow e^z$$

$$= 1 - p e^{-\frac{y-t}{\sigma}}$$

Exponential CDF

# PDF

$$f(y) = \frac{p}{\sigma} \left( 1 + \frac{y-t}{\sigma} \right)^{-\frac{1}{\alpha} - 1}$$

$$\sigma > 0$$

scale parameter

$$-\infty < y < +\infty$$

shape "

$$1 + \frac{y-t}{\sigma} > 0$$

## Quantile

$$F(y) = q, \quad 0 < q < 1$$

$$\Rightarrow 1 - p \left( 1 + \frac{y-t}{\sigma} \right)^{-\frac{1}{\alpha}} = q$$

$$\Rightarrow \left( 1 + \frac{y-t}{\sigma} \right)^{-\frac{1}{\alpha}} = \frac{1-q}{p}$$

$$\Rightarrow 1 + \frac{y-t}{\sigma} = \left( \frac{1-q}{p} \right)^{-\alpha}$$

$$\Rightarrow y = t + \frac{\sigma}{\alpha} \left[ \left( \frac{1-q}{p} \right)^{-\alpha} - 1 \right]$$

$q$  th quantile

## Estimation

Data:  $x_1, x_2, \dots, x_n$

$$L(\sigma, \xi) = \prod_{i=1}^n \frac{p}{\sigma} \left(1 + \xi \frac{x_i - t}{\sigma}\right)^{-\frac{1}{\xi} - 1}$$
$$= \frac{p^n}{\sigma^n} \left[ \prod_{i=1}^n \left(1 + \xi \frac{x_i - t}{\sigma}\right) \right]^{-\frac{1}{\xi} - 1}$$

$$\log L(\sigma, \xi) = n \log p - n \log \sigma$$

$$- \left(1 + \frac{1}{\xi}\right) \sum_{i=1}^n \log \left(1 + \xi \frac{x_i - t}{\sigma}\right)$$

The MLEs of  $\sigma$  and  $\xi$  will be the solutions of

$$\frac{\partial \log L}{\partial \sigma} = 0$$

$$\frac{\partial \log L}{\partial \xi} = 0$$



## MLE equations for the GP distribution

The MLEs of  $\sigma$  and  $\xi$  are the simultaneous solutions of

$$\begin{aligned}\frac{\partial \log L}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{1 + \xi}{\sigma^2} \sum_{i=1}^n (x_i - t) \left(1 + \xi \frac{x_i - t}{\sigma}\right)^{-1} \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \xi} &= \frac{1}{\xi^2} \sum_{i=1}^n \log \left(1 + \xi \frac{x_i - t}{\sigma}\right) \\ &\quad - \frac{1 + \xi}{\xi \sigma} \sum_{i=1}^n (x_i - t) \left(1 + \xi \frac{x_i - t}{\sigma}\right)^{-1} \\ &= 0.\end{aligned}$$

fpot in  $\mathbb{R}$  solves these  
to give you  $\hat{\sigma}$  and  $\hat{\xi}$ .

Definition 1 }  
" 2 }

Level 3/4/6

Definition 3 }

Level 4/6

## Conditions

$$I: P\left(\frac{M_n - b_n}{a_n} < x\right) \rightarrow \Lambda(x)$$

if there exists  $\gamma(t) > 0$

such that  $\lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = e^{-x}$ .

$$II: P\left(\frac{M_n - b_n}{a_n} < x\right) \rightarrow \Phi_\alpha(x)$$

if  $w(F) = \infty$ ,  $\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}$

$$III: P\left(\frac{M_n - b_n}{a_n} < x\right) \rightarrow \Psi_\alpha(x)$$

if  $w(F) < \infty$ ,  $\lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^{-\alpha}$

If any only one of (I)-(III) will be satisfied

Is there a way to find out that none of (I) - (III) will hold? Yes.

Discrete Case

None of the conditions will be satisfied (ETT will fail) if

PMF  $\rightarrow$   $\frac{P(X = k)}{1 - F(k-1)} \not\rightarrow 0$  as  $k \rightarrow w(F)$ .

$\uparrow$  CDF

Continuous Case

PDF  $\rightarrow$   $\frac{f(x)}{1 - F(x^-)} \not\rightarrow 0$  as  $x \rightarrow w(F)$

$\uparrow$  CDF



ps 1

$$P(X = k) = (1 - p)^{k-1} p, \quad k = 1, 2, \dots$$

Geometric

$$\frac{P(X = k)}{1 - F(k-1)}$$

$$= \frac{(1 - p)^{k-1} p}{1 - \sum_{j=1}^{k-1} (1 - p)^{j-1} p}$$

$$= \frac{(1 - p)^{k-1} p}{1 - \sum_{j=1}^{k-1} (1 - p)^{j-1} p}$$

$$= \frac{(1 - p)^{k-1} p}{1 - [1 - (1 - p)^{k-1}]}$$

$$= \frac{(1 - p)^{k-1} p}{(1 - p)^{k-1} p} = p \neq 0$$

$$\Rightarrow ETT \text{ will not hold.}$$

eg 2

$$P(X=k) = \frac{1}{N}, \quad k=1, 2, \dots, N$$

Discrete Uniform

$$\lim_{k \rightarrow W(F)} \frac{P(X=k)}{1-F(k-1)}$$

$$= \lim_{k \rightarrow N} \frac{P(X=k)}{1-F(k-1)}$$

$$= \frac{P(X=N)}{1-F(N-1)}$$

$$= \frac{P(X=N)}{1-P(X \leq N-1)}$$

$$= \frac{P(X=N)}{P(X > N-1)}$$

$$= \frac{P(X=N)}{P(X=N)} = 1 \neq 0$$

ETT fails to hold.

**EXAMPLE CLASS**

**10 OCTOBER**

**10:00-11:00AM**

**MATH3/4/68181**

Q1

$$L(\sigma) = \prod_{i=1}^n \left[ \frac{1}{\sigma} e^{-\frac{x_i}{\sigma}} e^{-e^{-\frac{x_i}{\sigma}}} \right]$$
$$= \frac{1}{\sigma^n} e^{-\frac{1}{\sigma} \sum_{i=1}^n x_i} e^{-\sum_{i=1}^n e^{-\frac{x_i}{\sigma}}}$$

$$\log L(\sigma) = -n \log \sigma - \frac{1}{\sigma} \sum_{i=1}^n x_i - \sum_{i=1}^n e^{-\frac{x_i}{\sigma}}$$

$$\frac{d \log L}{d \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^n x_i - \frac{1}{\sigma^2} \sum_{i=1}^n x_i e^{-\frac{x_i}{\sigma}}$$

The MLE  $\hat{\sigma}$  is the root of

$$-n\sigma + \sum_{i=1}^n x_i - \sum_{i=1}^n x_i e^{-\frac{x_i}{\sigma}} = 0$$

$\hat{\sigma}$  can be computed numerically.



Q2

$$L(\lambda, \sigma) = \prod_{i=1}^n \left[ \lambda \sigma^\lambda x_i^{-\lambda-1} e^{-\left(\frac{\sigma}{x_i}\right)^\lambda} \right]$$
$$= \lambda^n \sigma^{n\lambda} \left( \prod_{i=1}^n x_i \right)^{-\lambda-1} e^{-\sum_{i=1}^n \left(\frac{\sigma}{x_i}\right)^\lambda}$$

$$\log L = n \log \lambda + n\lambda \log \sigma - (\lambda+1) \sum_{i=1}^n \log x_i - \sum_{i=1}^n \left(\frac{\sigma}{x_i}\right)^\lambda$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n \log \sigma - \sum_{i=1}^n \log x_i - \sum_{i=1}^n \left(\frac{\sigma}{x_i}\right)^\lambda \log \left(\frac{\sigma}{x_i}\right) = 0 \quad (1)$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} - \lambda \sum_{i=1}^n \frac{\sigma^{\lambda-1}}{x_i^\lambda} = 0 \quad (2)$$

$$\frac{d z^a}{d a} = z^a \log z$$

$$(2) \Rightarrow \frac{n}{\sigma} = \sum_{i=1}^n x_i^{-\lambda}$$

$$\Rightarrow \sigma = \left[ \frac{n}{\sum_{i=1}^n x_i^{-\lambda}} \right]^{\frac{1}{\lambda}} \quad (3)$$

Sub (3) into (1) gives

$$\frac{n}{\lambda} + \frac{n}{\lambda} \log \left[ \frac{n}{\sum_{i=1}^n x_i^{-\lambda}} \right] - \sum_{i=1}^n \log x_i$$

$$- \left[ \frac{n}{\sum x_i^{-\lambda}} \right] \cdot \frac{1}{\lambda} \cdot \log \left[ \frac{n}{\sum_{i=1}^n x_i^{-\lambda}} \right] \cdot \sum_{i=1}^n x_i^{-\lambda}$$

$$+ \left[ \frac{n}{\sum x_i^{-\lambda}} \right] \cdot \left[ \sum_{i=1}^n x_i^{-\lambda} \log x_i \right] = 0$$

(4)

$\hat{\lambda}$  is the root of (4)  
Sub  $\hat{\lambda}$  into (3) to obtain  $\hat{\sigma}$

Q3

$$L(\lambda, \sigma) = \prod_{i=1}^n \left[ \lambda \sigma^{-\lambda} x_i^{\lambda-1} e^{-\left(\frac{x_i}{\sigma}\right)^\lambda} \right]$$
$$= \lambda^n \sigma^{-n\lambda} \left( \prod_{i=1}^n x_i \right)^{\lambda-1} e^{-\sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^\lambda}$$

$$\log L(\lambda, \sigma) = n \log \lambda - n\lambda \log \sigma$$
$$+ (\lambda-1) \sum_{i=1}^n \log x_i - \sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^\lambda$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - n \log \sigma + \sum_{i=1}^n \log x_i$$
$$- \sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^\lambda \log \left(\frac{x_i}{\sigma}\right) = 0 \quad \text{--- (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n\lambda}{\sigma} + \lambda \sum_{i=1}^n \frac{x_i^\lambda}{\sigma^{\lambda+1}} = 0 \quad \text{--- (2)}$$

$$(2) \Rightarrow \sigma = \left( \frac{1}{n} \sum_{i=1}^n x_i^\lambda \right)^{\frac{1}{\lambda}} \quad \text{--- (3)}$$

Sub (3) into (1) to get

$$\frac{n}{\lambda} - \frac{n}{\lambda} \log \left( \frac{1}{n} \sum_{i=1}^n x_i^\lambda \right) - \left( \frac{1}{n} \sum_{i=1}^n x_i^\lambda \right)^{-1}$$
$$\cdot \left( \sum_{i=1}^n x_i^\lambda \log x_i \right)$$
$$+ \left( \frac{1}{n} \sum_{i=1}^n x_i^\lambda \right)^{-1} \cdot \frac{1}{\lambda} \cdot \log \left( \frac{1}{n} \sum_{i=1}^n x_i^\lambda \right) \sum_{i=1}^n x_i^\lambda$$
$$+ \sum_{i=1}^n \log x_i = 0 \quad \text{--- (4)}$$

$\hat{\lambda}$  is the root of (4). Sub into (3) to get

Q4

$$L(\lambda) = \prod_{i=1}^n (1 - \lambda x_i)^{\frac{1}{\lambda} - 1}$$

$$\log L(\lambda) = \left(\frac{1}{\lambda} - 1\right) \sum_{i=1}^n \log(1 - \lambda x_i)$$

$$\frac{d \log L}{d \lambda} = -\frac{1}{\lambda^2} \sum_{i=1}^n \log(1 - \lambda x_i)$$

$$+ \left(\frac{1}{\lambda} - 1\right) \sum_{i=1}^n \frac{(-x_i)}{1 - \lambda x_i} = 0$$

$$\Rightarrow \sum_{i=1}^n \log(1 - \lambda x_i) = -\lambda (1 - \lambda) \sum_{i=1}^n \frac{x_i}{1 - \lambda x_i}$$

$\lambda$  is the root of this equation. Can be solved numerically.

**LECTURE**

**12 OCTOBER**

**12:00-13:00PM**

**MATH4/68181**

# Financial Ratios

## Examples

$$1. \text{ Current ratio} = \frac{\text{Assets (x)}}{\text{Liabilities (y)}}$$

$$2. \text{ Sales margin} = \frac{\text{Sales (x)} - \text{Costs (y)}}{\text{Sales (x)}}$$

4. Interest cover

$$= \frac{\text{Earnings (x)} + \text{Interest (y)}}{\text{Earnings (x)}}$$

Assume  $X$  and  $Y$  are <sup>independent</sup> Pareto RVs.

Pareto is an Italian economist.

The first distribution used to model financial variables.

$X$  is a Pareto RV with CDF

$$F_X(x) = 1 - \left(\frac{k}{x}\right)^a, \quad x \geq k$$

$Y$  is also Pareto RV with CDF

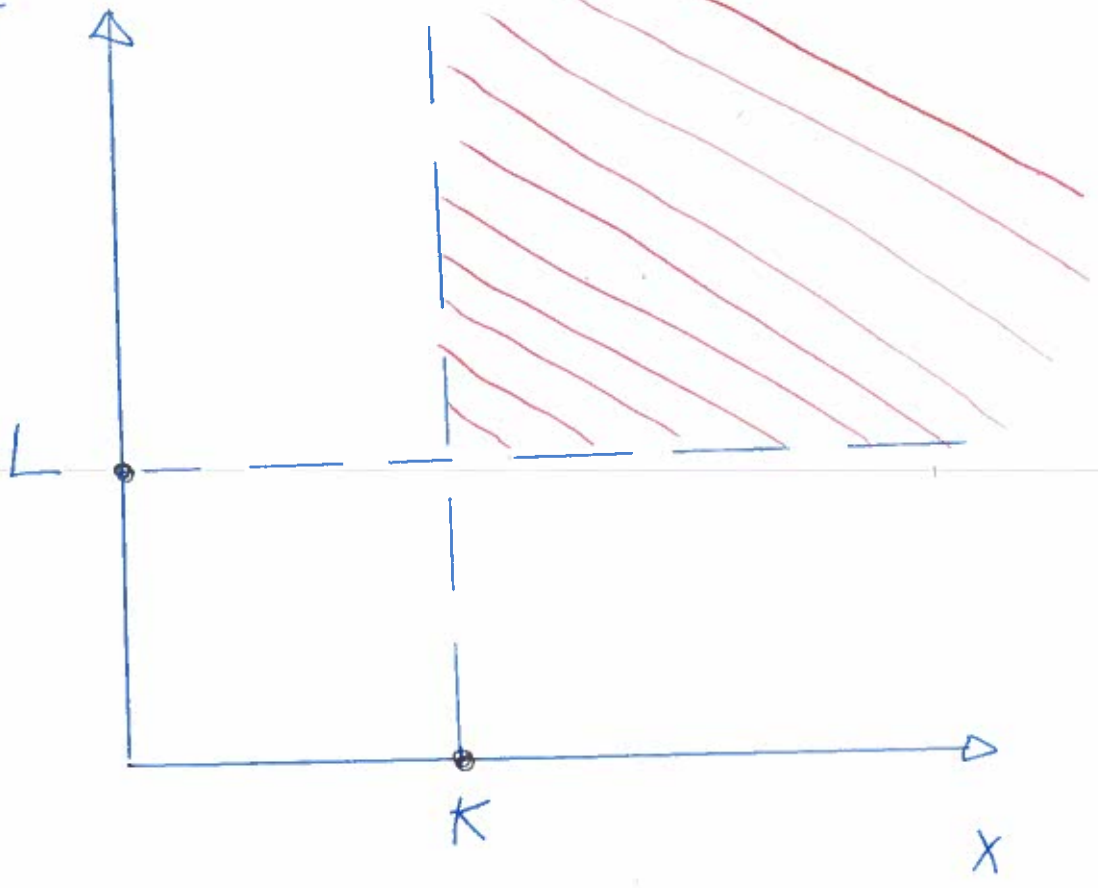
$$F_Y(y) = 1 - \left(\frac{L}{y}\right)^b, \quad y \geq L$$

$X$  and  $Y$  are indep RVs.

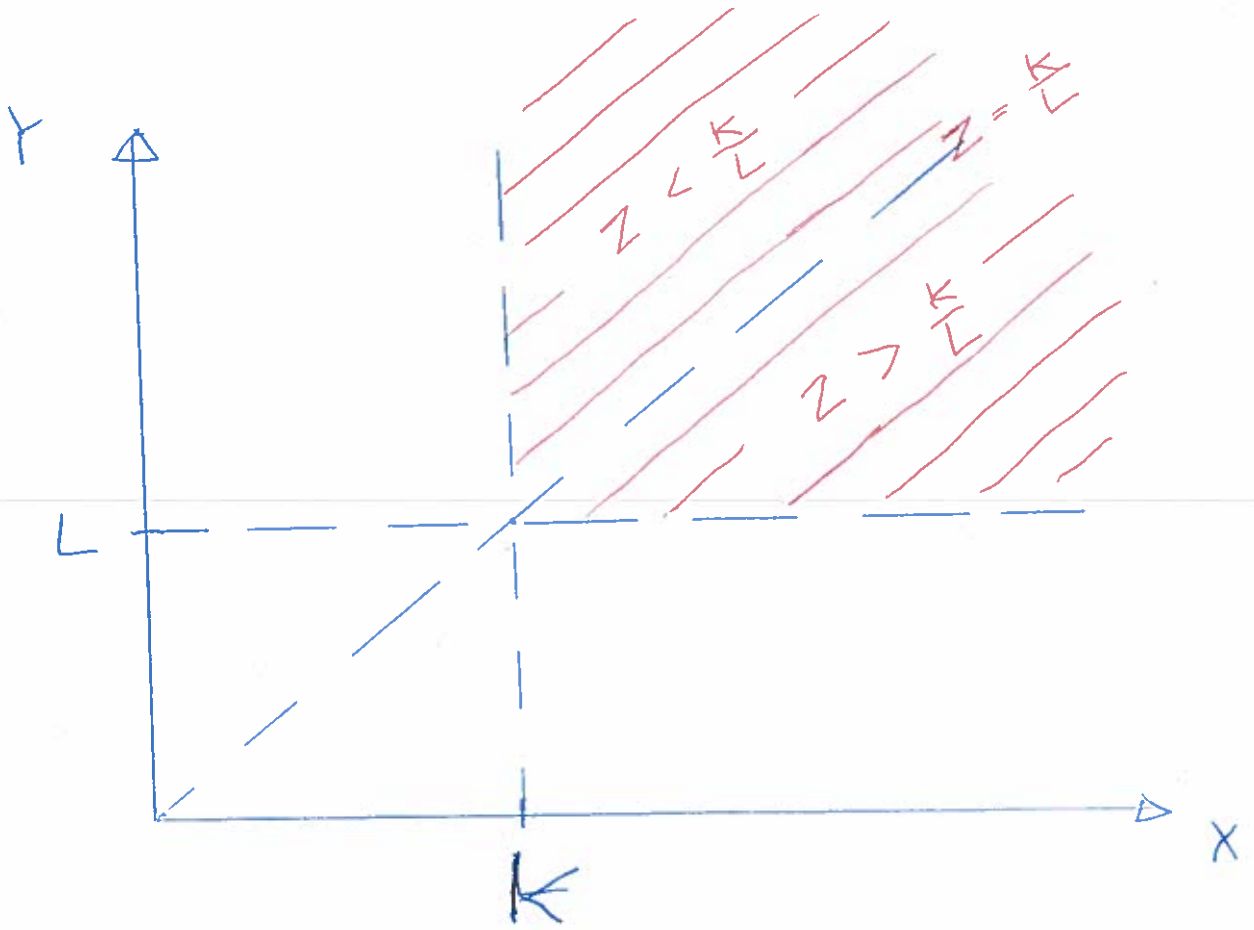
What is the distribution of

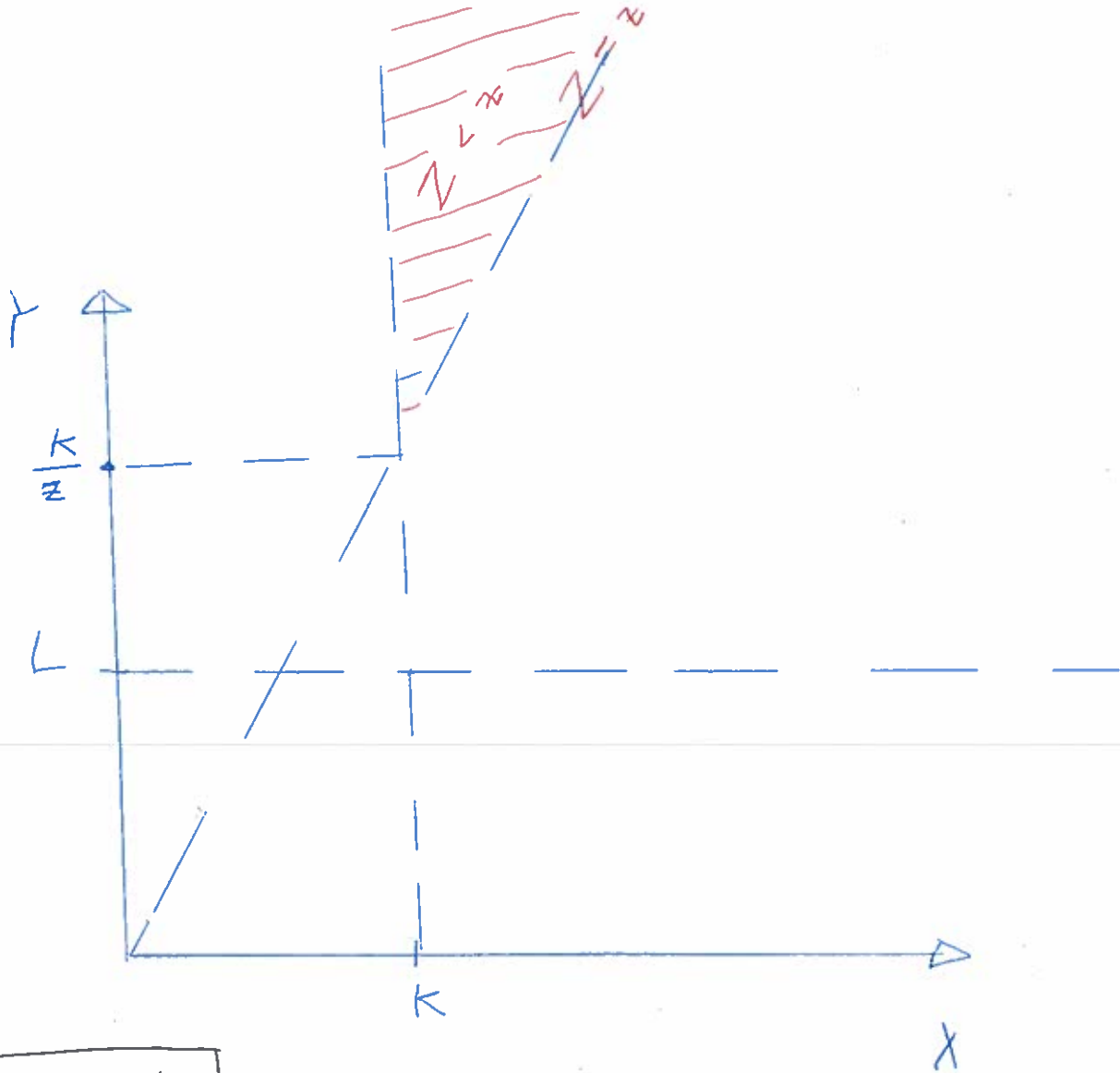
$$Z = \frac{X}{Y} \quad ?$$

$r$









$$Z < \frac{k}{L}$$

$$P(Z < z)$$

$$= \int \int f_{X,Y}(x,y) dy dx$$

$$\stackrel{\text{indep}}{=} \int \int f_X(x) f_Y(y) dy dx$$

$$= \int_k^\infty \int_{\frac{x}{z}}^\infty \frac{a k^a}{x^{a+1}} \cdot \frac{b L^b}{y^{b+1}} dy dx$$

$$= ab k^a L^b \int_k^\infty \frac{1}{x^{a+1}} \left[ \frac{y^{-b}}{-b} \right]_{\frac{x}{z}}^\infty dx$$

$$= ab k^a L^b \int_k^\infty \frac{1}{x^{a+1}} \left[ 0 + \frac{1}{b} \left( \frac{x}{z} \right)^{-b} \right] dx$$

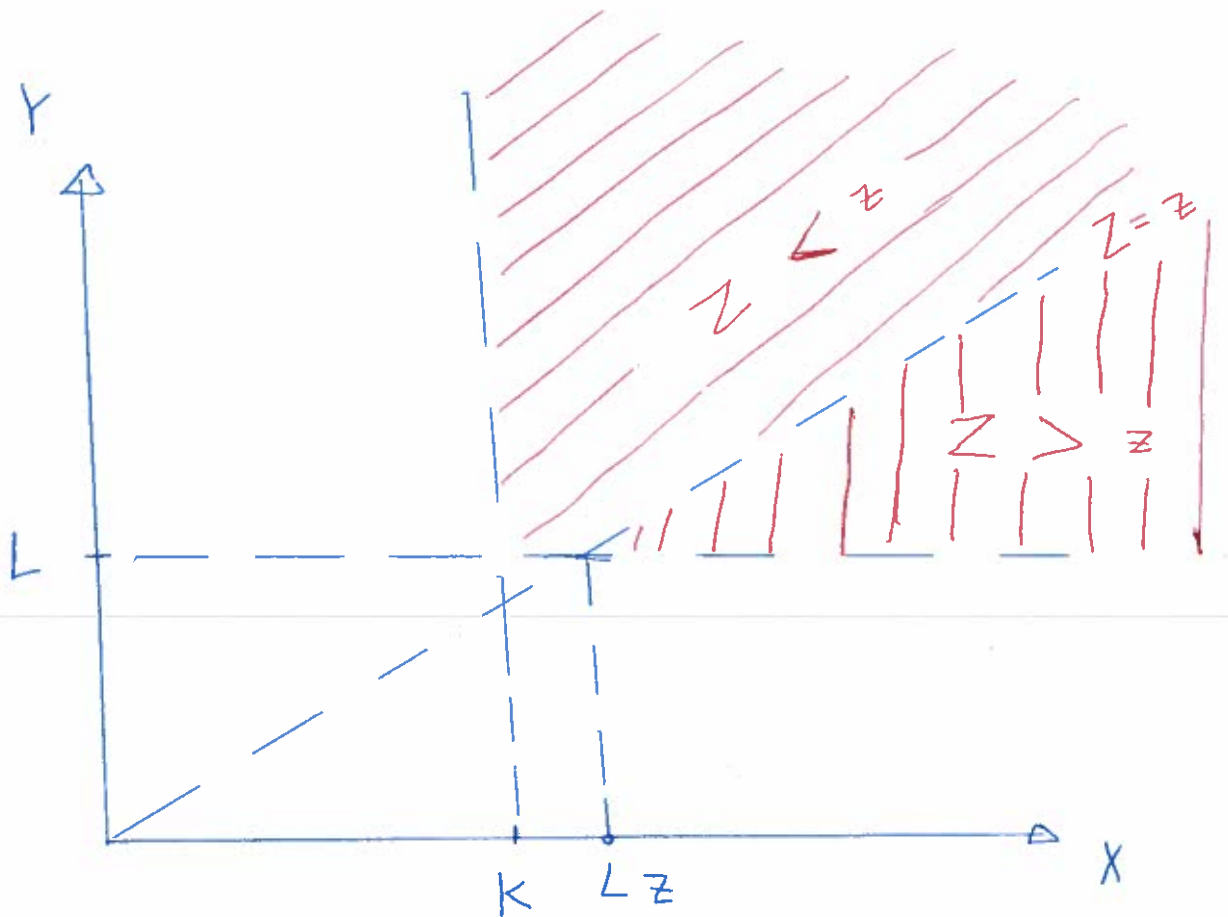
$$= a k^a L^b \int_k^\infty \frac{1}{x^{a+b+1}} \cdot z^b dx$$

$$= a k^a L^b z^b \left[ \frac{x^{-a-b}}{-a-b} \right]_k^\infty$$

$$= a k^a L^b z^b \left[ 0 + \frac{k^{-a-b}}{a+b} \right]$$

$$= \frac{a}{a+b} k^{-b} L^b z^b \cdot$$

---



$$\boxed{z > \frac{K}{L}} : P(Z < z) = 1 - P(Z > z)$$

$$= 1 - \int_L^{\infty} \int_{zy}^{\infty} \frac{a K^a}{x^{a+1}} \cdot \frac{b L^b}{y^{b+1}} dx dy$$

$$= 1 - a K^a b L^b \int_L^{\infty} \frac{1}{y^{b+1}} \left[ \frac{x^{-a}}{(-a)} \right]_{zy}^{\infty} dy$$

$$= 1 - a K^a b L^b \int_L^{\infty} \frac{1}{y^{b+1}} \frac{(zy)^{-a}}{a} dy$$

$$= 1 - \frac{K^a b L^b}{z^a} \int_L^{\infty} \frac{1}{y^{a+b+1}} dy$$

$$= 1 - \frac{K^a b L^b}{z^a} \left[ \frac{y^{-a-b}}{-a-b} \right]_L^{\infty}$$

$$= 1 - \frac{b}{a+b} \frac{K^a}{L^a} \cdot \frac{1}{z^a}$$

The CDF of  $Z$  is

$$F_Z(z) = \begin{cases} \frac{a}{a+b} \left(\frac{L}{k}\right)^b z^b, & z < \frac{k}{L} \\ 1 - \frac{b}{a+b} \left(\frac{k}{L}\right)^a \frac{1}{z^a}, & z > \frac{k}{L} \end{cases}$$

Ex 1

What is the current ratio to be expected once in every 100 times?

$$F_Z(z) = 1 - \frac{1}{100}$$

$$\Rightarrow 1 - \frac{b}{a+b} \left(\frac{k}{L}\right)^a \frac{1}{z^a} = 1 - \frac{1}{100}$$

$$\Rightarrow \frac{b}{a+b} \left(\frac{k}{L}\right)^a \frac{1}{z^a} = \frac{1}{100}$$

$$\Rightarrow z^{-a} = \frac{a+b}{b} \left(\frac{L}{k}\right)^a \frac{1}{100}$$

$$\Rightarrow z = \left[ \frac{a+b}{b} \cdot \left(\frac{L}{k}\right)^a \frac{1}{100} \right]^{-\frac{1}{a}}$$

## Ex 2

What is the average value of current ratio?

The PDF of  $Z$  is

$$f_Z(z) = \begin{cases} \frac{ab}{a+b} \left(\frac{L}{k}\right)^b z^{b-1}, & z < \frac{k}{L} \\ \frac{ab}{a+b} \left(\frac{k}{L}\right)^a \frac{1}{z^{a+1}}, & z > \frac{k}{L} \end{cases}$$

$$E(Z) = \frac{ab}{a+b} \left(\frac{L}{k}\right)^b \int_0^{k/L} z^b dz + \frac{ab}{a+b} \left(\frac{k}{L}\right)^a \int_{k/L}^{\infty} z^{-a} dz$$

$$= \frac{ab}{a+b} \left(\frac{L}{k}\right)^b \frac{1}{b+1} \left(\frac{k}{L}\right)^{b+1} + \frac{ab}{a+b} \left(\frac{k}{L}\right)^a \frac{1}{a-1} \left(\frac{k}{L}\right)^{1-a}$$

$$= \frac{ab}{(a+b)(b+1)} \cdot \frac{k}{L} + \frac{ab}{(a+b)(a-1)} \cdot \frac{k}{L}$$

**LECTURE**

**13 OCTOBER**

**9:00-10:00AM**

**MATH3/4/68181**

Is there a way to find out that none of (I) - (III) will hold? Yes.

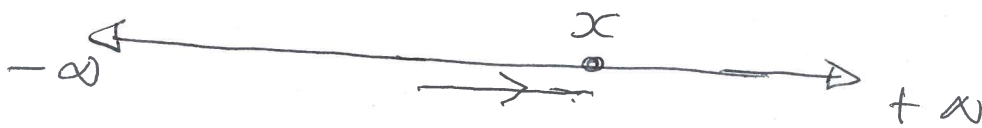
### Discrete Case

None of the conditions will be satisfied (ETT will fail) if

$$\frac{\text{PMF} \rightarrow P(X = k)}{1 - \text{CDF} \rightarrow F(k-1)} \not\rightarrow 0 \text{ as } k \rightarrow \omega(F).$$

### Continuous Case

$$\frac{\text{PDF} \rightarrow f(x)}{1 - \text{CDF} \rightarrow F(x^-)} \not\rightarrow 0 \text{ as } x \rightarrow \omega(F)$$



$x^-$



$x^+$



eg 3

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k},$$

$k = 0, 1, \dots, n$

Binomial

$$w(F) = n$$

$$\begin{aligned} \lim_{k \rightarrow n} \frac{P(X=k)}{1-F(k-1)} &= \frac{P(X=n)}{1-F(n-1)} \\ &= \frac{P(X=n)}{1-P(X \leq n-1)} = \frac{P(X=n)}{P(X > n-1)} \\ &= \frac{P(X=n)}{P(X=n)} = 1 \neq 0 \end{aligned}$$

$\Rightarrow$  ETT fails

eg 4

$$P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, 2, \dots$$

Poisson

$$w(F) = +\infty$$

$$\lim_{k \rightarrow \infty} \frac{P(X = k)}{1 - F(k-1)} = \lim_{k \rightarrow \infty} \frac{P(X = k)}{1 - P(X \leq k-1)}$$

$$= \lim_{k \rightarrow \infty} \frac{P(X = k)}{P(X \geq k)} = \lim_{k \rightarrow \infty} \frac{P(X = k)}{\sum_{j=k}^{\infty} P(X = j)}$$

$$= \lim_{k \rightarrow \infty} \frac{P(X = k)}{P(X = k) + \sum_{j=k+1}^{\infty} P(X = j)}$$

$$= \lim_{k \rightarrow \infty} \frac{\cancel{e^{-\lambda}} \lambda^k}{k!}{\cancel{e^{-\lambda}} \lambda^k / k! + \sum_{j=k+1}^{\infty} \cancel{e^{-\lambda}} \lambda^j / j!}$$

$$= \lim_{k \rightarrow \infty} \frac{\lambda^k / k!}{\lambda^k / k! + \sum_{j=k+1}^{\infty} \lambda^j / j!}$$

$$= \lim_{k \rightarrow \infty} \frac{1}{1 + \sum_{j=k+1}^{\infty} \frac{\lambda^{j-k} k!}{j!}}$$

$$= \lim_{k \rightarrow \infty} \frac{1}{1 + \sum_{j=k+1}^{\infty} \frac{\lambda^{j-k}}{(k+1)(k+2)\dots(j)}} \geq \frac{1}{k^{j-k}}$$

$$\geq \lim_{k \rightarrow \infty} \frac{1}{1 + \sum_{j=k+1}^{\infty} \left(\frac{\lambda}{k}\right)^{j-k}}$$

[  $m = j - k$  ]

$$= \lim_{k \rightarrow \infty} \frac{1}{1 + \sum_{m=1}^{\infty} \left(\frac{\lambda}{k}\right)^m}$$

$$= \lim_{k \rightarrow \infty} \frac{1}{1 + \frac{\lambda/k \rightarrow 0}{1 - \frac{\lambda}{k} \rightarrow 0}}$$

$$\sum_{m=1}^{\infty} a^m = \frac{a}{1-a}$$

$$= 1$$

$$\Rightarrow \lim_{k \rightarrow \infty} \frac{P(X=k)}{1 - F(k-1)} \geq 1$$

$$\Rightarrow \lim_{k \rightarrow \infty} \frac{P(X=k)}{1 - F(k-1)} \neq 0$$

$\Rightarrow$  ETT fails

# Portfolio

# Theory

Portfolio is a collection of investments.

Suppose there are  $n$  investments.

Let  $X_1 =$  loss in investment 1

$X_2 =$  " " " 2

$\vdots$

$X_n =$  " " "  $n$

## The six cases

- 1)  $X_1, \dots, X_n$  are IID RVs  
and  $n$  is fixed
- 2)  $X_1, \dots, X_n$  are IID RVs  
and  $n$  is a RV
- 3)  $X_1, \dots, X_n$  are INID RVs  
(independent but not identical)  
and  $n$  is a fixed
- 4)  $X_1, \dots, X_n$  are INID RVs  
and  $n$  is a RV
- 5)  $X_1, \dots, X_n$  are dependent RVs  
and  $n$  is fixed
- 6)  $X_1, \dots, X_n$  are dependent RVs  
and  $n$  is a RV.

## Variables of interest

1) Total portfolio loss

$$S = X_1 + X_2 + \dots + X_n$$

2) Maximum portfolio loss

$$U = \max(X_1, \dots, X_n)$$

3) Minimum portfolio loss

$$V = \min(X_1, \dots, X_n)$$

Case 1 The CDF of  $S'$  is

$$F_{S'}(s) = \int \int \dots \int_{\{x_1 + \dots + x_n \leq s\}} f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_n}(s - x_1 - \dots - x_{n-1}) dx_{n-1} \dots dx_2 dx_1$$

The PDF of  $S'$  is

$$f_{S'}(s) = \int \int \dots \int_{\{x_1 + \dots + x_n = s\}} f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_n}(s - x_1 - \dots - x_{n-1}) dx_{n-1} \dots dx_2 dx_1$$

The  $m^{\text{th}}$  moment of  $S'$  is

$$E(S'^m) = E[(X_1 + \dots + X_n)^m] \\ = \sum_{\{m_1 + \dots + m_n = m\}} \frac{m!}{m_1! \dots m_n!} E(X_1^{m_1}) \dots E(X_n^{m_n})$$

The CDF of  $U$  is

$$P(U \leq u) = P(\max(X_1, \dots, X_n) \leq u)$$

$$= P(X_1 \leq u, \dots, X_n \leq u)$$

$$\stackrel{\text{indep}}{=} P(X_1 \leq u) \dots P(X_n \leq u)$$

$$= F_{X_1}(u) \dots F_{X_n}(u)$$

$$= [F_X(u)]^n$$

The PDF of  $U$  is

$$f_U(u) = n [F_X(u)]^{n-1} f_X(u)$$

The  $m$ th moment of  $U$  is

$$E(U^m) = n \int_{-\infty}^{+\infty} u^m [F_X(u)]^{n-1} f_X(u) du$$



The CDF of  $V$  is

$$F_V(v) = P(\bar{V} \leq v)$$

$$= 1 - [1 - F_X(v)]^n$$

The PDF of  $V$  is

$$f_V(v) = n [1 - F_X(v)]^{n-1} f_X(v)$$

The  $m$ th moment  $\bar{V}$  is

$$E(V^m) = n \int_{-\infty}^{\infty} v^m [1 - F_X(v)]^{n-1} f_X(v) dv$$

# **EXAMPLE CLASS**

**16 OCTOBER**

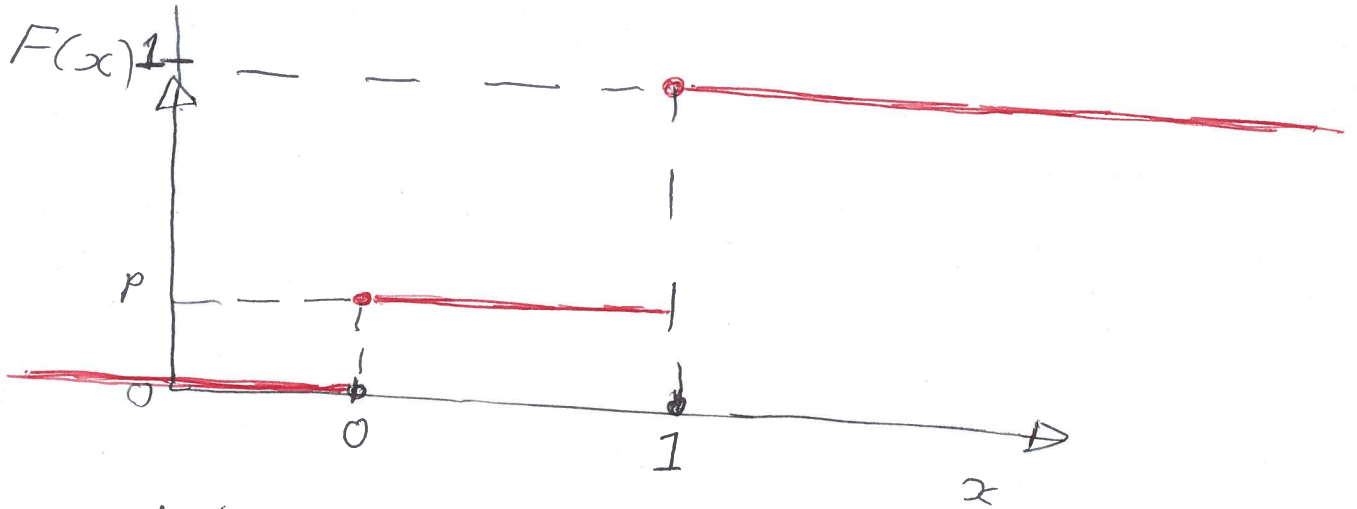
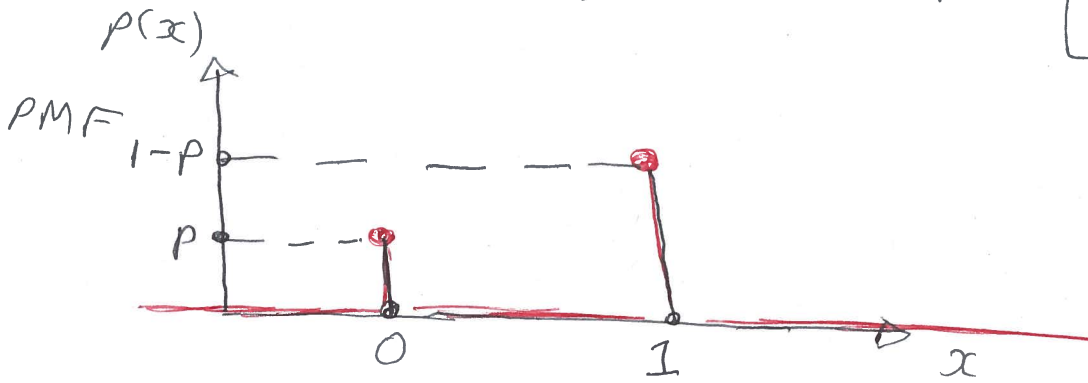
**12:00-13:00PM**

**MATH3/4/68181**

Q1

Bernoulli ( $p$ )

$$X = \begin{cases} 0 & \text{w.p. } p \\ 1 & \text{w.p. } 1-p \end{cases}$$

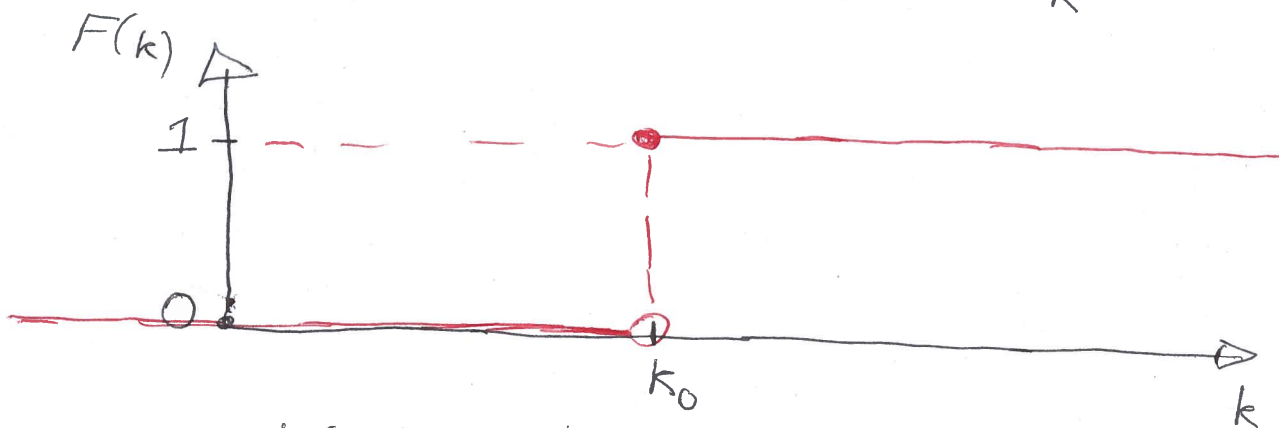
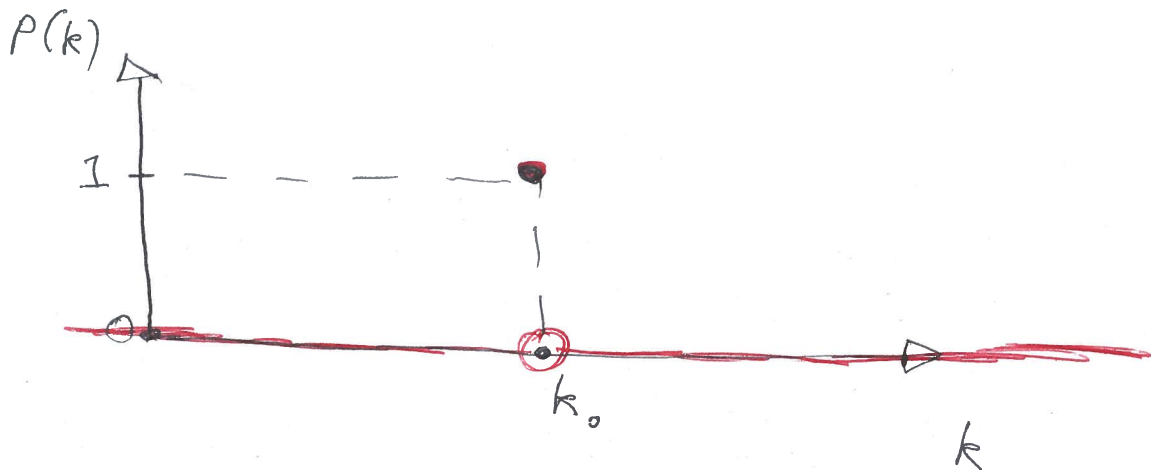


$$w(F) = 1$$

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{P(X=k)}{w(F)} &= \frac{P(X=1)}{1-F(0)} \\ &= \frac{1-p}{1-p} \\ &= 1 \neq 0 \end{aligned}$$

ETT fails.

Q2



$$w(F) = k_0$$

$$\begin{aligned} \lim_{k \rightarrow w(F)} \frac{P(X=k)}{1-F(k-1)} &= \frac{P(X=k_0)}{1-F(k_0-1)} \\ &= \frac{1}{1-0} = 1 \neq 0 \end{aligned}$$

ETT fails

Q4

$$P(k) = \frac{k^{-s}}{\zeta(s)}, \quad k \geq 1$$

$$w(F) = \infty.$$

$$\lim_{k \rightarrow \infty} \frac{P(X=k)}{1-F(k-1)} = \lim_{k \rightarrow \infty} \frac{P(X=k)}{1-P(X \leq k-1)}$$

$$= \lim_{k \rightarrow \infty} \frac{P(X=k)}{P(X \geq k)} = \lim_{k \rightarrow \infty} \frac{P(X=k)}{\sum_{j=k}^{\infty} P(X=j)}$$

$$= \lim_{k \rightarrow \infty} \frac{\frac{k^{-s}}{\zeta(s)}}{\sum_{j=k}^{\infty} \frac{j^{-s}}{\zeta(s)}}$$

$$= \lim_{k \rightarrow \infty} \frac{k^{-s}}{\sum_{j=k}^{\infty} j^{-s}}$$

$$= \lim_{k \rightarrow \infty} \frac{k^{-s}}{\int_k^{\infty} x^{-s} dx}$$

$$= \lim_{k \rightarrow \infty} \frac{k^{-s}}{\left[ \frac{x^{1-s}}{1-s} \right]_k^{\infty}}$$

$$= \lim_{k \rightarrow \infty} \frac{k^{-s}}{0 - \frac{k^{1-s}}{1-s}} = 0, \quad s > 1$$

$\Rightarrow$  ETT ... hold.

$$\frac{d}{dx} \log_2 x = \frac{1}{x \cdot \log 2}$$

Q5

$$P(k) = -\log_2 \left[ 1 - \frac{1}{(k+1)^2} \right], \quad k \geq 1$$

$$W(F) = +\infty$$

$$\lim_{k \rightarrow \infty} \frac{-\log_2 \left[ 1 - \frac{1}{(k+1)^2} \right]}{\cancel{1 - \left\{ \cancel{1 - \log_2 \left[ \frac{k+1}{k} \right] \right\}}}$$

$$= \lim_{k \rightarrow \infty} \frac{\log_2 \left[ \frac{k^2 + 2k}{(k+1)^2} \right]}{\log_2 \left[ \frac{k+1}{k} \right]}$$

$$= \lim_{k \rightarrow \infty} \frac{\log_2(k^2 + 2k) - 2 \log_2(k+1)}{\log_2(k+1) - \log_2 k}$$

$$\stackrel{LH}{=} \lim_{k \rightarrow \infty} \frac{\frac{2k+2}{k^2+2k} \cdot \cancel{\frac{1}{\log_2}} - \frac{2}{k+1} \cdot \cancel{\frac{1}{\log_2}}}{\frac{1}{k+1} \cdot \cancel{\frac{1}{\log_2}} - \frac{1}{k} \cdot \cancel{\frac{1}{\log_2}}}$$

$$= \lim_{k \rightarrow \infty} \frac{\frac{2(k+1)}{k(k+2)} - \frac{2}{k+1}}{\frac{1}{k(k+1)}}$$

$$= \lim_{k \rightarrow \infty} \frac{2(k+1)^2}{k+2} + 2k$$

$$= \lim_{k \rightarrow \infty} \frac{2 \left[ \frac{(k+1)^2 - k(k+2)}{(k+2)} \right]}{1}$$

= 0  $\Rightarrow$  EIT will hold.

**LECTURE**

**17 OCTOBER**

**9:00-10:00AM**

**MATH3/4/68181**



# Portfolio Theory

Portfolio is a collection of investments.

Suppose there are  $n$  investments.

Let  $X_1 =$  loss in investment 1  
 $X_2 =$  " " " 2  
:  
 $X_n =$  " " "  $n$

## The six cases

- 1)  $X_1, \dots, X_n$  are IID RVs  
and  $n$  is fixed
- 2)  $X_1, \dots, X_n$  are IID RVs  
and  $n$  is a RV
- 3)  $X_1, \dots, X_n$  are INIID RVs  
(independent but not identical)  
and  $n$  is a fixed
- 4)  $X_1, \dots, X_n$  are INIID RVs  
and  $n$  is a RV
- 5)  $X_1, \dots, X_n$  are dependent RVs  
and  $n$  is fixed
- 6)  $X_1, \dots, X_n$  are dependent RVs  
and  $n$  is a RV.

## Variables of interest

1) Total portfolio loss

$$S^t = X_1 + X_2 + \dots + X_n$$

2) Maximum portfolio loss

$$U = \max(X_1, \dots, X_n)$$

3) Minimum portfolio loss

$$V = \min(X_1, \dots, X_n)$$

**Case I** The CDF of  $S'$  is

$$F_{S'}(s) = \int \int \dots \int_{\{x_1 + \dots + x_n \leq s\}} f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_n}(s - x_1 - \dots - x_{n-1}) dx_{n-1} \dots dx_2 dx_1$$

The PDF of  $S'$  is

$$f_{S'}(s) = \int \int \dots \int_{\{x_1 + \dots + x_n = s\}} f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_n}(s - x_1 - \dots - x_{n-1}) dx_{n-1} \dots dx_2 dx_1$$

The  $m^{\text{th}}$  moment of  $S'$  is

$$E(S'^m) = E[(X_1 + \dots + X_n)^m] \\ = \sum_{\{m_1 + \dots + m_n = m\}} \frac{m!}{m_1! \dots m_n!} E(X_1^{m_1}) \dots E(X_n^{m_n})$$

The CDF of  $U$  is

$$\begin{aligned} P(U \leq u) &= P(\max(X_1, \dots, X_n) \leq u) \\ &= P(X_1 \leq u, \dots, X_n \leq u) \\ &\stackrel{\text{indep}}{=} P(X_1 \leq u) \cdot \dots \cdot P(X_n \leq u) \\ &= F_{X_1}(u) \cdot \dots \cdot F_{X_n}(u) \\ &= [F_X(u)]^n \end{aligned}$$

The PDF of  $U$  is

$$f_U(u) = n [F_X(u)]^{n-1} f_X(u)$$

The  $m$ th moment of  $U$  is

$$E(U^m) = n \int_{-\infty}^{+\infty} u^m [F_X(u)]^{n-1} f_X(u) du$$

The CDF of  $\bar{V}$  is

$$F_{\bar{V}}(v) = P(\bar{V} \leq v)$$

$$= 1 - [1 - F_X(v)]^n$$

The PDF of  $\bar{V}$  is

$$f_{\bar{V}}(v) = n [1 - F_X(v)]^{n-1} f_X(v)$$

The  $m$ th moment  $\bar{V}$  is

$$E(\bar{V}^m) = n \int_{-\infty}^{\infty} v^m [1 - F_X(v)]^{n-1} f_X(v) dv$$

Case 2

The CDF of  $S'$  is

$$P(S' \leq s)$$

total prob thm

$$= \sum_{n=1}^{\infty} P(S' \leq s | N=n) P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[ \int \int \dots \int_{\{x_1 + \dots + x_n \leq s\}} f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_n}(s - x_1 - \dots - x_{n-1}) dx_{n-1} \dots dx_2 dx_1 \right] \cdot P(N=n)$$

The PDF of  $S'$  is

$$f_{S'}(s) = \sum_{n=1}^{\infty} \left[ \int \int \dots \int_{\{x_1 + \dots + x_n = s\}} f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_n}(s - x_1 - \dots - x_{n-1}) dx_{n-1} \dots dx_2 dx_1 \right] \cdot P(N=n)$$

The  $m$ th moment of  $S$  is

$$E(S^m)$$

$$= E\left[ E(S^m | N) \right]$$

$$= E\left[ E\left( (X_1 + \dots + X_n)^m \mid N=n \right) \right]$$

$$= \sum_{n=1}^{\infty} \left[ \frac{m!}{m_1! \dots m_n!} E(X_1^{m_1}) \dots E(X_n^{m_n}) \right]$$

$$\{m_1 + \dots + m_n = m\} \cdot P(N=n)$$



The CDF of  $\bar{U}$  is

$$P(\bar{U} \leq u) = \sum_{n=1}^{\infty} \underbrace{P(\bar{U} \leq u | N=n)}_{[F_X(u)]^n} P(N=n)$$

$$= \sum_{n=1}^{\infty} [F_X(u)]^n P(N=n)$$

The PDF of  $\bar{U}$  is

$$f_{\bar{U}}(u) = \sum_{n=1}^{\infty} n [F_X(u)]^{n-1} f_X(u) \cdot P(N=n)$$

The  $m$ th moment of  $\bar{U}$  is

$$E(\bar{U}^m) = \sum_{n=1}^{\infty} n \cdot P(N=n) \cdot \int_{-\infty}^{\infty} u^m [F_X(u)]^{n-1} f_X(u) du$$

The CDF of  $V$  is

$$\begin{aligned} & P(V \leq v) \\ &= \sum_{n=1}^{\infty} \underbrace{P(V \leq v | N=n)} P(N=n) \\ &= \sum_{n=1}^{\infty} \{1 - [1 - F_X(v)]^n\} \cdot P(N=n) \\ &= \sum_{n=1}^{\infty} P(N=n) \\ &\quad - \sum_{n=1}^{\infty} [1 - F_X(v)]^n P(N=n) \\ &= 1 - \sum_{n=1}^{\infty} [1 - F_X(v)]^n P(N=n) \end{aligned}$$

The PDF of  $V$  is

$$f_V(v) = \sum_{n=1}^{\infty} n [1 - F_X(v)]^{n-1} f_X(v)$$

The  $m$ th moment of  $V$  is  $\cdot P(N=n)$

$$E(V^m) = \sum_{n=1}^{\infty} n \cdot P(N=n) \cdot \int_{-\infty}^{\infty} v^m [1 - F_X(v)]^{n-1} f_X(v) dv$$

Ex 1 Suppose  $X_1, \dots, X_N$  are IID representing losses on  $N$  investments. Suppose too  $N$  is Poisson ( $\lambda$ ), indep of  $X_1, X_2, \dots$ .

The CDF of  $U$  is

$$P(U \leq u) = \sum_{n=1}^{\infty} [F_X(u)]^n \frac{e^{-\lambda} \lambda^n}{n!}$$

$$= \left[ \sum_{n=1}^{\infty} \frac{[F_X(u) \lambda]^n}{n!} \right] e^{-\lambda}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$



$$= [e^{\lambda F_X(u)} - 1] e^{-\lambda}$$

$$f_U(u) = \lambda e^{\lambda F_X(u) - \lambda} f_X(u)$$

The CDF of  $\bar{V}$  is

$$P(\bar{V} \leq v) = 1 - \sum_{n=1}^{\infty} [1 - F_X(v)]^n \frac{e^{-\lambda} \lambda^n}{n!}$$

$$= 1 - e^{-\lambda} \left[ e^{\lambda [1 - F_X(v)]} - 1 \right]$$

by (\*)

$$= 1 - e^{-\lambda} F_X(v) - e^{-\lambda}$$

The PDF of  $\bar{V}$  is

$$f_{\bar{V}}(v) = \lambda e^{-\lambda} F_X(v) f_X(v).$$

Ex 2

Same as example 1 but  $N$  is Geometric ( $p$ ).

The CDF of  $U$  is

$$P(U \leq u) = \sum_{n=1}^{\infty} [F_X(u)]^n p(1-p)^{n-1}$$

$$= p F_X(u) \sum_{n=1}^{\infty} [F_X(u)]^{n-1} (1-p)^{n-1}$$

$$[k = n-1]$$

$$= p F_X(u) \sum_{k=0}^{\infty} [F_X(u)]^k (1-p)^k$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$$



$$= \frac{p F_X(u)}{1 - (1-p) F_X(u)}$$

$$1 - (1-p) F_X(u)$$

The PDF of  $U$  is

$$f_U(u) = \left\{ [1 - (1-p) F_X(u)] p f_X(u) + p F_X(u) (1-p) f_X(u) \right\}$$

$$/ [1 - (1-p) F_X(u)]^2$$

$$= \frac{p f_X(u)}{[1 - (1-p) F_X(u)]^2}$$

The CDF of  $\bar{V}$  is

$$F_{\bar{V}}(v) = 1 - \sum_{n=1}^{\infty} [1 - F_X(v)]^n p(1-p)^{n-1}$$
$$= 1 - \frac{p [1 - F_X(v)]}{1 - (1-p) [1 - F_X(v)]}$$

by 

$$= \frac{F_X(v)}{1 - (1-p) [1 - F_X(v)]}$$

The PDF of  $\bar{V}$  is

$$f_{\bar{V}}(v) = \left[ \left\{ 1 - (1-p) [1 - F_X(v)] \right\} f_X(v) - F_X(v) (1-p) f_X(v) \right] / \left\{ 1 - (1-p) [1 - F_X(v)] \right\}^2$$

$$= \frac{p f_X(v)}{\left\{ 1 - (1-p) [1 - F_X(v)] \right\}^2}$$

# **EXAMPLE CLASS**

**17 OCTOBER**

**10:00-11:00AM**

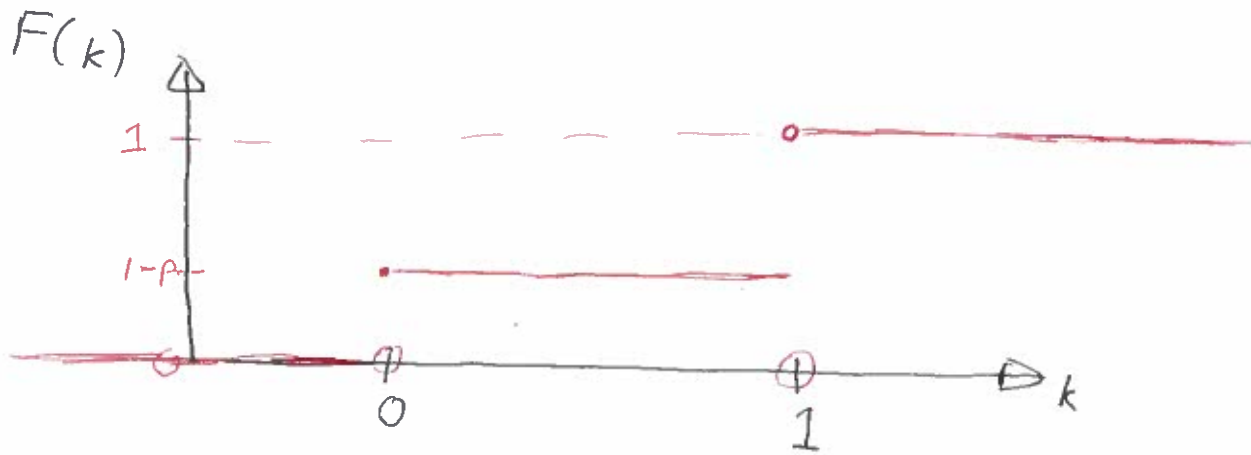
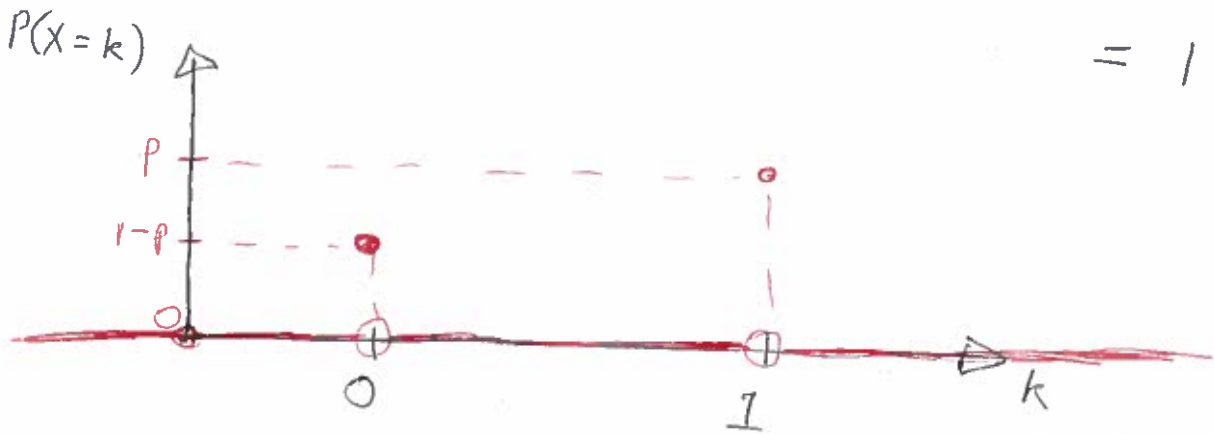
**MATH3/4/68181**

Q1

Bernoulli ( $p$ )

$$X = \begin{cases} 0 & \text{with prob } 1-p \\ 1 & \text{" " } p \end{cases}$$

$$\lim_{k \rightarrow \infty} \frac{P(X=k)}{1-F(k-1)} = \frac{P(X=1)}{1-F(0)} = \frac{p}{1-(1-p)} = 1 \neq 0$$

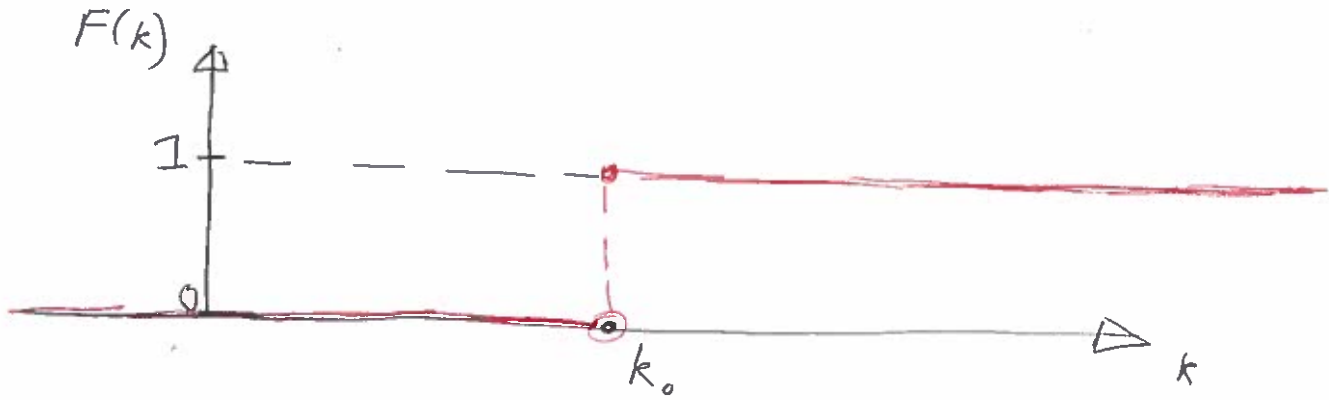
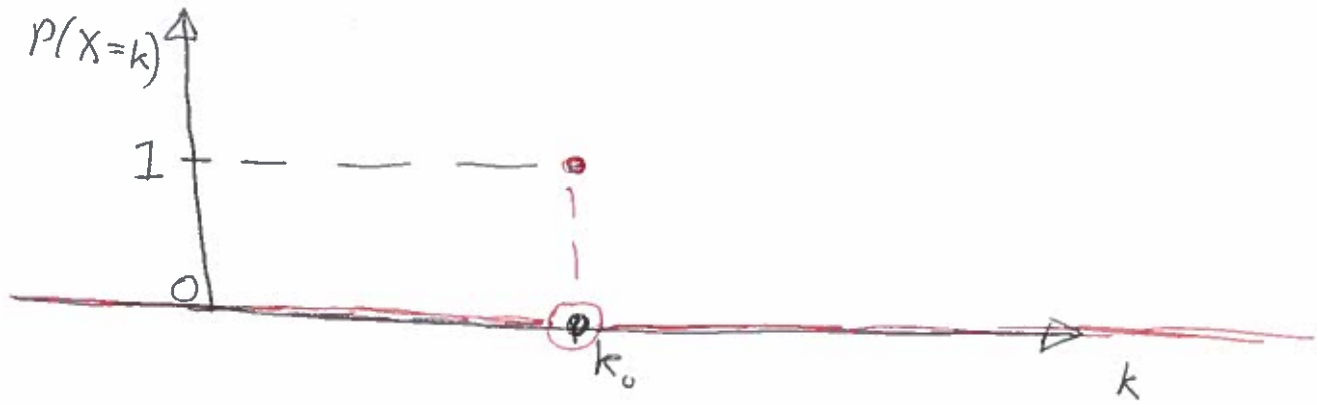


$$W(F) = 1$$

ETT fails to hold.



Q2



$$\lim_{k \rightarrow \infty} \frac{P(X=k)}{1-F(k-1)} = \frac{P(X=k_0)}{1-F(k_0-1)}$$
$$= \frac{1}{1-0} = 1 \neq 0$$

ETT fails to hold.

$$\underline{Q4} \quad p(k) = \frac{k^{-s}}{\zeta(s)}, \quad k \geq 1$$

$$w(F) = +\infty$$

$$\lim_{k \rightarrow \infty} \frac{P(X = k)}{1 - F(k-1)}$$

$$= \lim_{k \rightarrow \infty} \frac{\frac{k^{-s}}{\zeta(s)}}{1 - P(X \leq k-1)}$$

$$= \lim_{k \rightarrow \infty} \frac{\frac{k^{-s}}{\zeta(s)}}{P(X \geq k)}$$

$$= \lim_{k \rightarrow \infty} \frac{\frac{k^{-s}}{\zeta(s)}}{\sum_{j=k}^{\infty} P(X = j)}$$

$$= \lim_{k \rightarrow \infty} \frac{\cancel{\frac{k^{-s}}{\zeta(s)}}}{\sum_{j=k}^{\infty} \frac{j^{-s}}{\cancel{\zeta(s)}}}$$

$$= \lim_{k \rightarrow \infty} \frac{k^{-s}}{\sum_{j=k}^{\infty} j^{-s}}$$

$$= \lim_{k \rightarrow \infty} \frac{k^{-s}}{\int_k^{\infty} x^{-s} dx}$$

$$= \lim_{k \rightarrow \infty} \frac{k^{-s}}{\left[ \frac{x^{1-s}}{1-s} \right]_k^{\infty}}$$

$$= \lim_{k \rightarrow \infty} \frac{k^{-s}}{0 - \frac{k^{1-s}}{1-s}} \quad \text{if } s > 1$$

$= 0 \Rightarrow ETT$  will hold.

Q5

$$\lim_{k \rightarrow \infty} \frac{-\log_2 \left[ 1 - \frac{1}{(k+1)^2} \right]}{\cancel{1 - \left[ \cancel{1 - \log_2 \left[ \frac{k+1}{k} \right]} \right]}}$$

$$= \lim_{k \rightarrow \infty} \frac{\log_2 \left[ \frac{k^2 + 2k}{(k+1)^2} \right]}{\log_2 \left[ \frac{k+1}{k} \right]}$$

$$= - \lim_{k \rightarrow \infty} \frac{\log_2 (k^2 + 2k) - 2 \log_2 (k+1)}{\log_2 (k+1) - \log_2 k}$$

$$\stackrel{LH}{=} \lim_{k \rightarrow \infty} \frac{\frac{2k+2}{k^2+2k} \cdot \cancel{\frac{1}{\log 2}} - \frac{2}{k+1} \cdot \cancel{\frac{1}{\log 2}}}{\frac{1}{k+1} \cdot \cancel{\frac{1}{\log 2}} - \frac{1}{k} \cdot \cancel{\frac{1}{\log 2}}}$$

$$= - \lim_{k \rightarrow \infty} \frac{\frac{2(k+1)}{k(k+2)} - \frac{2}{k+1}}{\frac{1}{k(k+1)}}$$

$$= - \lim_{k \rightarrow \infty} \left[ \frac{2(k+1)^2}{k+2} - 2k \right]$$

$$= - \lim_{k \rightarrow \infty} 2 \cdot \left[ \frac{(k+1)^2 - k(k+2)}{k+2} \right]$$

$$= - \lim_{k \rightarrow \infty} \frac{2}{k+2} = 0$$

ETT will hold.

$$\frac{d}{dx} \log_2 x = \frac{1}{x} \cdot \frac{1}{\log 2}$$

**LECTURE**

**19 OCTOBER**

**12:00-13:00PM**

**MATH4/68181**

# Portfolio

# Theory

Portfolio is a collection of investments.

Suppose there are  $n$  investments.

Let

$X_1$	=	loss	in	investment	1
$X_2$	=	"	"	"	2
		⋮			
$X_n$	=	"	"	"	$n$

## The six cases

- 1)  $X_1, \dots, X_n$  are IID RVs  
and  $n$  is fixed
- 2)  $X_1, \dots, X_n$  are IID RVs  
and  $n$  is a RV
- 3)  $X_1, \dots, X_n$  are INID RVs  
(independent but not identical)  
and  $n$  is a fixed
- 4)  $X_1, \dots, X_n$  are INID RVs  
and  $n$  is a RV
- 5)  $X_1, \dots, X_n$  are dependent RVs  
and  $n$  is fixed
- 6)  $X_1, \dots, X_n$  are dependent RVs  
and  $n$  is a RV.



## Variables of interest

1) Total portfolio loss

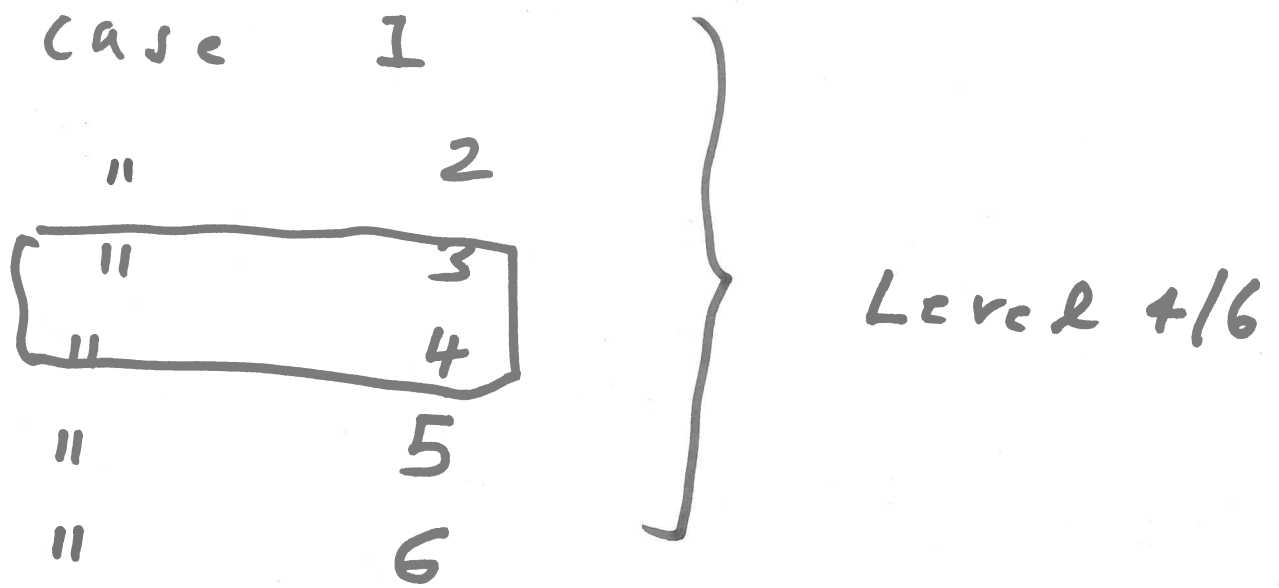
$$S = X_1 + X_2 + \dots + X_n$$

2) Maximum portfolio loss

$$U = \max(X_1, \dots, X_n)$$

3) Minimum portfolio loss

$$V = \min(X_1, \dots, X_n)$$



**Case 3** Losses  $X_1, X_2, \dots, X_n$   
are IID,  $n$  is fixed.

The CDF of  $S'$  is as in Case 1

" PDF " "  $S'$  " " " " "

" moments " "  $S'$  " " " " "

The CDF of  $U$  is

$$P(U \leq u)$$

$$= P(\max(X_1, \dots, X_n) \leq u)$$

$$= P(X_1 \leq u, \dots, X_n \leq u)$$

$$\stackrel{\text{indep}}{=} P(X_1 \leq u) \dots P(X_n \leq u)$$

$$= F_1(u) \dots F_n(u)$$

$$= \prod_{j=1}^n F_j(u).$$

The PDF of  $U$  is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \sum_{k=1}^n f_k(u) \prod_{\substack{j=1 \\ j \neq k}}^n f_j(u)$$

The  $m^{\text{th}}$  moment of  $U$  is

$$E(U^m) = \sum_{k=1}^n \int_{-\infty}^{\infty} u^m f_k(u) \prod_{\substack{j=1 \\ j \neq k}}^n f_j(u) du$$

The CDF of  $V$  is

$$P(V \leq v)$$

$$= 1 - P(V \geq v)$$

$$= 1 - P(\min(X_1, \dots, X_n) \geq v)$$

$$= 1 - P(X_1 > v, \dots, X_n > v)$$

indep

$$= 1 - P(X_1 > v) \dots P(X_n > v)$$

$$= 1 - [1 - P(X_1 \leq v)] \dots [1 - P(X_n \leq v)]$$

$$= 1 - [1 - F_1(v)] \dots [1 - F_n(v)]$$

$$= 1 - \prod_{j=1}^n [1 - F_j(v)]$$

The PDF of  $V$  is

$$f_V(v) = \sum_{k=1}^n f_k(v) \prod_{\substack{j=1 \\ j \neq k}}^n [1 - F_j(v)]$$

The  $m$ th moment of  $V$  is

$$E(V^m) = \sum_{k=1}^n \int_{-\infty}^{\infty} v^m f_k(v)$$

$$\cdot \prod_{\substack{j=1 \\ j \neq k}}^n [1 - F_j(v)] dv$$

**Case 4**  $X_1, \dots, X_N$  IID RVs  
 $N$  is a RV indep of  $X_1, \dots, X_N$ .

The CDF of  $S^t$  is the same as in Case 1,  
" PDF of " " " " " " "  
" moments of " " " " " " "

The CDF of  $U$  is

$$P(U \leq u)$$

total prob rule

$$= \sum_{n=1}^{\infty} \underbrace{P(U \leq u | N=n)}_{\text{total prob rule}} P(N=n)$$

$$= \sum_{n=1}^{\infty} \prod_{j=1}^n F_j(u) P(N=n)$$

The PDF of  $U$  is

$$f_U(u) = \sum_{n=1}^{\infty} \sum_{k=1}^n f_k(u)$$

$$\cdot \prod_{\substack{j=1 \\ j \neq k}}^n F_j(u) P(N=n)$$

The  $m$ th moment of  $U$  is

$$E(U^m) = \sum_{n=1}^{\infty} \sum_{k=1}^n \int_{-\infty}^{\infty} u^m f_k(u)$$

$$\cdot \prod_{\substack{j=1 \\ j \neq k}}^n F_j(u) \cdot P(N=n) du$$



The CDF of  $V$  is

$$P(V \leq v)$$

$$= 1 - P(V > v)$$

total  
prob  
rule

$$1 - \sum_{n=1}^{\infty} \underbrace{P(V > v | N=n)} P(N=n)$$

$$= 1 - \sum_{n=1}^{\infty} \prod_{j=1}^n [1 - F_j(v)] P(N=n)$$

The PDF of  $V$  is

$$f_V(v) = \sum_{n=1}^{\infty} \sum_{k=1}^n f_k(v)$$

$$\cdot \prod_{\substack{j=1 \\ j \neq k}}^n [1 - F_j(v)] P(N=n)$$

The  $m$ th moment of  $V$  is

$$E(V^m) = \sum_{n=1}^{\infty} \sum_{k=1}^n \int_{-\infty}^{\infty} v^m f_k(v)$$

$$\cdot \prod_{\substack{j=1 \\ j \neq k}}^n [1 - F_j(v)] P(N=n) dv$$

Ex 1 Suppose  $X_i \sim \text{Exp}(\lambda_i)$ ,  
 $i = 1, \dots, n$  are indep losses.

Suppose  $n$  is fixed.

The CDF of  $U$  is

$$F_U(u) = \prod_{j=1}^n F_j(u)$$

$$= \prod_{j=1}^n [1 - e^{-\lambda_j u}]$$

The PDF of  $U$  is

$$f_U(u) = \sum_{k=1}^n \lambda_k e^{-\lambda_k u} \prod_{\substack{j=1 \\ j \neq k}}^n [1 - e^{-\lambda_j u}]$$

The CDF of  $V$  is

$$F_V(v) = 1 - \prod_{j=1}^n [1 - F_j(v)]$$

$$= 1 - \prod_{j=1}^n e^{-\lambda_j v}$$

$$= 1 - e^{-\left(\sum_{j=1}^n \lambda_j\right) v}$$

The PDF of  $\bar{V}$  is

$$f_{\bar{V}}(v) = \left( \sum_{j=1}^n \lambda_j \right) e^{-\left( \sum_{j=1}^n \lambda_j \right) v}$$

$\Rightarrow \bar{V}$  is  $\text{Exp} \left( \sum_{j=1}^n \lambda_j \right)$ .

Ex 2 Suppose  ~~$X_i > 0$~~   $F_{X_i}(x_i) = e^{-\frac{k_i}{x_i}}$ ,  
 $x_i > 0$ . Suppose also  $n$  is fixed.

The CDF of  $U$  is

$$\begin{aligned} P(U \leq u) &= \prod_{j=1}^n F_j(u) \\ &= \prod_{j=1}^n e^{-\frac{k_j}{u}} = e^{-\frac{\sum_{j=1}^n k_j}{u}} \\ &\quad \text{[Fréchet distribution]} \end{aligned}$$

The PDF of  $U$  is

$$f_U(u) = \frac{\sum_{j=1}^n k_j}{u^2} e^{-\frac{\sum_{j=1}^n k_j}{u}}$$

The CDF of  $V$  is

$$\begin{aligned} P(V < v) &= 1 - \prod_{j=1}^n [1 - F_j(v)] \\ &= 1 - \prod_{j=1}^n \left[ 1 - e^{-\frac{K_j}{v}} \right] \end{aligned}$$

The PDF of  $V$  is

$$\begin{aligned} f_V(v) &= \sum_{k=1}^n \frac{1}{v^2} e^{-\frac{K_k}{v}} \\ &\quad \cdot \prod_{\substack{j=1 \\ j \neq k}}^n \left[ 1 - e^{-\frac{K_j}{v}} \right]. \end{aligned}$$

**LECTURE**

**20 OCTOBER**

**9:00-10:00AM**

**MATH3/4/68181**

# Portfolio

# Theory

Portfolio is a collection of investments.

Suppose there are  $n$  investments.

Let  $X_1 =$  loss in investment 1  
 $X_2 =$  " " " 2  
 $\vdots$   
 $X_n =$  " " "  $n$

## The six cases

- 1)  $X_1, \dots, X_n$  are IID RVs  
and  $n$  is fixed
- 2)  $X_1, \dots, X_n$  are IID RVs  
and  $n$  is a RV
- 3)  $X_1, \dots, X_n$  are INID RVs  
(independent but not identical)  
and  $n$  is a fixed
- 4)  $X_1, \dots, X_n$  are INID RVs  
and  $n$  is a RV
- 5)  $X_1, \dots, X_n$  are dependent RVs  
and  $n$  is fixed
- 6)  $X_1, \dots, X_n$  are dependent RVs  
and  $n$  is a RV.



## Variables of interest

1) Total portfolio loss

$$S^t = X_1 + X_2 + \dots + X_n$$

2) Maximum portfolio loss

$$U = \max(X_1, \dots, X_n)$$

3) Minimum portfolio loss

$$V = \min(X_1, \dots, X_n)$$

### Ex 3

Suppose  $X_1, X_2, \dots, X_n$   
are IID  $\text{Exp}(\lambda)$  RVs.  
Suppose  $n$  is fixed.

Derive

- (i) the CDF of  $S$
- (ii) " PDF of  $S$
- (iii) " CDF of  $U$
- (iv) " PDF of  $U$
- (v) " CDF of  $\bar{V}$
- (vi) " PDF of  $\bar{V}$

### Ex 3 (solutions)

$$S = X_1 + \dots + X_n$$

The MGF of  $S$  is  
(moment generating function)

$$M_{S'}(s) = E[e^{tS}]$$

$$= E[e^{t(X_1 + \dots + X_n)}]$$

$$= E[e^{tX_1} \dots e^{tX_n}]$$

indep

$$\rightarrow = E[e^{tX_1}] \dots E[e^{tX_n}]$$

$$= \left( \int_0^{\infty} e^{tx} \cdot \lambda e^{-\lambda x} dx \right) \dots \left( \int_0^{\infty} e^{tx} \cdot \lambda e^{-\lambda x} dx \right)$$

$$= \left( \lambda \int_0^{\infty} e^{-(\lambda-t)x} dx \right)^n$$

$$= \left( \lambda \left[ \frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^{\infty} \right)^n$$

$$= \left( \lambda \left( 0 - \frac{1}{-(\lambda-t)} \right) \right)^n$$

$$= \left( \frac{\lambda}{\lambda-t} \right)^n$$

the MGF of a Gamma RV with parameters  $n$  &  $\lambda$ .

$\Rightarrow S$  is a gamma RV with parameters  $n$  &  $\lambda$ .

$$\Rightarrow f_{S^X}(s) = \frac{\lambda^n s^{n-1} e^{-\lambda s}}{\Gamma(n)}$$

[Please look at Math 20802 notes]

The CDF of  $T$  is

$$\begin{aligned} F_T(u) &= [F_X(u)]^n \\ &= [1 - e^{-\lambda u}]^n \end{aligned}$$

The PDF of  $T$  is

$$f_T(u) = n \lambda e^{-\lambda u} [1 - e^{-\lambda u}]^{n-1}$$

The CDF of  $V$  is

$$\begin{aligned}F_V(v) &= 1 - [1 - F_X(v)]^n \\&= 1 - [e^{-\lambda v}]^n \\&= 1 - e^{-n\lambda v}\end{aligned}$$

The PDF of  $V$  is

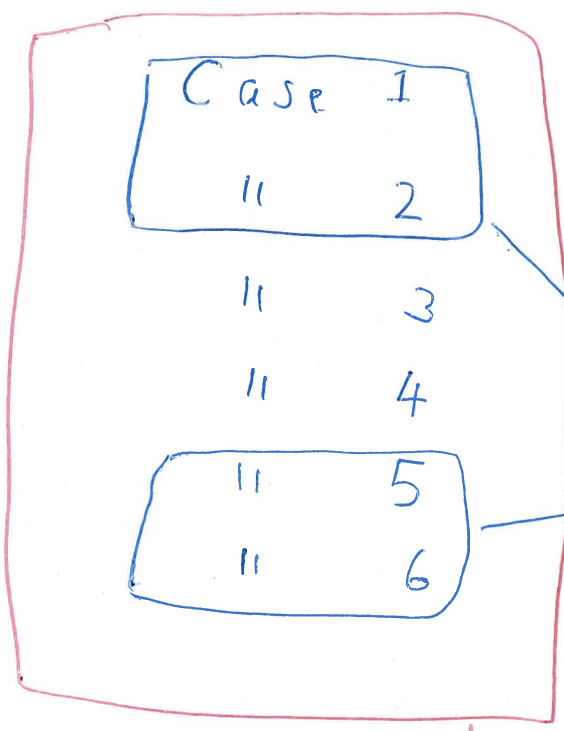
$$f_V(v) = n\lambda e^{-n\lambda v}$$

the PDF of  $\text{Exp}(n\lambda)$ .

$\Rightarrow V$  is an  $\text{Exp}(n\lambda)$ .

PORTFOLIO

THEORY



Level 3

Levels 4, 6

## Terminology

Suppose  $(X, Y)$  is a random vector.

$$F_{X, Y}(x, y) = P(X \leq x, Y \leq y)$$

- Joint CDF of  $(X, Y)$

$$\bar{F}_{X, Y}(x, y) = P(X > x, Y > y)$$

- Joint survival function of  $(X, Y)$

$$\left. \begin{aligned} F_X(x) &= F_{X, Y}(x, \infty) \\ F_X(x) &= 1 - \bar{F}_{X, Y}(x, -\infty) \end{aligned} \right\} \text{CDF of } X.$$

$$\left. \begin{aligned} F_Y(y) &= F_{X, Y}(\infty, y) \\ F_Y(y) &= 1 - \bar{F}_{X, Y}(-\infty, y) \end{aligned} \right\} \text{CDF of } Y$$



Relation between joint CDF & joint SF

$$F_{X, Y}(x, y) = 1 - \{ [1 - F_X(x)] + [1 - F_Y(y)] - \bar{F}_{X, Y}(x, y) \}$$

$$= -1 + F_X(x) + F_Y(y) + \bar{F}_{X, Y}(x, y)$$

$$\bar{F}_{X, Y}(x, y)$$

$$= 1 - F_X(x) - F_Y(y) + F_{X, Y}(x, y)$$

Suppose  $(X, Y, Z)$  is a random vector.

$$F_{X, Y, Z}(x, y, z) = P(X \leq x, Y \leq y, Z \leq z)$$

Joint CDF of  $(X, Y, Z)$

$$\bar{F}_{X, Y, Z}(x, y, z) = P(X > x, Y > y, Z > z)$$

Joint SF of  $(X, Y, Z)$

$$\begin{aligned} \bar{F}_{X, Y, Z}(x, y, z) &= 1 - F_X(x) - F_Y(y) \\ &\quad - F_Z(z) + F_{X, Y}(x, y) + F_{X, Z}(x, z) \\ &\quad + F_{Y, Z}(y, z) - F_{X, Y, Z}(x, y, z) \end{aligned}$$

Suppose  $(X_1, \dots, X_n)$  is a random vector.

$$F_{X_1, \dots, X_n}(x_1, \dots, x_n)$$

$$= P(X_1 < x_1, \dots, X_n < x_n)$$

Joint CDF of  $(X_1, \dots, X_n)$

$$\bar{F}_{X_1, \dots, X_n}(x_1, \dots, x_n)$$

$$= P(X_1 > x_1, \dots, X_n > x_n)$$

Joint SF of  $(X_1, \dots, X_n)$

$$\bar{\bar{F}}_{X_1, \dots, X_n}(x_1, \dots, x_n)$$

$$= 1 - \sum_{i=1}^n F_{X_i}(x_i)$$

$$+ \sum_{i < j} F_{X_i, X_j}(x_i, x_j)$$

$$- \sum_{i < j < k} F_{X_i, X_j, X_k}(x_i, x_j, x_k)$$

$$+ \dots + (-1)^n F_{X_1, \dots, X_n}(x_1, \dots, x_n)$$

**Case 5**  $X_1, \dots, X_n$  are dependent losses.  $n$  is fixed.

The CDF of  $S^l$  is

$$P(S^l \leq s) = \int \int \dots \int_{x_1 + \dots + x_n \leq s} \boxed{f_{X_1, \dots, X_n}}(y_1, \dots, y_n) dy_1 \dots dy_n$$

Joint PDF of  $(X_1, \dots, X_n)$

The PDF of  $S^l$  is

$$f_{S^l}(s) = \int \int \dots \int_{x_1 + \dots + x_n = s} f_{X_1, \dots, X_n}(y_1, \dots, y_n) dy_1 \dots dy_n$$

The  $m^{\text{th}}$  moment of  $S^l$  is

$$E(S^{l m}) =$$

$$\sum_{\substack{\uparrow \\ m_1 + \dots + m_n = m}} \frac{m!}{m_1! \dots m_n!} E(X_1^{m_1} \dots X_n^{m_n})$$

The CDF of  $\bar{U}$  is

$$F_{\bar{U}}(u) = P(\bar{U} \leq u)$$

$$= P(\max(X_1, \dots, X_n) \leq u)$$

$$= P(X_1 \leq u, \dots, X_n \leq u)$$

$$= F_{X_1, \dots, X_n}(u, \dots, u)$$

The PDF of  $\bar{U}$  is

$$f_{\bar{U}}(u) = \frac{d}{du} F_{X_1, \dots, X_n}(u, \dots, u)$$

The  $m$ th moment of  $\bar{U}$  is

$$E(\bar{U}^m) = \int_{-\infty}^{\infty} u^m \frac{d}{du} F_{X_1, \dots, X_n}(u, \dots, u) du$$

The CDF of  $\bar{V}$  is

$$\begin{aligned} & P(\bar{V} \leq v) \\ &= 1 - P(\bar{V} > v) \\ &= 1 - P(\min(X_1, \dots, X_n) > v) \\ &= 1 - P(X_1 > v, \dots, X_n > v) \\ &= 1 - \bar{F}_{X_1, \dots, X_n}(v, \dots, v) \end{aligned}$$

The PDF of  $\bar{V}$  is

$$f_{\bar{V}}(v) = -\frac{d}{dv} \bar{F}_{X_1, \dots, X_n}(v, \dots, v)$$

The  $m$ th moment of  $\bar{V}$  is

$$E(\bar{V}^m) = -\int_{-\infty}^{\infty} v^m \frac{d}{dv} \bar{F}_{X_1, \dots, X_n}(v, \dots, v) dv$$

**EXAMPLE CLASS**

**23 OCTOBER**

**12:00-13:00PM**

**MATH3/4/68181**

[1]

$$P(X \leq x)$$

$$= P(\max(X_1, \dots, X_\alpha) \leq x)$$

$$= P(X_1 \leq x, \dots, X_\alpha \leq x)$$

$$\stackrel{\text{indep}}{=} P(X_1 \leq x) \cdot \dots \cdot P(X_\alpha \leq x)$$

$$= [1 - e^{-\lambda x}] \cdot \dots \cdot [1 - e^{-\lambda x}]$$

$$= [1 - e^{-\lambda x}]^\alpha$$

[2]  $f_X(x) = \alpha [1 - e^{-\lambda x}]^{\alpha-1} \lambda e^{-\lambda x}$



3

$$E(X^n) = \alpha \lambda \int_0^{\infty} x^n e^{-\lambda x} [1 - e^{-\lambda x}]^{\alpha-1} dx$$

$$\begin{aligned} \text{Set } y &= e^{-\lambda x} \\ x &= -\frac{1}{\lambda} \log y \\ \frac{dx}{dy} &= -\frac{1}{\lambda y} \end{aligned}$$

$$= \alpha \lambda \int_1^0 \left(-\frac{1}{\lambda} \log y\right)^n y [1-y]^{\alpha-1} \left(-\frac{dy}{\lambda y}\right)$$

$$= \alpha \int_0^1 \left(-\frac{1}{\lambda} \log y\right)^n [1-y]^{\alpha-1} dy$$

$$= \alpha \left(-\frac{1}{\lambda}\right)^n \int_0^1 (\log y)^n [1-y]^{\alpha-1} dy$$

$$\stackrel{(*)}{=} \alpha \left(-\frac{1}{\lambda}\right)^n \int_0^1 \left(\frac{d^n}{db^n} y^b\right)_{b=0} [1-y]^{\alpha-1} dy$$

$$= \alpha \left(-\frac{1}{\lambda}\right)^n \frac{d^n}{db^n} \left(\int_0^1 y^b [1-y]^{\alpha-1} dy\right)_{b=0}$$

$$= \alpha \left(-\frac{1}{\lambda}\right)^n \frac{d^n}{db^n} B(b+1, \alpha) \Big|_{b=0}$$

$$\frac{d}{db} y^b = y^b \log y$$

$$\frac{d^2}{db^2} y^b = y^b (\log y)^2$$

$$\frac{d^n}{db^n} y^b \Big|_{b=0} = y^b (\log y)^n \Big|_{b=0}$$

$$\frac{d^n}{db^n} y^b \Big|_{b=0} = (\log y)^n \quad (*)$$

Beta Function

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt$$

4

$$E(X) = -\frac{\alpha}{\lambda} \frac{d}{db} B(b+1, \alpha) \Big|_{b=0}$$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$= -\frac{\alpha}{\lambda} \frac{d}{db} \frac{\Gamma(b+1) \Gamma(\alpha)}{\Gamma(b+1+\alpha)} \Big|_{b=0}$$

$$= -\frac{\alpha \Gamma(\alpha)}{\lambda} \left[ \frac{\Gamma(b+1+\alpha) \Gamma'(b+1) - \Gamma(b+1) \Gamma'(b+1+\alpha)}{\Gamma^2(b+1+\alpha)} \right] \Big|_{b=0}$$

$$= -\frac{\alpha \Gamma(\alpha)}{\lambda} \frac{\Gamma(1+\alpha) \Gamma'(1) - \Gamma'(1+\alpha)}{\Gamma^2(1+\alpha)} \Big|_{b=0}$$

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$$L(\alpha, \lambda) = \alpha \lambda e^{-\lambda x} [1 - e^{-\lambda x}]^{\alpha-1}$$

$$\log L = \log(\alpha \lambda) - \lambda x + (\alpha-1) \log[1 - e^{-\lambda x}]$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{1}{\alpha} + \log[1 - e^{-\lambda x}] = 0 \quad \text{--- (1)}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{1}{\lambda} - x + (\alpha-1) \frac{x e^{-\lambda x}}{1 - e^{-\lambda x}} = 0 \quad \text{--- (2)}$$

$$(1) \Rightarrow 1 - e^{-\lambda x} = e^{-1/\alpha}$$

$$\Rightarrow e^{-\lambda x} = 1 - e^{-1/\alpha}$$

$$\Rightarrow \lambda = -\frac{1}{x} \log\left(1 - e^{-1/\alpha}\right) \quad \text{--- (3)}$$

Sub (3) into (2)

$$-\frac{x}{\log\left(1 - e^{-1/\alpha}\right)} - x + (\alpha-1) \frac{x (1 - e^{-1/\alpha})}{e^{-1/\alpha}} = 0 \quad \text{--- (4)}$$

$\hat{\alpha}$  is the root of (4)

Sub  $\hat{\alpha}$  into (3) to get  $\hat{\lambda}$ .

**LECTURE**

**24 OCTOBER**

**9:00-10:00AM**

**MATH3/4/68181**

# Portfolio

# Theory

Portfolio is a collection of investments.

Suppose there are  $n$  investments.

Let  $X_1 =$  loss in investment 1  
 $X_2 =$  " " " 2  
:  
 $X_n =$  " " "  $n$

## The six cases

- 1)  $X_1, \dots, X_n$  are IID RVs  
and  $n$  is fixed
- 2)  $X_1, \dots, X_n$  are IID RVs  
and  $n$  is a RV
- 3)  $X_1, \dots, X_n$  are INIID RVs  
(independent but not identical)  
and  $n$  is a fixed
- 4)  $X_1, \dots, X_n$  are INIID RVs  
and  $n$  is a RV
- 5)  $X_1, \dots, X_n$  are dependent RVs  
and  $n$  is fixed
- 6)  $X_1, \dots, X_n$  are dependent RVs  
and  $n$  is a RV.

## Variables of interest

1) Total portfolio loss

$$S = X_1 + X_2 + \dots + X_n$$

2) Maximum portfolio loss

$$U = \max(X_1, \dots, X_n)$$

3) Minimum portfolio loss

$$V = \min(X_1, \dots, X_n)$$



**Case 5**  $X_1, \dots, X_n$  are dependent losses.  $n$  is fixed.

The CDF of  $S^l$  is

$$P(S^l \leq s) = \int \int \dots \int_{\substack{x_1 + \dots + x_n \leq s \\ y_1 + \dots + y_n}} f_{X_1, \dots, X_n}(y_1, \dots, y_n) dy_1 \dots dy_n$$

Joint PDF of  $(X_1, \dots, X_n)$

The PDF of  $S^l$  is

$$f_{S^l}(s) = \int \int \dots \int_{\substack{x_1 + \dots + x_n = s \\ y_1 + \dots + y_n}} f_{X_1, \dots, X_n}(y_1, \dots, y_n) dy_1 \dots dy_n$$

The  $m$ th moment of  $S^l$  is

$$E(S^l)^m =$$

$$\sum_{\substack{\uparrow \\ m_1 + \dots + m_n = m}} \frac{m!}{m_1! \dots m_n!} E(X_1^{m_1} \dots X_n^{m_n})$$

The CDF of  $\bar{U}$  is

$$\begin{aligned}F_{\bar{U}}(u) &= P(\bar{U} \leq u) \\&= P(\max(X_1, \dots, X_n) \leq u) \\&= P(X_1 \leq u, \dots, X_n \leq u) \\&= F_{X_1, \dots, X_n}(u, \dots, u)\end{aligned}$$

The PDF of  $\bar{U}$  is

$$f_{\bar{U}}(u) = \frac{d}{du} F_{X_1, \dots, X_n}(u, \dots, u)$$

The  $m$ th moment of  $\bar{U}$  is

$$E(\bar{U}^m) = \int_{-\infty}^{\infty} u^m \frac{d}{du} F_{X_1, \dots, X_n}(u, \dots, u) du$$

The CDF of  $\bar{V}$  is

$$\begin{aligned} & P(\bar{V} \leq v) \\ &= 1 - P(\bar{V} > v) \\ &= 1 - P(\min(X_1, \dots, X_n) > v) \\ &= 1 - P(X_1 > v, \dots, X_n > v) \\ &= 1 - \bar{F}_{X_1, \dots, X_n}(v, \dots, v) \end{aligned}$$

The PDF of  $\bar{V}$  is

$$f_{\bar{V}}(v) = -\frac{d}{dv} \bar{F}_{X_1, \dots, X_n}(v, \dots, v)$$

The  $m$ th moment of  $\bar{V}$  is

$$E(\bar{V}^m) = -\int_{-\infty}^{\infty} v^m \frac{d}{dv} \bar{F}_{X_1, \dots, X_n}(v, \dots, v) dv$$

**CASE 6** :  $X_1, \dots, X_N$  are dep random variables and  $N$  is a random variable indep of  $X_1, X_2, \dots$

The CDF of  $\sum^1$  is

total prob rule

$$P(\sum^1 \leq s) = \sum_{n=1}^{\infty} P(\sum^1 \leq s | N=n) P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[ \int \int \dots \int_{y_1 + \dots + y_n \leq s} f_{X_1, \dots, X_n}(y_1, \dots, y_n) dy_1 \dots dy_n \right] P(N=n)$$

The PDF of  $\sum^1$  is

$$f_{\sum^1}(s) = \sum_{n=1}^{\infty} \left[ \int \int \dots \int_{y_1 + \dots + y_n = s} f_{X_1, \dots, X_n}(y_1, \dots, y_n) dy_1 \dots dy_n \right] P(N=n)$$

The  $m$ th moment

$$E(\sum^1{}^m) = \sum_{n=1}^{\infty} E(\sum^1{}^m | N=n) P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[ \sum_{\substack{m_1 + \dots + m_n = m}} \frac{m!}{m_1! \dots m_n!} E(X_1^{m_1} \dots X_n^{m_n}) \right] P(N=n)$$

The CDF of  $\bar{U}$  is

$$\begin{aligned} F_{\bar{U}}(u) &= P(\bar{U} \leq u) \\ &= \sum_{n=1}^{\infty} \underbrace{P(\bar{U} \leq u | N=n)}_{F_{X_1, \dots, X_n}(u, \dots, u)} P(N=n) \\ &= \sum_{n=1}^{\infty} F_{X_1, \dots, X_n}(u, \dots, u) P(N=n) \end{aligned}$$

The PDF of  $\bar{U}$  is

$$f_{\bar{U}}(u) = \sum_{n=1}^{\infty} \frac{\partial}{\partial u} F_{X_1, \dots, X_n}(u, \dots, u) \cdot P(N=n)$$

The  $m$ th moment of  $\bar{U}$  is

$$E(\bar{U}^m) = \sum_{n=1}^{\infty} \left[ \int_{-\infty}^{\infty} u^m \frac{\partial}{\partial u} F_{X_1, \dots, X_n}(u, \dots, u) du \right] \cdot P(N=n)$$

The CDF of  $\bar{V}$  is

$$F_{\bar{V}}(v) = P(\bar{V} \leq v)$$

$$= 1 - P(\bar{V} > v)$$

$$= 1 - \sum_{n=1}^{\infty} \underbrace{P(\bar{V} > v | N=n)}_{\bar{F}_{X_1, \dots, X_n}(v, \dots, v)} P(N=n)$$

$$= 1 - \sum_{n=1}^{\infty} \bar{F}_{X_1, \dots, X_n}(v, \dots, v) P(N=n)$$

The PDF of  $\bar{V}$  is

$$f_{\bar{V}}(v) = - \sum_{n=1}^{\infty} \frac{\partial}{\partial v} \bar{F}_{X_1, \dots, X_n}(v, \dots, v)$$

$$\cdot P(N=n)$$

The  $m^{\text{th}}$  moment of  $\bar{V}$  is

$$E(\bar{V}^m) = - \sum_{n=1}^{\infty} \left[ \int_{-\infty}^{\infty} v^m \frac{\partial}{\partial v} \bar{F}_{X_1, \dots, X_n}(v, \dots, v) dv \right]$$

$$\cdot P(N=n)$$

Ex 1 Suppose a portfolio has 2 assets. Their losses say  $X$  and  $Y$  have the joint survival function

$$\bar{F}_{X,Y}(x,y) = \left[ 1 + \frac{x}{a} + \frac{y}{b} \right]^{-c},$$

$$x > 0, y > 0$$

Find the following

- (i) cdf of  $U$
- (ii) pdf "  $U$
- (iii) cdf of  $\bar{V}$
- (iv) pdf "  $\bar{V}$
- (v) " "  $\bar{V}$
- (vi) cdf "  $\bar{V}$

$$(i) \quad F_U(u) = F_{X,Y}(u, u)$$

$$= 1 - [1 - F_X(u)]$$

$$- [1 - F_Y(u)]$$

$$+ \bar{F}_{X,Y}(u, u)$$

$$= -1 + F_X(u) + F_Y(u)$$

$$+ \bar{F}_{X,Y}(u, u)$$

$$= -1 + [1 - \bar{F}_{X,Y}(u, 0)]$$

$$+ [1 - \bar{F}_{X,Y}(0, u)]$$

$$+ \bar{F}_{X,Y}(u, u)$$

$$= 1 - \bar{F}_{X,Y}(u, 0) - \bar{F}_{X,Y}(0, u)$$

$$+ \bar{F}_{X,Y}(u, u)$$

$$= 1 - \left[1 + \frac{u}{a}\right]^{-c} - \left[1 + \frac{u}{b}\right]^{-c}$$

$$+ \left[1 + \frac{u}{a} + \frac{u}{b}\right]^{-c}$$



$$(ii) f_U(u) = \frac{c}{a} \left[ 1 + \frac{u}{a} \right]^{-c-1}$$

$$+ \frac{c}{b} \left[ 1 + \frac{u}{b} \right]^{-c-1}$$

$$- c \left( \frac{1}{a} + \frac{1}{b} \right) \left[ 1 + \frac{u}{a} + \frac{u}{b} \right]^{-c-1}$$

$$(iii) F_U(u) = 1 - \overline{F}_{X,Y}(u, u)$$

$$= 1 - \left[ 1 + \frac{u}{a} + \frac{u}{b} \right]^{-c}$$

$$(iv) f_U(u) = c \left( \frac{1}{a} + \frac{1}{b} \right) \left[ 1 + \frac{u}{a} + \frac{u}{b} \right]^{-c-1}$$

(v)

The joint PDF of  $(X, Y)$

$$f_{X, Y}(x, y) = \frac{\partial^2}{\partial x \partial y} \bar{F}_{X, Y}(x, y)$$

$$= \frac{\partial}{\partial x} \left( \frac{-c}{b} \right) \left[ 1 + \frac{x}{a} + \frac{y}{b} \right]^{-c-1}$$

$$= \frac{c(c+1)}{ab} \left[ 1 + \frac{x}{a} + \frac{y}{b} \right]^{-c-2}$$

$$f_S(s) = \int_{x+y=s} \frac{c(c+1)}{ab} \left[ 1 + \frac{x}{a} + \frac{y}{b} \right]^{-c-2} dx dy$$

$$= \int_0^s \frac{c(c+1)}{ab} \left[ 1 + \frac{x}{a} + \frac{s-x}{b} \right]^{-c-2} dx$$

$$= \frac{c(c+1)}{ab} \int_0^s \left[ 1 + \frac{s}{b} + \left( \frac{1}{a} - \frac{1}{b} \right) x \right]^{-c-2} dx$$

$$= \frac{c(c+1)}{ab} \left\{ \frac{\left[ 1 + \frac{s}{b} + \left( \frac{1}{a} - \frac{1}{b} \right) x \right]^{-c-1}}{(-c-1) \left( \frac{1}{a} - \frac{1}{b} \right)} \right\}_0^s$$

$$= \frac{c(c+1)}{ab} \left\{ \frac{\left(1 + \frac{s}{a}\right)^{-c-1}}{(-c-1)\left(\frac{1}{a} - \frac{1}{b}\right)} + \frac{\left(1 + \frac{s}{b}\right)^{-c-1}}{(c+1)\left(\frac{1}{a} - \frac{1}{b}\right)} \right\}$$

$$= \frac{c}{b-a} \left\{ \left(1 + \frac{s}{b}\right)^{-c-1} - \left(1 + \frac{s}{a}\right)^{-c-1} \right\}$$

$$(vi) F_{\sqrt{1}}(s) = \int_0^s f_{\sqrt{1}}(t) dt$$

$$= \frac{c}{b-a} \left\{ \int_0^s \left(1 + \frac{t}{b}\right)^{-c-1} dt \right.$$

$$\left. - \int_0^s \left(1 + \frac{t}{a}\right)^{-c-1} dt \right\}$$

$$= \frac{c}{b-a} \left\{ \left[ \frac{b \left(1 + \frac{t}{b}\right)^{-c}}{(-c)} \right]_0^s \right.$$

$$\left. - \left[ \frac{a \left(1 + \frac{t}{a}\right)^{-c}}{(-c)} \right]_0^s \right\}$$

$$= \frac{c}{b-a} \left\{ \frac{b \left(1 + \frac{s}{b}\right)^{-c}}{(-c)} + \frac{b}{c} \right.$$

$$\left. - \frac{a \left(1 + \frac{s}{a}\right)^{-c}}{(-c)} - \frac{a}{c} \right\}$$

$$= \frac{1}{b-a} \left\{ -b \left( 1 + \frac{s}{b} \right)^{-c} + b \right. \\ \left. + a \left( 1 + \frac{s}{a} \right)^{-c} - a \right\}$$

Financial

Risk

Measures

**EXAMPLE CLASS**

**24 OCTOBER**

**10:00-11:00AM**

**MATH3/4/68181**

[1]

$$\begin{aligned}F_X(x) &= P(X \leq x) \\&= P[\max(X_1, \dots, X_n) \leq x] \\&= P[X_1 \leq x, \dots, X_n \leq x] \\&\stackrel{\text{indep}}{=} P(X_1 \leq x) \cdots P(X_n \leq x) \\&= [1 - e^{-\lambda x}] \cdots [1 - e^{-\lambda x}] \\&= [1 - e^{-\lambda x}]^n\end{aligned}$$

[2]

$$\begin{aligned}f_X(x) &= n [1 - e^{-\lambda x}]^{n-1} \lambda e^{-\lambda x} \\&= n \lambda e^{-\lambda x} [1 - e^{-\lambda x}]^{n-1}\end{aligned}$$

$$\frac{d y^b}{d b} = y^b \log y$$

$$\frac{d^2}{d b^2} y^b = y^b (\log y)^2$$

$$\frac{d^n}{d b^n} y^b \Big|_{b=0} = y^b (\log y)^n \Big|_{b=0}$$

$$\frac{d^n}{d b^n} y^b \Big|_{b=0} = (\log y)^n \cdot \text{(*)}$$

Beta Function

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt$$



3

$$E(X^n) = \alpha \lambda \int_0^{\infty} x^n e^{-\lambda x} [1 - e^{-\lambda x}]^{\alpha-1} dx$$

Set  $y = e^{-\lambda x}$   
 $x = -\frac{1}{\lambda} \log y$   
 $\frac{dx}{dy} = -\frac{1}{\lambda y}$

$$= \alpha \lambda \int_1^0 \left(-\frac{1}{\lambda} \log y\right)^n y (1-y)^{\alpha-1} \frac{dy}{(-\lambda y)}$$

$$= \alpha \left(-\frac{1}{\lambda}\right)^n \int_1^0 (\log y)^n (1-y)^{\alpha-1} (-dy)$$

$$= \alpha \left(-\frac{1}{\lambda}\right)^n \int_0^1 (\log y)^n (1-y)^{\alpha-1} dy$$

$\otimes$

$$\alpha \left(-\frac{1}{\lambda}\right)^n \int_0^1 \frac{d^n}{db^n} y^b \Big|_{b=0} (1-y)^{\alpha-1} dy$$

$$= \alpha \left(-\frac{1}{\lambda}\right)^n \frac{d^n}{db^n} \int_0^1 y^b (1-y)^{\alpha-1} dy \Big|_{b=0}$$

$$= \alpha \left(-\frac{1}{\lambda}\right)^n \frac{d^n}{db^n} B(b+1, \alpha) \Big|_{b=0}$$

4

$$E(x) = -\frac{\alpha}{\lambda} \frac{d}{db} B(b+1, \alpha) \Big|_{b=0}$$

$$B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$$

$$= -\frac{\alpha}{\lambda} \frac{d}{db} \frac{\Gamma(b+1) \Gamma(\alpha)}{\Gamma(b+1+\alpha)} \Big|_{b=0}$$

$$= -\frac{\alpha \Gamma(\alpha)}{\lambda} \frac{\Gamma(b+1+\alpha) \Gamma'(b+1) - \Gamma(b+1) \Gamma'(b+1+\alpha)}{\Gamma^2(b+1+\alpha)} \Big|_{b=0}$$

$$= -\frac{\alpha \Gamma(\alpha)}{\lambda} \frac{\Gamma(1+\alpha) \Gamma'(1) - \Gamma'(1+\alpha)}{\Gamma^2(1+\alpha)}$$

5

$$\text{Var}(X)$$

$$= E(X^2) - (E(X))^2$$

$$= \frac{\alpha}{\lambda^2} \frac{d^2}{db^2} B(b+1, \alpha) \Big|_{b=0} - (E(X))^2$$

8 Suppose  $X$  is the only observation.

$$L(\alpha, \lambda) = \alpha \lambda e^{-\lambda x} [1 - e^{-\lambda x}]^{\alpha-1}$$

$$\begin{aligned} \log L(\alpha, \lambda) &= \log(\alpha \lambda) - \lambda x \\ &\quad + (\alpha-1) \log[1 - e^{-\lambda x}] \end{aligned}$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{1}{\alpha} + \log[1 - e^{-\lambda x}] = 0 \quad \text{--- (1)}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{1}{\lambda} - x + (\alpha-1) \frac{x e^{-\lambda x}}{1 - e^{-\lambda x}} = 0 \quad \text{--- (2)}$$

$$(1) \Rightarrow 1 - e^{-\lambda x} = e^{-\frac{1}{\alpha}}$$

$$\Rightarrow \lambda x = -\frac{1}{x} \log\left(1 - e^{-\frac{1}{\alpha}}\right) \quad \text{--- (3)}$$

Sub (3) into (2)

$$-\frac{x}{\log\left(1 - e^{-\frac{1}{\alpha}}\right)} - x + (\alpha-1) \frac{x \cdot \left(1 - e^{-\frac{1}{\alpha}}\right)}{e^{-\frac{1}{\alpha}}} = 0 \quad \text{--- (4)}$$

$\hat{\alpha}$  is the root of (4)  
Sub  $\hat{\alpha}$  into (3) to get  $\hat{\lambda}$ .

**LECTURE**

**26 OCTOBER**

**12:00-13:00PM**

**MATH4/68181**

**MATH3/4/68181: Extreme values and financial risk**  
**Semester 1**  
**Problem sheet 15**

Suppose a portfolio is made up of  $n$  assets, where  $n$  is known. Suppose also that the prices of the assets, say  $X_i, i = 1, 2, \dots, n$ , have the joint survival function

$$\bar{F}_{X_1, \dots, X_n}(x_1, \dots, x_n) = \left[ 1 + \frac{1}{a} \sum_{i=1}^n x_i \right]^{-c}$$

for  $x_i > 0, i = 1, 2, \dots, n$ . Find the following:

1. the pdf of  $S$ ;
2. the cdf of  $S$ ;
3. the mean of  $S$ ;
4. the variance of  $S$ ;
5. the cdf of  $V$ ;
6. the pdf of  $V$ ;
7. the mean of  $V$ ;
8. the variance of  $V$ .

II

$$\bar{F}_{X_1, \dots, X_n}(x_1, \dots, x_n)$$

$$= \left[ 1 + \frac{1}{a} \sum_{i=1}^n x_i \right]^{-c}$$

The joint PDF is

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n)$$

$$= (-1)^n \frac{\partial^n}{\partial x_1 \dots \partial x_n} \bar{F}_{X_1, \dots, X_n}(x_1, \dots, x_n) \quad (*)$$

$$\frac{\partial}{\partial x_n} \bar{F} = \left( -\frac{c}{a} \right) \left[ 1 + \frac{1}{a} \sum_{i=1}^n x_i \right]^{-c-1}$$

$$\frac{\partial^2}{\partial x_{n-1} \partial x_n} \bar{F} = \left( -\frac{c}{a} \right) \left( -\frac{c-1}{a} \right) \left[ 1 + \frac{1}{a} \sum_{i=1}^n x_i \right]^{-c-2}$$

$$\frac{\partial^3}{\partial x_{n-2} \partial x_{n-1} \partial x_n} \bar{F} = \left( -\frac{c}{a} \right) \left( -\frac{c-1}{a} \right) \left( -\frac{c-2}{a} \right) \left[ 1 + \frac{1}{a} \sum_{i=1}^n x_i \right]^{-c-3}$$

...

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} \bar{F} = \left( -\frac{c}{a} \right) \left( -\frac{c-1}{a} \right) \dots \left( -\frac{c-n+1}{a} \right) \left[ 1 + \frac{1}{a} \sum_{i=1}^n x_i \right]^{-c-n}$$

By (\*),

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n)$$

$$= \left( \frac{c}{a} \right) \left( \frac{c+1}{a} \right) \dots \left( \frac{c+n-1}{a} \right) \left[ 1 + \frac{1}{a} \sum_{i=1}^n x_i \right]^{-c-n}$$

$$f_{\Sigma}(s) = \int \int \dots \int_{x_1 + \dots + x_n = s} f_{X_1, \dots, X_n}(x_1, \dots, x_n) dx_n \dots dx_1$$

$$= \left(\frac{c}{a}\right) \cdot \left(\frac{c+1}{a}\right) \dots \left(\frac{c+n-1}{a}\right)$$

$$\int \int \dots \int_{x_1 + \dots + x_n = s} \left[1 + \frac{1}{a} \sum_{i=1}^n x_i\right]^{-c-n} dx_n \dots dx_1$$

$$= \left(\frac{c}{a}\right) \cdot \left(\frac{c+1}{a}\right) \dots \left(\frac{c+n-1}{a}\right)$$

$$\int \int \dots \int_{x_1 + \dots + x_n = s} \left[1 + \frac{s}{a}\right]^{-c-n} dx_n \dots dx_1$$

$$= \left(\frac{c}{a}\right) \cdot \left(\frac{c+1}{a}\right) \dots \left(\frac{c+n-1}{a}\right) \cdot \left[1 + \frac{s}{a}\right]^{-c-n}$$

$$\int \int \dots \int_{x_1 + \dots + x_n = s} 1 \cdot dx_n \dots dx_1 = \frac{s^n}{n!}$$

$$= \left(\frac{c}{a}\right) \cdot \left(\frac{c+1}{a}\right) \dots \left(\frac{c+n-1}{a}\right) \cdot \left[1 + \frac{s}{a}\right]^{-c-n} \cdot \frac{s^n}{n!}$$



[2]

$$F_{\mathcal{L}}(s) = \int_0^s f_{\mathcal{L}}(t) dt$$

$$= \frac{1}{n!} \left(\frac{c}{a}\right) \cdot \left(\frac{c+1}{a}\right) \cdots \left(\frac{c+n-1}{a}\right) \int_0^s t^n \left[1 + \frac{t}{a}\right]^{-c-n} dt$$

Set  $y = \left[1 + \frac{t}{a}\right]^{-1}$   
 $t = a \frac{1-y}{y}$   
 $\frac{dt}{dy} = -\frac{a}{y^2}$

$$= \frac{1}{n!} \left(\frac{c}{a}\right) \cdot \left(\frac{c+1}{a}\right) \cdots \left(\frac{c+n-1}{a}\right) \int_1^{\frac{1+s}{a}} a^n \left(\frac{1-y}{y}\right)^n y^{c+n} \left(-\frac{a dy}{y^2}\right)$$

$$= \frac{1}{n!} \left(\frac{c}{a}\right) \cdot \left(\frac{c+1}{a}\right) \cdots \left(\frac{c+n-1}{a}\right) a^{n+1} \int_{\frac{a}{s+a}}^1 y^{c-2} (1-y)^n dy$$

$$= \frac{1}{n!} \left(\frac{c}{a}\right) \cdot \left(\frac{c+1}{a}\right) \cdots \left(\frac{c+n-1}{a}\right) a^{n+1}$$

$$\cdot \left[ \int_0^1 y^{c-2} (1-y)^n dy \right]$$

$$- \int_{\frac{a}{s+a}}^1 y^{c-2} (1-y)^n dy$$

$$= \frac{1}{n!} \left(\frac{c}{a}\right) \left(\frac{c+1}{a}\right) \cdots \left(\frac{c+n-1}{a}\right) a^{n+1} \text{ Incomplete Beta fn}$$

$$\cdot \left[ \underbrace{B(c-1, n+1)}_{\text{Beta fn}} - \underbrace{B_{\frac{a}{s+a}}(c-1, n+1)}_{\text{Beta fn}} \right]$$

[3]

$$E(s^r)$$

$$= \int_0^{\infty} s^r f_{s^r}(s) ds$$

$$= \frac{1}{n!} \left(\frac{c}{a}\right) \cdot \left(\frac{c+1}{a}\right) \cdots \left(\frac{c+n-1}{a}\right)$$

$$\cdot \int_0^{\infty} s^{n+1} \left[1 + \frac{s}{a}\right]^{-c-n} ds$$

Set  $y = \left[1 + \frac{s}{a}\right]^{-1} \Rightarrow s = a \frac{1-y}{y}$

$\Rightarrow \frac{ds}{dy} = -\frac{a}{y^2}$

$$= \frac{1}{n!} \left(\frac{c}{a}\right) \cdot \left(\frac{c+1}{a}\right) \cdots \left(\frac{c+n-1}{a}\right)$$

$$\int_1^0 a^{n+1} \left(\frac{1-y}{y}\right)^{n+1} y^{c+n} \left(-\frac{a}{y^2}\right) dy$$

$$= \frac{1}{n!} \left(\frac{c}{a}\right) \cdot \left(\frac{c+1}{a}\right) \cdots \left(\frac{c+n-1}{a}\right) \cdot a^{n+2}$$

$$\int_0^1 y^{c-3} (1-y)^{n+1} dy$$

$$= B(c-2, n+2)$$

$$= \frac{1}{n!} \left(\frac{c}{a}\right) \cdot \left(\frac{c+1}{a}\right) \cdots \left(\frac{c+n-1}{a}\right) \cdot a^{n+2} B(c-2, n+2)$$

4

$\text{Var}[S']$

$$= E(S'^2) - (E(S'))^2$$

$$= \int_0^{\infty} s^2 f_{S'}(s) ds$$

$$- (E(S'))^2$$

$$\begin{aligned}
 \boxed{5} \quad P(\bar{V} \leq v) & \\
 &= F_{\bar{V}}(v) \\
 &= 1 - \bar{F}_{X_1, \dots, X_n}(v, \dots, v) \\
 &= 1 - \left[ 1 + \frac{1}{a} \sum_{i=1}^n v \right]^{-c}
 \end{aligned}$$

$$= 1 - \left[ 1 + \frac{n \cdot v}{a} \right]^{-c}$$

$$\begin{aligned}
 \boxed{6} \quad f_{\bar{V}}(v) & \\
 &= \frac{nc}{a} \left[ 1 + \frac{n \cdot v}{a} \right]^{-c-1}
 \end{aligned}$$

$$\begin{aligned}
 \boxed{7} \quad E(\bar{V}) & \\
 &= \frac{nc}{a} \int_0^{\infty} v \cdot \left[ 1 + \frac{n \cdot v}{a} \right]^{-c-1} dv
 \end{aligned}$$

$$\begin{aligned}
 \boxed{8} \quad \text{Var}(\bar{V}) &= E(\bar{V}^2) - (E(\bar{V}))^2 \\
 &= \frac{nc}{a} \int_0^{\infty} v^2 \left[ 1 + \frac{n \cdot v}{a} \right]^{-c-1} dv - (E(\bar{V}))^2
 \end{aligned}$$

**LECTURE**

**27 OCTOBER**

**9:00-10:00AM**

**MATH3/4/68181**

Next week  
is Reading Week

Mon to Fri

11am to 5pm

ATB 2, 223

0161 275 5912

FINANCIAL

RISK

MEASURES





What is a risk measure?

A risk measure gives probabilities associated with a given loss.

Ex

$$X = \text{Loss}$$

$$P(X > \text{£1 million}) > 0.9$$

$\Rightarrow$  will not invest

$$P(X > \text{£1 million}) < 10^{-10}$$

$\Rightarrow$  may be ok to invest

Definition of a risk measure

A risk measure  $\rho(\cdot)$  from a class of random variables to  $(0, \infty)$  must satisfy

$$(i) \quad \rho(0) = 0 \quad [\text{normalised property}]$$

$$(ii) \quad \rho(X + c) = \rho(X) + c \quad [\text{translative property}]$$

$$(iii) \quad X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$$

[monotone property]

## Definition of a coherent risk measure

A risk measure  $\rho$  is coherent if it satisfies (i) - (iii) and

$$(iv) \rho(cX) = c\rho(X)$$

[positive homogeneity]

$$(v) \rho(X + Y) \leq \rho(X) + \rho(Y)$$

[sub additive]

The two most popular risk measures

i) Value at Risk (VaR)  
introduced by J. P. Morgan  
in the 1980s.

Suppose  $X$  has CDF  $F_X(\cdot)$

Then

$$\boxed{\text{VaR}_p(X)} = \inf \{u : F_X(u) \geq p\}$$

Interpretation:  $\Delta$  is the loss exceeded with probability  $1-p$ .

## 2) Expected Shortfall (ES)

introduced in the 1990s by Artzner et al.

$$ES_p(X) = \frac{1}{p} \left[ E(X I\{X \leq VaR_p(X)\}) + p VaR_p(X) \right]$$

$$- VaR_p(X) \cdot P(X \leq VaR_p(X))$$

$$I\{A\} = \begin{cases} 1 & A \text{ is true} \\ 0 & A \text{ is false} \end{cases}$$

“indicator function”

Interpretation:  $ES_p(X)$  is the average loss given that the loss does not exceed  $VaR_p(X)$ .

Are VaR & ES coherent risk measures?

VaR

(i) ✓

(ii) ✓

(iii) ✓

(iv) ✓

(v) ✗

⇒ VaR is not a coherent risk measure

ES

(i) ✓

(ii) ✓

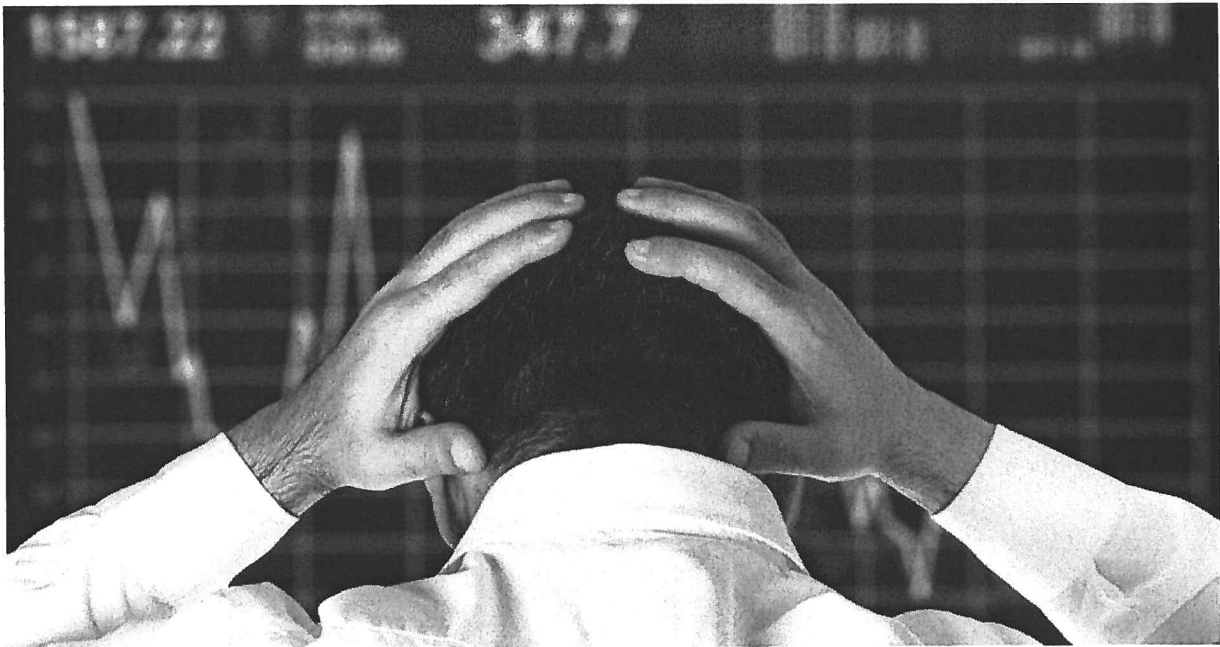
(iii) ✓

(iv) ✓

(v) ✓

⇒ ES is a coherent risk measure.

## News - Did Value at Risk cause the crisis it was meant to avert?



## News

### Did Value at Risk cause the crisis it was meant to avert?

12 May 2016

What were the causes of the crisis of 2008? We show that managing risk using the procedure recommended by Basel II, which is called *Value at Risk*, may have played a central role. We make a very simple model for the banking system that captures the key elements of risk management under Value at Risk. Providing the banks' only take modest risks, the financial system remains stable. But if they take higher risks, or if the banking sector gets larger, the market begins to spontaneously oscillate, in a way that resembles the period leading up to and including the Global Financial Crisis. For about 10 - 15 years prices and leverage slowly rise while volatility slowly falls, then prices and leverage suddenly crash and volatility

# VaR and its Role in the Credit Crisis

Page 1 of 3

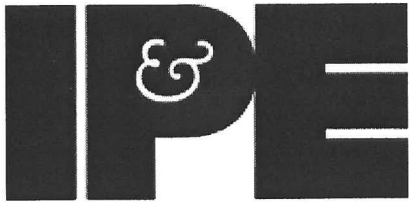
by Mark Kirkland, VP Treasury, Bombardier Transportation

The causes of the credit crisis of 2009 will be discussed by many for numerous years to come, although probably for fewer years than we now think. People have a unique ability to forget, perhaps black out, the worst episodes. I have sat down on a number of occasions and tried to think, what were the possible causes of the crisis? An inherent weakness in accounting of results, large numbers of over the counter derivatives with large fair values, weak governance by regulatory bodies or even that bankers were paid too much? In the end, I believe that none of the above was a key contributor to the crisis. In my mind there are two unrelated causes.

The first is the mode of compensation in the financial industry. Not the amounts. Most bankers receive a kind of option pay out. If the firm makes a large profit (based on the mark to market of future uncertain cash flows), the employees receive large cash bonuses. If the firm makes a loss, in the worst case, staff may receive no bonus. Clearly, for a betting man, this gives carte blanche to load up the company with significant risk. Since most bonuses are not discussed with the owners of the company (the shareholders) but set by a compensation committee, often chaired by senior employees, there is a tendency to overpay since this justifies the compensation of the very people making the decisions. I will not dwell on this cause much longer – except to stress that the whole model encourages large risk taking.

It is now clear that very few shareholders of banks understood the risks that some banks were in fact taking.

The second is the point of this article. Risk was and still is, very badly understood, managed and reported. It is now clear that very few shareholders of banks understood the risks that some banks were in fact taking. In part, this is because disclosure of risk is unclear. A more fundamental issue, however, is that it appears that some of the banks did not fully comprehend the risk and actually outsourced much of their risk assessment to the rating agencies and then used flawed measures such as Value at Risk (VaR) not only to manage risk but also to report to management and shareholders alike.



## Is VaR to blame for the downturn?

IP Asia May 2009 By **Richard Newell**

Author and derivatives specialist Nassim Nicholas Taleb was recently quoted in a *New York Times* article entitled "Risk Mis-management". He made some valid points with regard to the usefulness of risk metrics at times of extreme market behaviour. But while VaR certainly has its laundry list of problems, Taleb takes VaR out of context by focusing on only one version of it; the Gaussian based parametric VaR, which he rightly points out is severely constrained by the dangerous assumption that asset returns follow a normal bell-shaped distribution.

In fact, he even goes so far as to state that VaR was highly responsible for the current financial crises. This is rather disturbing, as his claims seem to have gained a wider currency, thus detracting from the infinitely more important issues behind the crisis. If we look back in history, we can see quite clearly that most "blow-ups" were not due to poor allocation decisions based on an over-reliance on risk measurement and optimisation models, but were about leverage, unchecked greed, operational disaster and outright fraud.

While VaR is a requirement for a bank, most traders and fund managers would laugh if you asked them if they took VaR seriously. The reality, alarmingly, is that risk managers have hardly any clout when it comes to strong-arming a trader or liquidity. Risk manager warnings are often ignored or overridden as senior management tends to focus purely on profitability, not risk. This is not a risk model problem, but a corporate governance problem. Instead of bashing risk managers, we should be giving them more independence, capabilities and authority to identify and limit excessive risk taking.

Long Term Capital Management was leveraged 100 times at one point and Bear Stearns' credit hedge funds over 40 times. A simple cap on gross exposure would have helped to avoid the problems they encountered with leverage. Of course, this would have interfered with a strategy that depended heavily on leverage to 'boost' minuscule returns. Back in the 1990s, Nick Leeson at Barings, the Orange County debacle, events in Mexico and Korea - all of these events had excessive leverage in common. The problems that lie within VaR are its inability to fully capture leverage and liquidity risk. Good risk managers are fully aware of this shortcoming and, as a result, VaR is only one in a whole repertoire of tools, both quantitative and qualitative, that risk managers use to get a sense of the risks they are taking on.

Taleb gives the impression that risk managers are only managing risk according to Gaussian principles, where probabilities are assumed to be normally distributed. There is more to the story than he lets on. Interestingly enough, Taleb seems to be a big fan of Monte Carlo simulations (a method that does not need to assume normality in asset return distributions) as seen in his use of Monte Carlo in the book 'Fooled by Randomness'. Taleb suggests Monte Carlo simulators allow us to learn from the simulated future which is superior to learning from the past, because the past has a survivorship bias, and we also tend to denigrate the past by claiming misfortune had by others will not happen to us. Most sophisticated risk managers use Monte Carlo very much in the same way he does.

To his credit, though, managers at LTCM were proponents of parametric VaR, which severely underestimated risk and was largely responsible for its catastrophic collapse. But even in the





Follow Futures

# VaR: The number that killed us

By Pablo Triana

December 1, 2010 • Reprints

FROM THE ARCHIVES



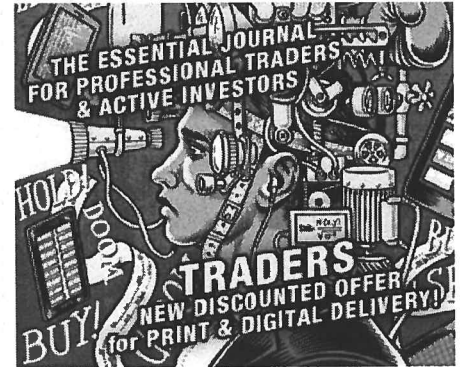
On Sept. 10, 2009 former trader and bestselling author Nassim Taleb did something that he very seldom does: he wore a tie. Taleb has oftentimes publicly expressed his distaste for the blood-constraining artifacts, as well as for those who tend to do them, so the Lebanese-American let the world know that was a very special day for him by betraying a sacred personal disposition.

So what prompted the composer of “The Black Swan” to button his shirt all the way up on that fall date? He had been invited to a very solemn

venue by very distinguished hosts. And that was an invitation that Taleb had every intention of accepting. In fact, he had been waiting and expecting it for more than a decade. The *raison d'être* of the event for which his company was now being required had been close to Taleb’s heart for most of his professional and intellectual life. It represented a central theme in his actions and ideas, close to an obsession. He had through the years incessantly warned as to the havoc that might be wreaked should others massively act in a manner counter to his convictions. Such concerns typically went unheeded (to the detriment, it turned out, of society), but now he was being offered a pulpit that seemed irresistible. This time, the world would have no option but to listen attentively.

As Taleb entered the Rayburn Building of the U.S. House of Representatives on Capitol Hill that September morning, he must have felt vindication. As he approached the sober room where several men and women awaited the start of the House Committee on Science and Technology’s hearing on the responsibility of mathematical model Value at Risk (VaR) for the terrible economic and financial crisis that had caused so much misery, Taleb probably reflected proudly on all those times when, indefatigably and in the face of harsh opposition, he alerted us of the lethal threat to the system posed by the widespread use of VaR in finance. Now that the damage wrought by VaR seemed so inescapably obvious that lawmakers had been motivated into investigating the device, Taleb no longer seemed like a lone wolf howling at the moon.

What is so wrong about VaR, and why was Taleb so concerned about its impact? More importantly, why should VaR be held responsible for the crisis? VaR is a number that purports to estimate future losses derived from a portfolio of financial assets, and presents two major problems: 1) it is doomed to being a very wrong estimate, because of its analytical foundations and the realities of real-life markets; 2) in spite of such (well-known) deficiencies, it has for the past two decades become an ubiquitously influential force in the financial world, capable of directing decision-making inside the most important banks. In other words, by letting trading activity be guided by VaR, we have essentially exposed our economic fate to a deeply flawed mechanism. Such flawedness, as was the case not only in this crisis but also before, can yield untold malaise.



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[Equities](#)
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[Forex](#)



Symbol	Last	Change	(%)
Emini	2576.50	+3.00	(+0.12%)
Euro	1.17715	-0.00485	(-0.41%)
Gold	1274.2	-7.6	(-0.59%)
Oil	52.19	+0.72	(+1.40%)
Gas	2.991	+0.081	(+2.78%)
Corn	344¢	+0¢	(+0.15%)

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## Simpler expressions for VaR & ES

Suppose  $X$  is an absolutely continuous random variable.

Then

$$\text{VaR}_p(X) = F_X^{-1}(p)$$

$$\text{ES}_p(X) = \frac{1}{p} \int_0^p F_X^{-1}(t) dt$$

Ex 1 Suppose  $X$  is  $N(\mu, \sigma^2)$ .

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

CDF of  $N(0, 1)$

$$\text{Set } F_X(x) = p$$

$$\Rightarrow \Phi\left(\frac{x-\mu}{\sigma}\right) = p$$

$$\Rightarrow \frac{x-\mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow x = \mu + \sigma \Phi^{-1}(p)$$

$$\Rightarrow \text{VaR}_p(X) = \mu + \sigma \Phi^{-1}(p)$$

$$ES_p(X) = \frac{1}{p} \int_0^p \left[ \mu + \sigma \Phi^{-1}(t) \right] dt$$

$$= \frac{1}{p} \left[ \mu p + \sigma \int_0^p \Phi^{-1}(t) dt \right]$$

$$= \mu + \frac{\sigma}{p} \int_0^p \Phi^{-1}(t) dt$$

Ex 2

Suppose  $X$  is Uni  $[-\theta, \theta]$ .

$$F_X(x) = \frac{x + \theta}{2\theta}$$

$$\text{Set } F_X(x) = p$$

$$\Rightarrow \frac{x + \theta}{2\theta} = p$$

$$\Rightarrow x = 2\theta p - \theta$$

$$\Rightarrow \text{VaR}_p(X) = 2\theta p - \theta$$

$$ES_p(X) = \frac{1}{p} \int_0^p (2\theta t - \theta) dt$$

$$= \frac{1}{p} \left[ \theta t^2 - \theta t \right]_0^p$$

$$= \frac{1}{p} (\theta p^2 - \theta p)$$

$$= \theta p - \theta$$

Properties satisfied by VAR

(i)  $\text{VaR}_p(0) = 0$

(ii)  $\text{VaR}_p(X+c) = \text{VaR}_p(X) + c$

(iii)  $X \leq Y \Rightarrow \text{VaR}_p(X) \leq \text{VaR}_p(Y)$

(iv)  $\text{VaR}_p(cX) = c \text{VaR}_p(X)$

Homework: Try and prove these.

**EXAMPLE CLASS**

**6 NOVEMBER**

**12:00-13:00PM**

**MATH3/4/68181**

If  $X$  is an absolutely  
continuous RV then

$$\text{VaR}_p(X) = F^{-1}(p)$$

and

$$\text{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(t) dt$$

Q1

$$F(x) = 1 - e^{-\lambda x}$$

$$1 - e^{-\lambda x} = p$$

$$\Rightarrow e^{-\lambda x} = 1 - p$$

$$\Rightarrow -\lambda x = \log(1 - p)$$

$$\Rightarrow x = -\frac{1}{\lambda} \log(1 - p)$$

$$\Rightarrow \text{VaR}_p(x) = -\frac{1}{\lambda} \log(1 - p)$$

$$ES_p(x) = \frac{1}{p} \int_0^p \left( -\frac{1}{\lambda} \log(1 - t) \right) dt$$

$$= -\frac{1}{\lambda p} \int_0^p \log(1 - t) dt$$

Int by  
Parts

$$= -\frac{1}{\lambda p} \left\{ \left[ t \cdot \log(1 - t) \right]_0^p - \int_0^p \frac{t \cdot (-1)}{1 - t} dt \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \log(1 - p) - 0 + \int_0^p \frac{t}{1 - t} dt \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \log(1 - p) + \int_0^p \frac{(t - 1) + 1}{1 - t} dt \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \log(1 - p) + \int_0^p \left( -1 + \frac{1}{1 - t} \right) dt \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \log(1 - p) + \left[ -t - \log(1 - t) \right]_0^p \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \log(1 - p) - p - \log(1 - p) \right\}.$$



Q2

$$F(x) = x^a$$

$$x^a = p$$

$$\Rightarrow x = p^{\frac{1}{a}}$$

$$\Rightarrow \text{VaR}_p(x) = p^{\frac{1}{a}}$$

$$ES_p(x) = \frac{1}{p} \int_0^p t^{\frac{1}{a}} dt$$

$$= \frac{1}{p} \left[ \frac{t^{\frac{1}{a} + 1}}{\frac{1}{a} + 1} \right]_0^p$$

$$= \frac{1}{p} \frac{p^{\frac{1}{a} + 1} - 0}{\frac{1}{a} + 1}$$

$$= \frac{p^{\frac{1}{a}}}{\frac{1}{a} + 1}$$

Q3

$$F(x) = \frac{x - a}{b - a}$$

$$\frac{x - a}{b - a} = p$$

$$\Rightarrow x = a + p(b - a)$$

$$\Rightarrow \text{VaR}_p(x) = a + p(b - a)$$

$$\begin{aligned} ES_p(x) &= \frac{1}{p} \int_0^p [a + t(b - a)] dt \\ &= \frac{1}{p} \left[ at + \frac{t^2}{2} (b - a) \right]_0^p \end{aligned}$$

$$= \frac{1}{p} \left[ ap + \frac{p^2}{2} (b - a) \right]$$

$$= a + \frac{p}{2} (b - a)$$

Q4

$$F(x) = 1 - \left(\frac{k}{x}\right)^a$$

$$1 - \left(\frac{k}{x}\right)^a = p$$

$$\Rightarrow \frac{k}{x} = (1-p)^{\frac{1}{a}}$$

$$\Rightarrow x = k (1-p)^{-\frac{1}{a}}$$

$$\Rightarrow \text{VaR}_p(x) = k (1-p)^{-\frac{1}{a}}$$

$$ES_p(x) = \frac{k}{p} \int_0^p (1-t)^{-\frac{1}{a}} dt$$

$$= \frac{k}{p} \left[ \frac{(1-t)^{1-\frac{1}{a}}}{(-1)\left(1-\frac{1}{a}\right)} \right]_0^p$$

$$= \frac{k}{p} \frac{(1-p)^{1-\frac{1}{a}} - 1}{\left(\frac{1}{a} - 1\right)}$$

Q6

$$F(x) = \left[ 1 + \left( \frac{x}{a} \right)^{-b} \right]^{-1}$$

$$F(x) = p$$

$$\Rightarrow 1 + \left( \frac{x}{a} \right)^{-b} = \frac{1}{p}$$

$$\Rightarrow \left( \frac{x}{a} \right)^{-b} = \frac{1-p}{p}$$

$$\Rightarrow \frac{x}{a} = \left( \frac{1-p}{p} \right)^{-\frac{1}{b}}$$

$$\Rightarrow x = a \left( \frac{1-p}{p} \right)^{-\frac{1}{b}}$$

$$\Rightarrow \text{VaR}_p(X) = a \left( \frac{1-p}{p} \right)^{-\frac{1}{b}}$$

$$ES_p(X) = \frac{a}{p} \int_0^p t^{\frac{1}{b}} (1-t)^{-\frac{1}{b}} dt$$

Incomplete Beta Function

$$B_x(\alpha, \beta) = \int_0^x t^{\alpha-1} (1-t)^{\beta-1} dt$$

$$= \frac{a}{p} B_p \left( \frac{1}{b} + 1, 1 - \frac{1}{b} \right)$$

Q7

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}$$

$$F(x) = p$$

$$\Rightarrow \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = 1 - p$$

$$\Rightarrow 1 + \frac{x}{\lambda} = (1 - p)^{-\frac{1}{\alpha}}$$

$$\Rightarrow x = \lambda \left[ (1 - p)^{-\frac{1}{\alpha}} - 1 \right]$$

$$\Rightarrow \text{VaR}_p(X) = \lambda \left[ (1 - p)^{-\frac{1}{\alpha}} - 1 \right]$$

$$ES_p(X) = \frac{\lambda}{p} \int_0^p \left[ (1 - t)^{-\frac{1}{\alpha}} - 1 \right] dt$$

$$= \frac{\lambda}{p} \left[ \frac{(1 - t)^{1 - \frac{1}{\alpha}}}{\left(1 - \frac{1}{\alpha}\right)(-1)} - t \right]_0^p$$

$$= \frac{\lambda}{p} \left[ \frac{(1 - p)^{1 - \frac{1}{\alpha}}}{\left(\frac{1}{\alpha} - 1\right)} - p - \frac{1}{\left(\frac{1}{\alpha} - 1\right)} \right]$$

Q8

$$F(x) = e^{-\left(\frac{\sigma}{x}\right)^\alpha}$$

$$F(x) = p$$

$$\Rightarrow -\left(\frac{\sigma}{x}\right)^\alpha = \log p$$

$$\Rightarrow \frac{\sigma}{x} = \left(-\log p\right)^{\frac{1}{\alpha}}$$

$$\Rightarrow x = \sigma \left(-\log p\right)^{-\frac{1}{\alpha}}$$

$$\Rightarrow \text{VaR}_p(X) = \sigma \left(-\log p\right)^{-\frac{1}{\alpha}}$$

$$E S_p(X) = \frac{\sigma}{p} \int_0^p \left(-\log t\right)^{-\frac{1}{\alpha}} dt$$

$$\text{Set } y = -\log t \Rightarrow t = e^{-y}$$
$$\frac{dt}{dy} = -e^{-y}$$

$$= \frac{\sigma}{p} \int_{-\log p}^{-\log p} y^{-\frac{1}{\alpha}} (-e^{-y}) dy$$

$$= \frac{\sigma}{p} \int_{-\log p}^{\infty} y^{-\frac{1}{\alpha}} e^{-y} dy$$

Comp Incomplete Gamma Function

$$\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t} dt$$

$$= \frac{\sigma}{p} \Gamma\left(1 - \frac{1}{\alpha}, -\log p\right).$$

**LECTURE**

**7 NOVEMBER**

**9:00-10:00AM**

**MATH3/4/68181**

IN - CLASSES

TEST

14 NOV

TUESDAY

9:00 - 10:00 AM

(LEVEL 3)

9:00 - 10:30 AM

(LEVEL 4, 6)



## Proof of (ii)

Assume  $X$  is absolutely cont RV.

$$\text{VaR}_p(X+c) = \text{VaR}_p(X) + c \quad (*)$$

$$\Leftrightarrow F_{X+c}^{-1}(p) = F_X^{-1}(p) + c$$

$$\Leftrightarrow F_{X+c}(F_{X+c}^{-1}(p)) = F_{X+c}(F_X^{-1}(p) + c)$$

$$\Leftrightarrow p = F_{X+c}(F_X^{-1}(p) + c)$$

$$\Leftrightarrow p = P(X+c \leq F_X^{-1}(p) + c)$$

$$\Leftrightarrow p = P(X \leq F_X^{-1}(p))$$

$$\Leftrightarrow p = F_X(F_X^{-1}(p))$$

$$\Leftrightarrow p = p$$

Hence, (\*) must hold.

## Proof of (iv)

Assume that  $X$  is absolutely cont RV.

$$\text{VaR}_p(cX) = c \text{VaR}_p(X) \quad (*)$$

$$\Leftrightarrow F_{cX}^{-1}(p) = c F_X^{-1}(p)$$

$$\Leftrightarrow F_{cX}(F_{cX}^{-1}(p)) = F_{cX}(c F_X^{-1}(p))$$

$$\Leftrightarrow p = F_{cX}(c F_X^{-1}(p))$$

$$\Leftrightarrow p = P(cX \leq c F_X^{-1}(p))$$

$$\Leftrightarrow p = P(X \leq F_X^{-1}(p))$$

$$\Leftrightarrow p = F_X(F_X^{-1}(p))$$

$$\Leftrightarrow p = p$$

Hence,  $(*)$  must hold.

Estimation      methods      for      VAR

a) Parametric estimation

Level 3, 4, 6

b) Non-parametric

||  
Level 3, 4, 6

c) Semi-parametric

||  
Level 4, 6

a) Parametric estimation of VaR

1) Assume  $X = \text{Loss}$  has  $N(\mu, \sigma^2)$  distribution

$$\text{VaR}_p(X) = \mu + \sigma \Phi^{-1}(p)$$

Suppose  $x_1, x_2, \dots, x_n$  are observed losses. Assumed to be IID from  $N(\mu, \sigma^2)$ .

By the method of maximum likelihood, the estimators of  $\mu$  and  $\sigma$  are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

and

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2.$$

The MLE of  $\text{Var}_P(X)$  is

$$\widehat{\text{Var}}_P(X) = \hat{\mu} + \hat{\sigma} \Phi^{-1}(p)$$

$$= \bar{x} + \sqrt{\frac{1}{n} \sum (x_i - \bar{x})^2} \Phi^{-1}(p)$$

$\Phi$  is the CDF of  $N(0,1)$

Suppose  $\hat{\theta}$  is an estimator of  $\theta$ .

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

$$\text{MSE}(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$$

(mean squared error)

$\hat{\theta}$  is unbiased if  $\text{Bias}(\hat{\theta}) = 0$

$\hat{\theta}$  is consistent if  $\text{MSE}(\hat{\theta}) \rightarrow 0$   
as  $n \rightarrow \infty$

Homework: Show that  $\widehat{\text{Var}}_P(X)$   
is biased.

2) Suppose  $X = \text{Loss}$  has the  $\text{Uni}[a, b]$  distribution

$$\text{VaR}_p(X) = a + p(b - a)$$

Suppose  $x_1, \dots, x_n$  are observed losses IID from  $\text{Uni}[a, b]$ .

By MLE, the estimators of  $a$  and  $b$  are

$$\hat{a} = \min(x_1, \dots, x_n)$$

and

$$\hat{b} = \max(x_1, \dots, x_n).$$

So, the MLE of  $\text{VaR}_p(X)$  is

$$\begin{aligned} \widehat{\text{VaR}}_p(X) &= \hat{a} + p(\hat{b} - \hat{a}) \\ &= \min(x_1, \dots, x_n) + p \left[ \max(x_1, \dots, x_n) - \min(x_1, \dots, x_n) \right] \end{aligned}$$

Homework : Show that  $\widehat{\text{VaR}}_p(X)$  is a biased estimator.

3) Assume  $X = \text{Loss}$  has the Weibull distribution with CDF

$$F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^\beta}, \quad x > 0.$$

$$F(x) = p$$

$$\Rightarrow 1 - e^{-\left(\frac{x}{\theta}\right)^\beta} = p$$

$$\Rightarrow e^{-\left(\frac{x}{\theta}\right)^\beta} = 1 - p$$

$$\Rightarrow \left(\frac{x}{\theta}\right)^\beta = -\log(1-p)$$

$$\Rightarrow x = \theta \left[-\log(1-p)\right]^{\frac{1}{\beta}}$$

$$\Rightarrow \text{VaR}_p(X) = \theta \left[-\log(1-p)\right]^{\frac{1}{\beta}}.$$

Suppose  $x_1, \dots, x_n$  are observed losses assumed to be IID from the Weibull distribution.

The estimators  $\hat{\theta}$  and  $\hat{\beta}$  are given by

$$\hat{\theta} = \frac{\bar{x}}{\Gamma\left(1 + \frac{1}{\hat{\beta}}\right)},$$

$$\frac{\bar{x}^2}{s^2} = \frac{\left[\Gamma\left(1 + \frac{1}{\hat{\beta}}\right)\right]^2}{\Gamma\left(1 + \frac{2}{\hat{\beta}}\right) - \left[\Gamma\left(1 + \frac{1}{\hat{\beta}}\right)\right]^2}.$$

Hence, the estimator of  $\text{VaR}_p(X)$  is

$$\widehat{\text{VaR}}_p(X) = \hat{\theta} \left[-\log(1-p)\right]^{\frac{1}{\hat{\beta}}}.$$

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2.$$



**EXAMPLE CLASS**

**7 NOVEMBER**

**10:00-11:00AM**

**MATH3/4/68181**

Q1

$$F(x) = 1 - e^{-\lambda x}$$

$$1 - e^{-\lambda x} = p$$

$$\Rightarrow e^{-\lambda x} = 1 - p$$

$$\Rightarrow -\lambda x = \log(1 - p)$$

$$\Rightarrow x = -\frac{1}{\lambda} \log(1 - p)$$

$$\Rightarrow \text{VaR}_p(x) = -\frac{1}{\lambda} \log(1 - p)$$

$$ES_p(x) = \frac{1}{p} \int_0^p F^{-1}(t) dt$$

$$= -\frac{1}{\lambda p} \int_0^p \log(1 - t) dt$$

int  
by =  
parts

$$= -\frac{1}{\lambda p} \left\{ \left[ t \cdot \log(1 - t) \right]_0^p - \int_0^p \frac{(-t)}{1 - t} dt \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1 - p) - 0 + \int_0^p \frac{(t-1)+1}{1-t} dt \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1 - p) + \int_0^p \left( -1 + \frac{1}{1-t} \right) dt \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1 - p) + \left[ -t - \log(1 - t) \right]_0^p \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1 - p) - p - \log(1 - p) \right\}$$

Q2

$$F(x) = x^a$$

$$x^a = p$$

$$\Rightarrow x = p^{\frac{1}{a}}$$

$$\Rightarrow \text{Var}_p(X) = p^{\frac{1}{a}}$$

$$E S_p(X) = \frac{1}{p} \int_0^p t^{\frac{1}{a}} dt$$

$$= \frac{1}{p} \left[ \frac{t^{\frac{1}{a} + 1}}{\frac{1}{a} + 1} \right]_0^p$$

$$= \frac{1}{p} \frac{p^{\frac{1}{a} + 1} - 0}{\frac{1}{a} + 1}$$

$$= \frac{p^{\frac{1}{a}}}{\frac{1}{a} + 1} \cdot$$

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$$F(x) = \frac{x-a}{b-a}$$

$$F(x) = p$$

$$\Rightarrow \frac{x-a}{b-a} = p$$

$$\Rightarrow x = a + p(b-a)$$

$$\Rightarrow \text{Var}_p(x) = a + p(b-a)$$

$$E_{S_p}(X) = \frac{1}{p} \int_0^p [a + t(b-a)] dt$$

$$= \frac{1}{p} \left[ at + \frac{t^2}{2}(b-a) \right]_0^p$$

$$= \frac{1}{p} \left[ ap + \frac{p^2}{2}(b-a) \right]$$

$$= a + \frac{p}{2}(b-a)$$

Q4

$$F(x) = 1 - \left(\frac{k}{x}\right)^a$$

$$1 - \left(\frac{k}{x}\right)^a = p$$

$$\Rightarrow \left(\frac{k}{x}\right)^a = 1 - p$$

$$\Rightarrow \frac{k}{x} = (1-p)^{\frac{1}{a}}$$

$$\Rightarrow x = k (1-p)^{-\frac{1}{a}}$$

$$ES_p(x) = \frac{k}{p} \int_0^p (1-t)^{-\frac{1}{a}} dt$$

$$= \frac{k}{p} \left[ \frac{(1-t)^{1-\frac{1}{a}}}{\left(1-\frac{1}{a}\right)(-1)} \right]_0^p$$

$$= \frac{k}{p} \frac{(1-p)^{1-\frac{1}{a}} - 1}{\left(\frac{1}{a} - 1\right)}$$

Q6

$$F(x) = \left[ 1 + \left( \frac{x}{a} \right)^{-b} \right]^{-1}$$

$$F(x) = p$$

$$\Rightarrow \left[ 1 + \left( \frac{x}{a} \right)^{-b} \right]^{-1} = p$$

$$\Rightarrow \left( \frac{x}{a} \right)^{-b} = \frac{1-p}{p}$$

$$\Rightarrow x = a \left( \frac{1-p}{p} \right)^{-\frac{1}{b}}$$

$$\Rightarrow \text{VaR}_p(X) = a \left( \frac{1-p}{p} \right)^{-\frac{1}{b}}$$

$$\begin{aligned} \text{ES}_p(X) &= \frac{a}{p} \int_0^p \left( \frac{1-t}{t} \right)^{-\frac{1}{b}} dt \\ &= \frac{a}{p} \int_0^p t^{\frac{1}{b}} (1-t)^{-\frac{1}{b}} dt \end{aligned}$$

$$B_x(\alpha, \beta) = \int_0^x t^{\alpha-1} (1-t)^{\beta-1} dt$$

Incomplete Beta Function

$$= \frac{a}{p} B_p \left( 1 + \frac{1}{b}, 1 - \frac{1}{b} \right)$$

Q7

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}$$

$$1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = p$$

$$\Rightarrow \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = 1 - p$$

$$\Rightarrow 1 + \frac{x}{\lambda} = (1 - p)^{-\frac{1}{\alpha}}$$

$$\Rightarrow x = \lambda \left[ (1 - p)^{-\frac{1}{\alpha}} - 1 \right]$$

$$\Rightarrow \text{VaR}_p(x) = \lambda \left[ (1 - p)^{-\frac{1}{\alpha}} - 1 \right]$$

$$E\mathcal{J}_p(x) = \frac{\lambda}{p} \int_0^p \left[ (1 - t)^{-\frac{1}{\alpha}} - 1 \right] dt$$

$$= \frac{\lambda}{p} \left[ \frac{(1 - t)^{1 - \frac{1}{\alpha}}}{(-1) \left(1 - \frac{1}{\alpha}\right)} - t \right]_0^p$$

$$= \frac{\lambda}{p} \left[ \frac{(1 - p)^{1 - \frac{1}{\alpha}}}{\frac{1}{\alpha} - 1} - p + \frac{1}{1 - \frac{1}{\alpha}} \right]$$

Estimation      methods      for      VAR

a) Parametric estimation  
Level 3, 4, 6

b) Non-parametric      ||  
Level 3, 4, 6

c) Semi-parametric      ||  
Level 4, 6



# c) Semi-parametric Estimation Methods for VaR

## 1) Extreme Value Method

Let  $M_n = \max(X_1, \dots, X_n)$

[Definition 1]

We know that the distribution of  $M_n$  can be approximated by

$$P(M_n < x) \approx e^{-\left(1 + \xi \frac{x - \mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

[the CDF of the  
GEV distribution]

$$-\infty < \xi < \infty$$

$$\sigma > 0$$

$$-\infty < \mu < \infty$$

"shape parameter"

"scale" "

"location" "

We estimate  $\mu$  and  $\sigma$  by maximum likelihood. Let  $\hat{\mu}$  and  $\hat{\sigma}$  be the MLEs of  $\mu$  and  $\sigma$ .

Suppose  $X_1, X_2, \dots, X_n$  are IID

Then

$$\hat{\mu} = \frac{1}{k} \sum_{i=1}^k \log \frac{X_{(i)}}{X_{(k+1)}}$$

or

$$\hat{\mu} = \frac{1}{\log 2} \log \frac{X_{(k+1)} - X_{(2k+1)}}{X_{(2k+1)} - X_{(4k+1)}}$$

where  $1 \leq k \leq n$

and  $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$

are the data arranged in increasing order.

To find  $\text{VaR}_p$ , set

$$e^{-\left(1 + \sum \frac{x - \mu}{\sigma}\right)^{-\frac{1}{\sum}}} = p$$

$$\Rightarrow \left(1 + \sum \frac{x - \mu}{\sigma}\right)^{-\frac{1}{\sum}} = -\log p$$

$$\Rightarrow 1 + \sum \frac{x - \mu}{\sigma} = (-\log p)^{-\sum}$$

$$\Rightarrow x = \mu + \frac{\sigma}{\sum} \left[ (-\log p)^{-\sum} - 1 \right].$$

Hence, a semi-parametric estimate of VaR is

$$\widehat{\text{VaR}}_p(x) = \hat{\mu} + \frac{\hat{\sigma}}{\hat{F}(x)} \left[ (-\log p) - 1 \right]$$

parametric estimates

non-parametric estimate

## 2. Generalized Pareto method

Suppose Definition 2 of extreme values. From the notes,

$$F(x) = 1 - p \left( 1 + \xi \frac{x-t}{\sigma} \right)^{-\frac{1}{\xi}}$$

Where  $p = P(X > t)$ ,  $t$  is the threshold,  $\xi$  is the shape parameter and  $\sigma$  is the scale parameter.

~~Take~~ Estimate  $\sigma$  by the method of maximum likelihood. Let  $\hat{\sigma}$  denote the MLE of  $\sigma$ .

Suppose  $X_1, X_2, \dots, X_n$  are the data assumed to be IID. We estimate  $\xi$  by

$$\hat{\xi} = \frac{1}{\log 2} \log \frac{X_{(n-k+1)} - X_{(n-2k+1)}}{X_{(n-2k+1)} - X_{(n-4k+1)}}$$

where  $1 \leq k \leq n$  and  $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$  are data arranged in increasing order.

Set

$$1 - p \left( 1 + \sum \frac{x-t}{\sigma} \right)^{-\frac{1}{\sum}} = q$$

$$\Rightarrow \left( 1 + \sum \frac{x-t}{\sigma} \right)^{-\frac{1}{\sum}} = \frac{1-q}{p}$$

$$\Rightarrow 1 + \sum \frac{x-t}{\sigma} = \left( \frac{1-q}{p} \right)^{-\sum}$$

$$\Rightarrow x = t + \frac{\sigma}{\sum} \left[ \left( \frac{1-q}{p} \right)^{-\sum} - 1 \right]$$

Hence, a semi-parametric estimate of  $\text{VaR}_q$  is

$$\text{VaR}_q = t + \underbrace{\frac{\sigma}{\sum}}_{\text{parametric estimate}} \left[ \underbrace{\left( \frac{1-q}{p} \right)^{-\sum}}_{\text{Non-parametric estimate}} - 1 \right]$$

parametric estimate

Non-parametric estimate.

### 3. A hybrid method

The GP distribution does not usually give good fits to the lower part of the data. An alternative estimate for VaR is

$$\widehat{\text{VaR}}_q = \begin{cases} X_{(i)} & \text{if } q \in \left(\frac{i-1}{n}, \frac{i}{n}\right], \quad q < q_0 \\ t^* \frac{\sigma}{\mu} \left[ \left(\frac{1-q}{p}\right)^{-\frac{1}{\alpha}} - 1 \right] & , \quad q \geq q_0 \end{cases}$$

$$\lim_{t \rightarrow \infty} \left[ \frac{1 - G^{\circ}(tx)}{1 - G^{\circ}(t)} \right]^4$$

$$= \lim_{t \rightarrow \infty} \left\{ \frac{1 - [1 - (1 - G(tx))]^{\circ}}{1 - [1 - (1 - G(t))]^{\circ}} \right\}^4$$

$$= \lim_{t \rightarrow \infty} \left\{ \frac{1 - [1 - (1 - G(tx))]}{1 - [1 - (1 - G(t))]} \right\}^4$$

$$= \lim_{t \rightarrow \infty} \left[ \frac{1 - G(tx)}{1 - G(t)} \right]^4$$

**LECTURE**

**10 NOVEMBER**

**9:00-10:00AM**

**MATH3/4/68181**



Estimation      methods      for      VAR

a) Parametric estimation  
Level 3, 4, 6

b) Non-parametric      ||  
Level 3, 4, 6

c) Semi-parametric      ||  
Level 4, 6

#### 4) Power function method

Suppose  $X = \text{Loss}$  has cdf  $F(x) = x^a$ ,  
 $0 < x < 1$ .

$$x^a = p$$

$$\Rightarrow x = p^{\frac{1}{a}}$$

$$\Rightarrow \text{VaR}_p(x) = p^{\frac{1}{a}}$$

Suppose  $x_1, \dots, x_n$  are observed losses assumed to be IID with cdf  $F(x) = x^a$ . The MLE for  $a$  is

$$\hat{a} = - \left( \frac{1}{n} \sum_{i=1}^n \log x_i \right)^{-1}$$

So, the MLE of  $\text{VaR}_p(x)$  is

$$\widehat{\text{VaR}}_p(x) = p^{\left( \frac{n}{\sum_{i=1}^n \log x_i} \right)^{-1}}$$

## 5) Variance Covariance Method

Suppose the loss of a portfolio is

$$X = \sum_{i=1}^P w_i X_i$$

Weights
loss on  $i^{\text{th}}$  investment

Suppose  $X_i$  are independent  $N(\mu_i, \sigma_i^2)$ . Then

$$X \sim N\left(\sum_{i=1}^P w_i \mu_i, \sum_{i=1}^P w_i^2 \sigma_i^2\right).$$

The VaR of  $X$  must satisfy

$$F_X(x) = q$$

$$\Rightarrow \Phi\left(\frac{x - \sum_{i=1}^P w_i \mu_i}{\sqrt{\sum_{i=1}^P w_i^2 \sigma_i^2}}\right) = q$$

$$\Rightarrow x = \sum_{i=1}^P w_i \mu_i + \sqrt{\sum_{i=1}^P w_i^2 \sigma_i^2} \Phi^{-1}(q)$$

$$\Rightarrow \text{VaR}_q(X) = \sum_{i=1}^P w_i \mu_i + \sqrt{\sum_{i=1}^P w_i^2 \sigma_i^2} \Phi^{-1}(q).$$

Suppose

$x_{1,1}, x_{1,2}, \dots, x_{1,n_1}$  are IID  
obsns on  $X_1$

$x_{2,1}, x_{2,2}, \dots, x_{2,n_2}$  " "  $X_2$   
" " " "

⋮

$x_{p,1}, x_{p,2}, \dots, x_{p,n_p}$  " "  $X_p$   
" " " "

The MLEs of  $\mu_i$  and  $\sigma_i^2$  are

$$\hat{\mu}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{i,j}$$

$$\hat{\sigma}_i^2 = \frac{1}{n_i} \sum_{j=1}^{n_i} (x_{i,j} - \hat{\mu}_i)^2$$

Hence, the MLE of  $\text{Var}_q(X)$  is

$$\widehat{\text{Var}}_q(X) = \sum_{i=1}^p w_i \hat{\mu}_i + \sqrt{\sum_{i=1}^p w_i^2 \hat{\sigma}_i^2} \Phi^{-1}(q)$$

## b) Non-parametric Estimation of VaR

### 1) Historical Method

Suppose  $x_1, x_2, \dots, x_n$  are the losses. Arrange the data as

$$\boxed{x_{(1)}} \leq x_{(2)} \leq \dots \leq \boxed{x_{(n)}}$$

smallest loss largest loss

Then

$$\widehat{\text{VaR}}_p(X) = x_{(i)}$$

if  $p \in \left( \frac{i-1}{n}, \frac{i}{n} \right]$ .

#### Example

Losses:  $-5, 5, -8, 1, 0$

-8	-5	0	1	5
$x_{(1)}$	$x_{(2)}$	$x_{(3)}$	$x_{(4)}$	$x_{(5)}$

$$\widehat{\text{VaR}}_{0.3} = x_{(2)} = -5$$

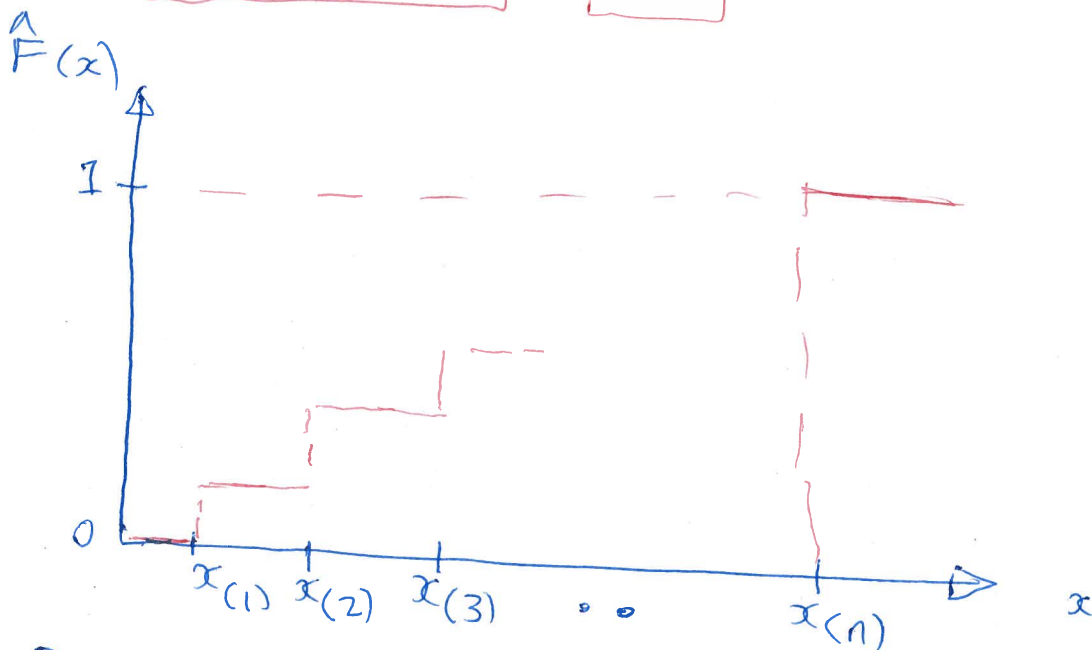
$$\widehat{\text{VaR}}_{0.9} = x_{(5)} = 5$$

## 2) Bootstrap Method

Suppose  $x_1, x_2, \dots, x_n$  are the observed losses. Let  $\hat{F}$  denote the empirical cdf of the data.

$$\hat{F}(x) = \frac{1}{n} \sum_{j=1}^n I\{x_{(j)} \leq x\}$$

Empirical CDF



Bootstrap method:

- i) simulate  $B$  samples each of size  $n$  from  $\hat{F}$
- ii) estimate  $\text{VaR}_p$  by the historical method for each of the  $B$  samples, giving  $\widehat{\text{VaR}}_p^{(1)}, \widehat{\text{VaR}}_p^{(2)}, \dots, \widehat{\text{VaR}}_p^{(B)}$ ,
- iii) compute  $\widehat{\text{VaR}}_p$  as the mean or median of  $\widehat{\text{VaR}}_p^{(1)}, \dots, \widehat{\text{VaR}}_p^{(B)}$ .

### 3) Jock knife Method

- i) estimate  $\text{VaR}_p$  by the historical method for  $x_2, x_3, \dots, x_n$ , giving  $\widehat{\text{VaR}}_p^{(1)}$
- ii) estimate  $\text{VaR}_p$  by the historical method for  $x_1, x_3, \dots, x_n$ , giving  $\widehat{\text{VaR}}_p^{(2)}$ .
- ⋮
- iii) estimate  $\text{VaR}_p$  by the historical method for  $x_1, x_2, \dots, x_{(n-1)}$ , giving  $\widehat{\text{VaR}}_p^{(n)}$ .
- iv) Compute  $\widehat{\text{VaR}}_p$  as the mean or median of  $\widehat{\text{VaR}}_p^{(1)}, \dots, \widehat{\text{VaR}}_p^{(n)}$ .

#### 4) Kernel Method

Suppose  $x_1, x_2, \dots, x_n$  are the observed losses. The kernel estimate of the cdf of the data is

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^n G\left(\frac{x - x_i}{h}\right)$$

where

$$G(x) = \int_{-\infty}^x K(u) du \quad \text{band width}$$

$K(\cdot)$  is a symmetric PDF, usually

$$K(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \quad \text{kernel}$$

of  $\text{VaR}_p$  is given by the root of

$$\hat{F}(x) = p$$

or

$$\widehat{\text{VaR}}_p(x) = \frac{\sum_{i=1}^n \hat{F}\left(\frac{i - \frac{1}{2}}{n} - p\right) x_{(i)}}{\sum_{i=1}^n \hat{F}\left(\frac{i - \frac{1}{2}}{n} - p\right)}$$



# **EXAMPLE CLASS**

**13 NOVEMBER**

**12:00-13:00PM**

**MATH3/4/68181**

2014 Q2 (iv)

$$\lim_{k \rightarrow w(F)} \frac{P(X=k)}{1-F(k-1)}$$

$$w(F) = 2$$

$$= \frac{P(X=2)}{1-F(1)}$$

$$= \frac{P(X=2)}{1-P(X \leq 1)}$$

$$= \frac{P(X=2)}{P(X > 1)}$$

$$= \frac{P(X=2)}{P(X \geq 2)} = \frac{1}{1} = 1 \neq 0$$

FTT will not hold.

2014 Test Q2 (iii)

$$w(F) = +\infty$$

$$\frac{[-e^{-2x}]^2}{\frac{1}{2} + \frac{1}{2}[1-e^{-2x}]^2} = 1$$

$$\Rightarrow [1-e^{-2x}]^2 = \frac{1}{2} + \frac{1}{2}[1-e^{-2x}]^2$$

$$\Rightarrow \frac{1}{2}[1-e^{-2x}]^2 = \frac{1}{2}$$

$$\Rightarrow [1-e^{-2x}]^2 = 1$$

$$\Rightarrow 1 - e^{-2x} = 1$$

$$\Rightarrow e^{-2x} = 0$$

$$\Rightarrow 2x = +\infty$$

$$\Rightarrow x = +\infty$$

$$(I) : \lim_{t \rightarrow \infty} \frac{1 - F(t + x \gamma(t))}{1 - F(t)}$$

$$= \lim_{t \rightarrow \infty} \frac{1 - \left[ \frac{1}{2} + \frac{1}{2} \left[ 1 - e^{-2(t + x \gamma(t))} \right]^2 \right]^2}{1 - \left[ 1 - e^{-2t} \right]^2}$$

$$\frac{\left[ \frac{1}{2} + \frac{1}{2} \left[ 1 - e^{-2t} \right]^2 \right]^2}{1 - \left[ 1 - e^{-2t} \right]^2}$$

$$= \lim_{t \rightarrow \infty} \frac{1 - \left[ 1 - e^{-2(t + x \gamma(t))} \right]^2}{1 - \left[ 1 - e^{-2t} \right]^2}$$

$$\stackrel{||}{=} \lim_{t \rightarrow \infty} \frac{\left[ (1-x)^x \approx 1 - \alpha x \right]}{\left[ 1 - 2 \cdot e^{-2(t + x \gamma(t))} \right]} \quad \text{as } x \rightarrow 0$$

$$= \lim_{t \rightarrow \infty} \frac{e^{-2(t + x \gamma(t))}}{e^{-2t}}$$

$$= \lim_{t \rightarrow \infty} e^{-2x \gamma(t)}$$

$$= e^{-x} \quad \text{if } \gamma(t) = \frac{1}{2}.$$

2016 Test Q1

Suppose  $G$  belongs to Grubel domain.

$$\lim_{t \rightarrow w(F)} \frac{1 - F(t + x\delta(t))}{1 - F(t)}$$

$$= \lim_{t \rightarrow w(F)} \frac{\{1 - \{1 - [1 - G(t + x\delta(t))]^2\}^3\}^4}{\{1 - \{1 - [1 - G(t)]^2\}^3\}^4}$$

$w(F)$        $w(G)$

$(1-z)^\alpha \approx 1 - \alpha z$  as  $z \rightarrow 0$

$$= \lim_{t \rightarrow w(G)} \frac{\{1 - \{1 - 3[1 - G(t + x\delta(t))]^2\}^3\}^4}{\{1 - \{1 - 3[1 - G(t)]^2\}^3\}^4}$$

$$= \lim_{t \rightarrow w(G)} \frac{[1 - G(t + x\delta(t))]^8}{[1 - G(t)]^8}$$

2015 Test Q2 (iii)

$$P(k) = p(1-p)^{k-1}, \quad k = 1, 2, \dots$$

$$F(k) = \sum_{i=1}^k p(1-p)^{i-1}$$

$$= p \sum_{i=1}^k (1-p)^{i-1}$$

$$= 1 - (1-p)^k \quad \text{by sum of a geometric series}$$

$$\frac{p(k)}{1 - F(k-1)} = \frac{p(1-p)^{k-1}}{1 - [1 - (1-p)^{k-1}]}$$

$$= p$$

$$\neq 0$$

ETT does not hold.

2014/15 Test Q1

i) Assume  $G$  belongs to Gumbel domain. Then there exist  $\gamma(t) > 0$  such that

$$\lim_{t \rightarrow w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x} \dots (*)$$

Note

$$\lim_{t \rightarrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$\stackrel{LH}{=} \lim_{t \rightarrow w(F)} \frac{-f(t + x\gamma(t)) (1 + x\gamma'(t))}{-f(t)}$$

$$= \lim_{t \rightarrow w(F)} \frac{\frac{1}{2} [-\log(1 - G(t + x\gamma(t)))]^2 g(t + x\gamma(t)) \cdot (1 + x\gamma'(t))}{\frac{1}{2} [-\log(1 - G(t))]^2 g(t)}$$

$$= \lim_{t \rightarrow w(F)} \left[ \frac{\log(1 - G(t + x\gamma(t)))}{\log(1 - G(t))} \right]^2 \cdot \frac{g(t + x\gamma(t)) (1 + x\gamma'(t))}{g(t)}$$

$$\stackrel{\text{LH}}{=} \lim_{t \rightarrow w(F)} \left[ \frac{-g(t+x\delta(t))(1+x\delta'(t))}{1-G(t+x\delta(t))} \right]^2 \cdot \frac{g(t+x\delta(t))(1+x\delta'(t))}{g(t)}$$

$$= \lim_{t \rightarrow w(F)} \left[ \frac{1-G(t)}{1-G(t+x\delta(t))} \right]^2 \cdot \left[ \frac{g(t+x\delta(t))(1+x\delta'(t))}{g(t)} \right]^3$$

$$\stackrel{\text{LH}}{=} \lim_{\substack{t \rightarrow w(F) \\ \text{in reverse}}} \left[ \frac{1-G(t)}{1-G(t+x\delta(t))} \right]^2 \left[ \frac{1-G(t+x\delta(t))}{1-G(t)} \right]^3$$

$$\stackrel{\text{LH}}{=} \lim_{t \rightarrow w(G)} \frac{1-G(t+x\delta(t))}{1-G(t)}$$

$$= e^{-x} \quad \text{by } (*)$$

Hence,  $F$  also belong to the Gumbel domain.



2015 Test, Q2 (iv)

$$F(x) = 1$$

$$\Rightarrow \Phi^2(x) = 1$$

$$\Rightarrow \bar{\Phi}(x) = 1$$

$$\Rightarrow x = \bar{\Phi}^{-1}(1) = +\infty$$

$$(I) : \lim_{t \rightarrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \rightarrow \infty} \frac{1 - \bar{\Phi}^2(t + x\gamma(t))}{1 - \bar{\Phi}^2(t)}$$

$$\stackrel{LH}{=} \lim_{t \rightarrow \infty} \frac{-2 \bar{\Phi}(t + x\gamma(t)) \phi(t + x\gamma(t)) (1 + x\gamma'(t))}{-2 \bar{\Phi}(t) \phi(t)}$$

$$= \lim_{t \rightarrow \infty} \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(t+x\gamma(t))^2}{2}} (1+x\gamma'(t))}{\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}}$$

$$= \lim_{t \rightarrow \infty} e^{\frac{t^2}{2} - \frac{(t+x\gamma(t))^2}{2}} (1+x\gamma'(t))$$

$$= \lim_{t \rightarrow \infty} e^{-\frac{tx\gamma(t)}{2} - \frac{x^2(\gamma(t))^2}{2}} (1+x\gamma'(t))$$

$$\text{Let } \gamma(t) = \frac{1}{t} \circ \gamma''(t) = -\frac{1}{t^2}$$

$$= \lim_{t \rightarrow \infty}$$

$$e^{-x} \cdot \left(1 - \frac{x}{t^2}\right)$$

The fraction  $\frac{x^2}{2t^2}$  is circled in red, with an arrow pointing to a red circle below it. The fraction  $\frac{x}{t^2}$  is also circled in red, with an arrow pointing to a red circle below it.

$$= e^{-x}$$

2015 test  
Q2 (ii)

$$f(x) = 6x(1-x), \quad 0 < x < 1$$

$$W(F) = 1$$

(III)

$$\lim_{t \rightarrow 0} \frac{1 - F(1-tx)}{1 - F(1-t)}$$

$$\stackrel{\text{LH}}{=} \lim_{t \rightarrow 0} \frac{-x f(1-tx)}{-f(1-t)}$$

$$= \lim_{t \rightarrow 0} \frac{x \cdot \cancel{6} \cdot (1 - \cancel{t}x) \cdot \cancel{x}}{\cancel{6} \cdot (1 - \cancel{t}) \cdot \cancel{t}}$$

$$= x^2$$

**LECTURE**

**23 NOVEMBER**

**12:00-13:00PM**

**MATH4/68181**

Estimation                      Methods                      for                      ~~VAR~~ ES

- a) Parametric estimation methods  
Levels 3, 4, 6
- b) Non-parametric                      "                      "  
Levels 3, 4, 6
- c) Semi-parametric                      "                      "  
Levels 4, 6

c) Semi-parametric estimation methods  
for ES

i) Heavy tailed method

Let  $X_t =$  financial variable  
at time  $t$

Then

$$r_t = X_t - X_{t-1}$$

is the return at time  $t$

Assume that

$$P(r_t < -x) \sim x^{-\alpha} L(x) \quad (**)$$

behaves as

where  $\alpha > 0$  and  $L(\cdot)$  is a  
slowly varying function.

A function  $L(\cdot)$  is slowly varying  
if

$$\frac{L(tx)}{L(x)} \rightarrow 1$$

as  $t \rightarrow \infty$  for all  $x$

Suppose we have

$$\boxed{r_{(1)}} \leq r_{(2)} \leq \dots \leq \boxed{r_{(n)}}$$

smallest  largest.

Then

$$\widehat{ES}_p = \frac{1}{p} \int_0^p \exp \left[ \frac{\widehat{\alpha}_{l_{n,p}}}{nq} \right] dq \quad (*)$$

- 1

where  $l_{n,p} = \lfloor n(p + 0.05) \rfloor$

and  $\widehat{\alpha}_{l,n} = \left[ \frac{1}{l} \sum_{i=1}^l \left( \frac{r_{(i)}}{r_{(l)}} \right) \right]^{-1}$

(\*) is a semi parametric estimator of  $ES$ .

## ii) Necir et al's estimator

Suppose  $r_1, r_2, \dots, r_n$   
are observed returns.

Order them as

$$r_{(1)} \leq r_{(2)} \leq \dots \leq r_{(n)}$$

Assume (\*\*). Then

$$\widehat{ES}_p = \frac{1}{p} \int_{\frac{k}{n}}^p (\widehat{F})^{-1}(t) dt + \frac{k r_{(n-k)}}{n p (1 - \widehat{\alpha})}$$

where

$$\widehat{\alpha} = \frac{1}{k} \sum_{i=1}^k \log \frac{r_{(n-i+1)}}{r_{(n-k)}}$$

$$1 \leq k \leq n,$$

$\widehat{F}(\cdot)$  is the empirical CDF  
of the returns.



Q3, Prob Sheet 14

$$C(u_1, u_2) = u_1 u_2 e^{-\theta (\log u_1) (\log u_2)}$$

$$\begin{aligned} \text{i) } C(0, u) &= 0 \cdot u \cdot e^{-(\log 0) (\log u)} \\ &= 0 \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{ii) } C(u, 0) &= u \cdot 0 \cdot e^{-(\log u) (\log 0)} \\ &= 0 \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{iii) } C(1, u) &= 1 \cdot u \cdot e^{-\theta (\log 1) (\log u)} \\ &= u \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{iv) } C(u, 1) &= u \cdot 1 \cdot e^{-\theta (\log u) (\log 1)} \\ &= u \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{v) } \frac{\partial}{\partial u_1} C(u_1, u_2) &= u_2 e^{-\theta (\log u_1) (\log u_2)} \\ &\quad + \cancel{u_1} u_2 e^{-\theta (\log u_1) (\log u_2)} \\ &\quad \cdot \left( -\frac{\theta}{\cancel{u_1}} \right) \cdot (\log u_2) \end{aligned}$$

$$= \underbrace{u_2}_{\checkmark 0} \underbrace{e^{-\theta (\log u_1) (\log u_2)}}_{\checkmark 0} \underbrace{\left[ 1 - \theta \log u_2 \right]}_{\checkmark 0} \checkmark$$

$$\text{vi) } \frac{\partial}{\partial u_2} C(u_1, u_2) \\ = \underbrace{u_1}_{\substack{\vee \\ 0}} \underbrace{e^{-\theta (\log u_1) (\log u_2)}}_{\substack{\vee \\ 0}} \underbrace{[1 - \theta \log u_1]}_{\substack{\vee \\ 0}} \quad \checkmark$$

Hence,  $C$  is a copula.

Q4, Sheet 14

- i)  $C(0, u) = 0 \cdot u [1 + \phi(1-0)(1-u)] = 0 \checkmark$
- ii)  $C(u, 0) = u \cdot 0 [1 + \phi(1-u)(1-0)] = 0 \checkmark$
- iii)  $C(1, u) = 1 \cdot u [1 + \phi(1-1)(1-u)] = u \checkmark$
- iv)  $C(u, 1) = u \cdot 1 [1 + \phi(1-u)(1-1)] = u \checkmark$
- v)  $\frac{\partial C(u_1, u_2)}{\partial u_1} = u_2 [1 + \phi(1-u_1)(1-u_2)]$

$$+ u_1 u_2 \cdot \phi \cdot (-1)(1-u_2)$$

$$= u_2 [1 + \phi(1-u_1)(1-u_2) - \phi u_1(1-u_2)]$$

$$= u_2 [1 + \underbrace{\phi(1-2u_1)}_{-1 \leq 1-1 \leq -1 \leq 1 \leq 0 \leq 1 \leq 1} (1-u_2)] \geq 0 \checkmark$$

$-1 \leq \leq 1$

$$vi) \frac{\partial C(u_1, u_2)}{\partial u_2} = u_1 [1 + \underbrace{\phi(1-u_1)(1-2u_2)}_{-1 \leq 1-1 \leq 0 \leq 1 \leq -1 \leq 1} ] \geq 0 \checkmark$$

$-1 \leq \leq 1$

Hence,  $C$  is a copula.

$$\begin{aligned} \text{(i)} \quad C(0, u) &= 0 + u - 1 + \left[ (1-0)^{-\frac{1}{\alpha}} + (1-u)^{-\frac{1}{\alpha}} - 1 \right]^{-\alpha} \\ &= u - 1 + \left[ (1-u)^{-\frac{1}{\alpha}} \right]^{-\alpha} \\ &= u - 1 + 1 - u = 0 \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad C(u, 0) &= u + 0 - 1 + \left[ (1-u)^{-\frac{1}{\alpha}} + (1-0)^{-\frac{1}{\alpha}} - 1 \right]^{-\alpha} \\ &= u - 1 + 1 - u \\ &= 0 \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad C(1, u) &= 1 + u - 1 + \left[ \underbrace{(1-1)^{-\frac{1}{\alpha}}}_{\substack{=0 \\ +\infty}} + (1-u)^{-\frac{1}{\alpha}} - 1 \right]^{-\alpha} \\ &= u \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad C(u, 1) &= u + 1 - 1 + \left[ (1-u)^{-\frac{1}{\alpha}} + \underbrace{(1-1)^{-\frac{1}{\alpha}}}_{\substack{=0 \\ +\infty}} - 1 \right]^{-\alpha} \\ &= u \quad \checkmark \end{aligned}$$

**LECTURE**

**17 NOVEMBER**

**9:00-10:00AM**

**MATH3/4/68181**

## Expected Shortfall

$$X = \text{Loss}$$

$$ES_p(X) = \frac{1}{p} \left[ E(X I \{X \leq VaR_p(X)\}) + p VaR_p(X) - VaR_p(X) Pr(X \leq VaR_p(X)) \right]$$

where  $I\{\cdot\}$  denotes the indicator function

$$I\{A\} = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false} \end{cases}$$

If  $X$  is absolutely continuous

$$ES_p(X) = \frac{1}{p} \int_0^p F^{-1}(t) dt$$

where  $F$  denotes the CDF of  $X$ .

## Properties of ES

(i) if  $X \leq Y$  then  $ES_p(X) \leq ES_p(Y)$

$$(ii) \quad ES_p(X+c) = ES_p(X) + c$$

where  $c$  is a constant

$$(iii) \quad ES_p(cX) = c \quad ES_p(X)$$

where  $c$  is a constant

Proof of (i)

Suppose  $X, Y$  ~~is~~ are absolutely continuous random variables. If  $X \leq Y$

then

$$\text{VaR}_q(X) \leq \text{VaR}_q(Y) \quad \forall q \in (0, 1)$$

$$\Rightarrow F_X^{-1}(q) \leq F_Y^{-1}(q) \quad \forall q \in (0, 1)$$

$$\Rightarrow \int_0^p F_X^{-1}(q) dq \leq \int_0^p F_Y^{-1}(q) dq \quad \forall p \in (0, 1)$$

$$\Rightarrow \frac{1}{p} \int_0^p F_X^{-1}(q) dq \leq \frac{1}{p} \int_0^p F_Y^{-1}(q) dq \quad \forall p \in (0, 1)$$

$$\Rightarrow ES_p(X) \leq ES_p(Y) \quad \forall p \in (0, 1)$$



Proof of (ii)

Suppose  $X$  is an absolutely continuous RV. Then

$$\text{VaR}_q(X+c) = \text{VaR}_q(X) + c \quad \forall q \in (0,1)$$

$$\Rightarrow F_{X+c}^{-1}(q) = F_X^{-1}(q) + c \quad \forall q \in (0,1)$$

$$\Rightarrow \int_0^p F_{X+c}^{-1}(q) dq = \int_0^p [F_X^{-1}(q) + c] dq \quad \forall p \in (0,1)$$

$$\Rightarrow \frac{1}{p} \int_0^p F_{X+c}^{-1}(q) dq = \frac{1}{p} \int_0^p [F_X^{-1}(q) + c] dq \quad \forall p \in (0,1)$$

$$\Rightarrow ES_p(X+c) = ES_p(X) + c$$

Proof of (iii)

Suppose  $X$  is an absolutely continuous RV.

Then

$$\text{VaR}_q(cX) = c \text{VaR}_q(X) \quad \forall q \in (0, 1)$$

$$\Rightarrow F_{cX}^{-1}(q) = c F_X^{-1}(q) \quad \forall q \in (0, 1)$$

$$\Rightarrow \int_0^p F_{cX}^{-1}(q) dq = c \int_0^p F_X^{-1}(q) dq$$

$\forall p \in (0, 1)$

$$\Rightarrow \frac{1}{p} \int_0^p F_{cX}^{-1}(q) dq = \frac{c}{p} \int_0^p F_X^{-1}(q) dq$$

$\forall p \in (0, 1)$

$$\Rightarrow ES_p(cX) = c ES_p(X).$$

Estimation                      Methods                      for                      ~~VAR~~ ES

a) Parametric estimation methods

Levels 3, 4, 6

b) Non-parametric                      "                      "

Levels 3, 4, 6

c) Semi-parametric                      "                      "

Levels 4, 6

## a) Parametric estimation of ES

### i) Normal distribution method

$$X = \text{Loss} \sim N(\mu, \sigma^2)$$

$$\text{VaR}_p(X) = \mu + \sigma \Phi^{-1}(p)$$

$$\text{ES}_p(X) = \frac{1}{p} \int_0^p [\mu + \sigma \Phi^{-1}(t)] dt$$

$$= \mu + \frac{\sigma}{p} \int_0^p \Phi^{-1}(t) dt$$

Suppose  $x_1, \dots, x_n$  are IID observed losses on  $X$ . Then the MLEs of  $\mu$  and  $\sigma$  are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

and

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2}$$

So, the MLE of  $\text{ES}_p(X)$  is

$$\hat{\text{ES}}_p(X) = \bar{x} + \frac{1}{p} \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2} \int_0^p \Phi^{-1}(t) dt$$

ii) Uniform distribution method

$$X = \text{Loss} \sim \text{Uni}[a, b]$$

$$\text{Var}_p(X) = a + (b-a)p$$

$$ES_p(X) = \frac{1}{p} \int_0^p [a + (b-a)t] dt$$

$$= a + \frac{p}{2} (b-a).$$

Suppose  $x_1, \dots, x_n$  are IID observed losses on  $X$ . Then the MLEs of  $a$  and  $b$  are

$$\hat{a} = \min(x_1, \dots, x_n)$$

$$\hat{b} = \max(x_1, \dots, x_n).$$

So, the MLE of  $ES_p(X)$  is

$$\hat{ES}_p(X) = \min(x_1, \dots, x_n)$$

$$+ \frac{p}{2} \left[ \max(x_1, \dots, x_n) - \min(x_1, \dots, x_n) \right]$$

(iii) Power function distribution method

$X =$  Loss has CDF  $F(x) = x^a$ ,  
 $0 < x < 1$

$$\text{Var}_p(X) = p \frac{1}{a}$$

$$\begin{aligned} E S_p(X) &= \frac{1}{p} \int_0^p t^{\frac{1}{a}} dt \\ &= \frac{1}{p} \left[ \frac{t^{\frac{1}{a} + 1}}{\frac{1}{a} + 1} \right]_0^p \\ &= \frac{a p^{\frac{1}{a}}}{a + 1} \end{aligned}$$

Suppose  $x_1, \dots, x_n$  are IID observed losses on  $X$ . Then the MLE of  $a$  is

$$\hat{a} = - \frac{n}{\sum_{i=1}^n \log x_i}$$

So, the MLE of  $E S_p(X)$  is

$$\widehat{E S_p(X)} = \frac{p - \frac{\sum_{i=1}^n \log x_i}{n}}{1 - \frac{\sum_{i=1}^n \log x_i}{n}}$$

# (iv) Weibull distribution method

$$X = \text{Loss} \sim F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^\beta}$$

$$\text{Set } F(x) = p$$

$$\Rightarrow 1 - e^{-\left(\frac{x}{\theta}\right)^\beta} = p$$

$$\Rightarrow \left(\frac{x}{\theta}\right)^\beta = -\log(1-p)$$

$$\Rightarrow \text{VaR}_p(x) = \theta \left[-\log(1-p)\right]^{\frac{1}{\beta}}$$

$$E S_p(x) = \frac{\theta}{p} \int_0^p \left[-\log(1-t)\right]^{\frac{1}{\beta}} dt$$

$$\text{Set } x = -\log(1-t)$$

$$\Rightarrow e^{-x} = 1-t$$

$$\Rightarrow t = 1 - e^{-x}$$

$$\Rightarrow \frac{dt}{dx} = e^{-x}$$

$$= \frac{\theta}{p} \int_0^{-\log(1-p)} x^{\frac{1}{\beta}} e^{-x} dx$$

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt$$

Incomplete gamma function

$$= \frac{\theta}{p} \gamma\left(\frac{1}{\beta} + 1, -\log(1-p)\right)$$

Suppose  $x_1, \dots, x_n$  are IID observed losses on  $X$ . The estimators of  $\theta$  and  $\beta$  are given by

$$\hat{\theta} = \frac{\bar{x}}{\Gamma\left(1 + \frac{1}{\hat{\beta}}\right)}$$

and

$$\frac{\bar{x}^2}{s^2} = \frac{\left(\Gamma\left(1 + \frac{1}{\hat{\beta}}\right)\right)^2}{\Gamma\left(1 + \frac{2}{\hat{\beta}}\right) - \left(\Gamma\left(1 + \frac{1}{\hat{\beta}}\right)\right)^2}$$

where

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

and

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2.$$

The estimator of  $ES_p(X)$  is

$$\widehat{ES}_p(X) = \frac{\hat{\theta}}{p} \gamma\left(\frac{1}{\hat{\beta}} + 1, -\log(1-p)\right).$$



**EXAMPLE CLASS**

**20 NOVEMBER**

**12:00-13:00PM**

**MATH3/4/68181**

Q1

$$X \sim \text{Exp}(\lambda)$$

$$\text{VaR}_p(X) = -\frac{1}{\lambda} \log(1-p)$$

$$\text{ES}_p(X) = -\frac{1}{\lambda p} \left[ p \log(1-p) - p - \log(1-p) \right]$$

$$L(\lambda) = \prod_{i=1}^n \left[ \lambda e^{-\lambda x_i} \right]$$

$$= \lambda^n e^{-\lambda \sum_{i=1}^n x_i}$$

$$\log L(\lambda) = n \log \lambda - \lambda \sum_{i=1}^n x_i$$

$$\frac{d \log L}{d \lambda} = \frac{n}{\lambda} - \sum_{i=1}^n x_i = 0$$

$$\Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^n x_i} = \frac{1}{\bar{x}}$$

Thus, the MLEs of VaR and ES are

$$\widehat{\text{VaR}}_p(X) = -\bar{x} \cdot \log(1-p)$$

$$\widehat{\text{ES}}_p(X) = -\frac{\bar{x}}{p} \left[ p \cdot \log(1-p) - p - \log(1-p) \right]$$

Q2

$$\text{Var}_p(X) = p \frac{1}{a}$$

$$E S_p(X) = \frac{p \frac{1}{a}}{\frac{1}{a} + 1}$$

$$L(a) = \prod_{i=1}^n [a x_i^{a-1}]$$

$$= a^n \left( \prod_{i=1}^n x_i \right)^{a-1}$$

$$\log L(a) = n \log a + (a-1) \sum_{i=1}^n \log x_i$$

$$\frac{d \log L}{d a} = \frac{n}{a} + \sum_{i=1}^n \log x_i = 0$$

$$\hat{a} = - \frac{n}{\sum_{i=1}^n \log x_i}$$

Thus, the MLEs of  $\text{Var}$  and  $ES$  are

$$\widehat{\text{Var}}_p(X) = p \frac{- \sum_{i=1}^n \log x_i}{n}$$

and

$$\widehat{E S}_p(X) = \frac{p \frac{- \sum_{i=1}^n \log x_i}{n}}{- \frac{\sum_{i=1}^n \log x_i}{n} + 1}$$

$$\underline{\underline{Q3}} \quad X \sim N(\mu, \sigma^2)$$

$$\text{Var}_p(X) = \mu + \sigma \Phi^{-1}(p)$$

$$E_{\Phi}^{-1}(X) = \mu + \frac{\sigma}{p} \int_0^p \Phi^{-1}(t) dt$$

$$L(\mu, \sigma) = \prod_{i=1}^n \left[ \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} \right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2}$$

$$\log L(\mu, \sigma) = -\frac{n}{2} \log(2\pi) - n \log \sigma$$

$$- \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2$$

$$\frac{\partial \log L}{\partial \mu} = -\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu) \cdot 2 \cdot (-1)$$

$$= \frac{1}{\sigma^2} \left( \left( \sum_{i=1}^n x_i \right) - n\mu \right) = 0 \quad \text{--- (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n (x_i - \mu)^2 = 0 \quad \text{--- (2)}$$

$$(1) \Rightarrow \hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

$$(2) \Rightarrow \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

Thus, the MLEs of VaR and ES are

$$\widehat{\text{VaR}}_p(x) = \bar{x} + \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2} \Phi^{-1}(p)$$

$$\widehat{\text{ES}}_p(x) = \bar{x} + \frac{1}{p} \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2}$$

$$\cdot \int_0^p \Phi^{-1}(t) dt$$

Q4

$$F(x) = \Phi \left( \frac{\log x - \mu}{\sigma} \right)$$

$$\text{Set } F(x) = p$$

$$\Rightarrow \Phi \left( \frac{\log x - \mu}{\sigma} \right) = p$$

$$\Rightarrow \frac{\log x - \mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow \log x = \mu + \sigma \Phi^{-1}(p)$$

$$\Rightarrow x = \boxed{e^{\mu + \sigma \Phi^{-1}(p)}}$$

$$E S_p(X) = \frac{1}{p} \int_0^p F^{-1}(t) dt$$

$$= \boxed{\frac{e^{\mu}}{p} \int_0^p e^{\sigma \Phi^{-1}(t)} dt}$$

$$L(\mu, \sigma) = \prod_{i=1}^n \left[ \frac{1}{\sqrt{2\pi} \sigma x_i} e^{-\frac{(\log x_i - \mu)^2}{2\sigma^2}} \right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n \left( \prod_{i=1}^n x_i \right)} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (\log x_i - \mu)^2}$$

$$\log L(\mu, \sigma) = -\frac{n}{2} \log(2\pi) - n \log \sigma$$

$$- \sum_{i=1}^n \log x_i - \frac{1}{2\sigma^2} \sum_{i=1}^n (\log x_i - \mu)^2$$

$$\frac{\partial \log L}{\partial \mu} = -\frac{1}{2\sigma^2} \sum_{i=1}^n (\log x_i - \mu) \cdot 2 \cdot (-1)$$

$$= \frac{1}{\sigma^2} \left[ \sum_{i=1}^n \log x_i - n\mu \right] = 0 \quad (1)$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n (\log x_i - \mu)^2 = 0$$

— (2)

$$(1) \Rightarrow \hat{\mu} = \frac{1}{n} \sum_{i=1}^n \log x_i$$

$$(2) \Rightarrow \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (\log x_i - \hat{\mu})^2$$

The MLEs of VaR and ES are

$$\widehat{\text{VaR}}_p(x) = e^{\frac{1}{n} \sum_{i=1}^n \log x_i + \sqrt{\frac{1}{n} \sum_{i=1}^n (\log x_i - \hat{\mu})^2} \Phi^{-1}(p)}$$

and

$$\widehat{\text{ES}}_p(x) = e^{\frac{\frac{1}{n} \sum_{i=1}^n \log x_i}{p} \int_0^p \sqrt{\frac{1}{n} \sum_{i=1}^n (\log x_i - \hat{\mu})^2} \Phi^{-1}(t) dt}$$



**LECTURE**

**21 NOVEMBER**

**9:00-10:00AM**

**MATH3/4/68181**

Estimation

Methods

for

~~VAR~~ ES

a) Parametric estimation methods

Levels 3, 4, 6

b) Non-parametric

"

"

Levels 3, 4, 6

c) Semi-parametric

"

"

Levels 4, 6

## b) Non-parametric estimation methods for ES

### i) Historical method

Suppose the data are  $x_1, x_2, \dots, x_n$

Order the data as

$$\boxed{x_{(1)}} \leq x_{(2)} \leq \dots \leq \boxed{x_{(n)}}$$

*smallest*  *largest*

The historical estimator of  $ES_p(X)$  is

$$\widehat{ES}_p(X) = \frac{\sum_{i=1}^{[np]} x_{(i)}}{[np]},$$

where  $[x]$  denotes the largest integer not greater than  $x$ .

Ex

$$[4.1] = 4$$

$$[5.6] = 5$$

Ex

Observed losses: -6, 7, 0, 1, 9

-6	0	1	7	9
$x_{(1)}$	$x_{(2)}$	$x_{(3)}$	$x_{(4)}$	$x_{(5)}$

$$\widehat{ES}_{0.2}(X) = \frac{\sum_{i=1}^1 x_{(i)}}{1} = -6$$

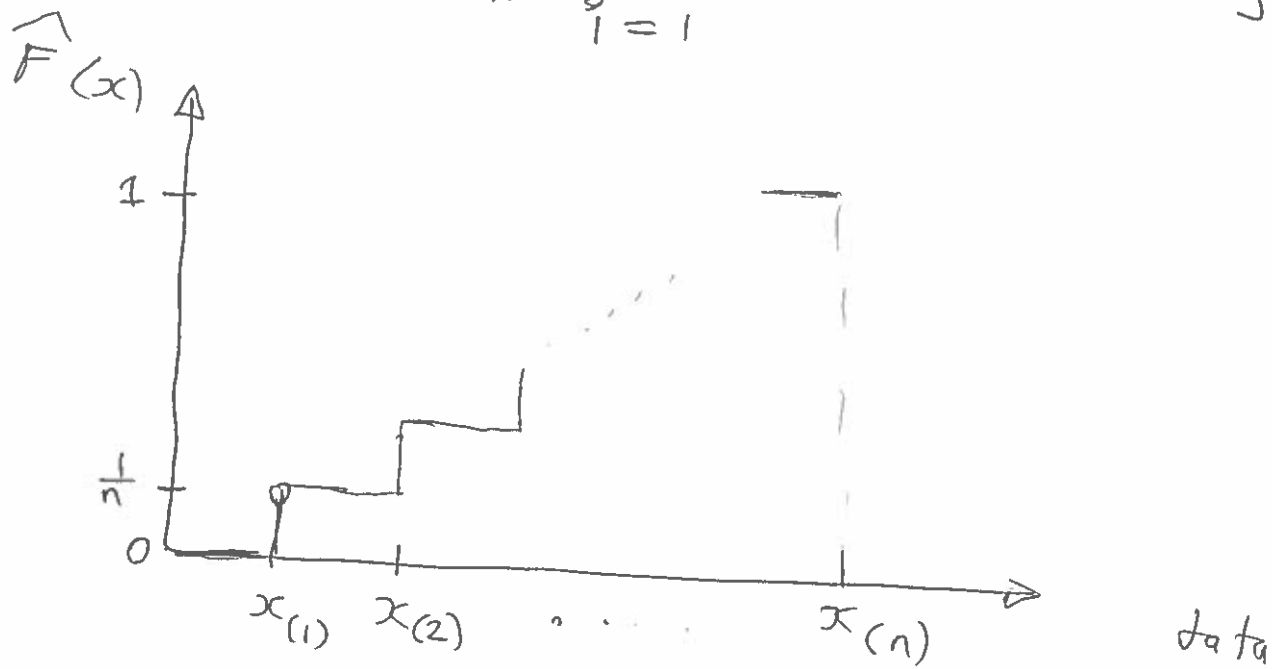
$$\begin{aligned}\widehat{ES}_{0.8}(X) &= \frac{\sum_{i=1}^4 x_{(i)}}{4} \\ &= \frac{-6 + 0 + 1 + 7}{4} \\ &= 0.5\end{aligned}$$

## (ii) Bootstrap method

Suppose the data are  $x_1, x_2, \dots, x_n$ .

The empirical cdf is

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^n I\{x_i \leq x\}$$



1. Simulate  $B$  random samples each of size  $n$  from  $\hat{F}$ .
2. Estimate  $ES_p(x)$  by the historical method for each of the  $B$  samples, resulting in  $\hat{ES}_p^{(1)}, \hat{ES}_p^{(2)}, \dots, \hat{ES}_p^{(B)}$ .
3. Compute  $\hat{ES}_p$  as the mean or median of  $\hat{ES}_p^{(1)}, \hat{ES}_p^{(2)}, \dots, \hat{ES}_p^{(B)}$ .

### iii) Jackknife method

Suppose the data are  $x_1, x_2, \dots, x_n$ .

1. Estimate  $ES_p(x)$  by the historical estimator based on  $x_2, x_3, \dots, x_n$ , resulting in  $\widehat{ES}_p^{(1)}$ .

2. Estimate  $ES_p(x)$  by the historical estimator based on  $x_1, x_3, \dots, x_n$ , resulting in  $\widehat{ES}_p^{(2)}$ .

•

•

•

3. Estimate  $ES_p(x)$  by the historical estimator based on  $x_1, x_2, \dots, x_{n-1}$ , resulting in  $\widehat{ES}_p^{(n)}$ .

4. Compute  $\widehat{ES}_p$  as the mean or median of  $\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \dots, \widehat{ES}_p^{(n)}$ .

## iv) Kernel method

Suppose the data are  $x_1, x_2, \dots, x_n$ .

Order the data as  $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ .

The kernel estimator  $\hat{\lambda}$  is

$$\widehat{ES}_p(x) = \frac{1}{n p} \sum_{i=1}^n x_i A\left(\frac{\hat{Q}(p) - x_i}{h}\right),$$

where

$$A(x) = \int_{-\infty}^x K(u) du$$

and

$$\hat{Q}(p) = \frac{1}{h} \sum_{i=1}^n \left[ \int_{\frac{i-1}{n}}^{\frac{i}{n}} K\left(\frac{t-p}{h}\right) dt \right] x_{(i)}$$

$K(\cdot)$  is the kernel, eg

$$K(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$$

$h$  = bandwidth

## v) Richardson's method

(Richardson was a professor in Manchester. His photo in ground floor of ATB).

1. Compute the empirical cdf  $\hat{F}$  of the observed data
2. simulate  $x_1, x_2, \dots, x_N$  from  $\hat{F}$
3. estimate  $ES_p$  by the historical method for  $x_1, x_2, \dots, x_N$
4. Repeat steps 2 and 3 1000 times and compute

$$m_N = \frac{1}{1000} \sum_{i=1}^{1000} \hat{ES}_{N,i}$$

estimate obtained in the  $i$ th iteration



5. set

$$S_q = m N_q$$

for  $q = 1, 2, \dots, k+1$

and for some  $N_1, N_2, \dots, N_{k+1}$ .

6. Compute

$$E S_p(x) = \frac{\begin{vmatrix} s_1 & s_2 & \dots & s_{k+1} \\ 1 & \frac{1}{2} & \dots & \frac{1}{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1^k & \left(\frac{1}{2}\right)^k & \dots & \left(\frac{1}{k+1}\right)^k \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & \frac{1}{2} & \dots & \frac{1}{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1^k & \left(\frac{1}{2}\right)^k & \dots & \left(\frac{1}{k+1}\right)^k \end{vmatrix}}$$

**EXAMPLE CLASS**

**21 NOVEMBER**

**10:00-11:00AM**

**MATH3/4/68181**

Q1

$$\text{VaR}_p(X) = -\frac{1}{\lambda} \log(1-p)$$

$$\text{ES}_p(X) = -\frac{1}{\lambda p} \left[ p \cdot \log(1-p) - p - \log(1-p) \right]$$

$$L(\lambda) = \prod_{i=1}^n \left[ \lambda e^{-\lambda x_i} \right]$$

$$= \lambda^n e^{-\lambda \sum_{i=1}^n x_i}$$

$$\log L(\lambda) = n \log \lambda - \lambda \sum_{i=1}^n x_i$$

$$\frac{d \log L}{d \lambda} = \frac{n}{\lambda} - \sum_{i=1}^n x_i = 0$$

$$\Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^n x_i} = \frac{1}{\bar{x}}$$

$$\frac{d^2 \log L}{d \lambda^2} = -\frac{n}{\lambda^2} < 0$$

So  $\hat{\lambda} = \frac{1}{\bar{x}}$  is an MLE

Hence, the MLEs of VaR and ES are

$$\widehat{\text{VaR}}_p(X) = -\bar{x} \log(1-p)$$

$$\widehat{\text{ES}}_p(X) = -\frac{\bar{x}}{p} \left[ p \log(1-p) - p - \log(1-p) \right]$$

Q2

$$\text{Var}_p(X) = p^{\frac{1}{a}}$$

$$\text{ES}_p(X) = \frac{p^{\frac{1}{a}}}{\frac{1}{a} + 1}$$

$$L(a) = \prod_{i=1}^n [a x_i^{a-1}] = a^n \left( \prod_{i=1}^n x_i \right)^{a-1}$$

$$\log L(a) = n \log a + (a-1) \sum_{i=1}^n \log x_i$$

$$\frac{d \log L}{d a} = \frac{n}{a} + \sum_{i=1}^n \log x_i = 0$$

$$\Rightarrow \hat{a} = - \frac{n}{\sum_{i=1}^n \log x_i}$$

$$\frac{d^2 \log L}{d a^2} = - \frac{n}{a^2} < 0$$

So,  $\hat{a}$  is an MLE. Hence, the MLEs of  $\text{Var}$  and  $\text{ES}$  are

$$\widehat{\text{Var}}_p(X) = p - \frac{\sum_{i=1}^n \log x_i}{n}$$

and

$$\widehat{\text{ES}}_p(X) = \frac{p - \frac{\sum_{i=1}^n \log x_i}{n}}{1 - \frac{\sum_{i=1}^n \log x_i}{n}}$$

Q3

$$\text{VaR}_p(X) = \mu + \sigma \Phi^{-1}(p)$$

$$\text{ES}_p(X) = \mu + \frac{\sigma}{p} \int_0^p \Phi^{-1}(t) dt$$

$$L(\mu, \sigma) = \prod_{i=1}^n \left[ \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} \right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2}$$

$$\log L(\mu, \sigma) = -\frac{n}{2} \log(2\pi) - n \log \sigma$$

$$- \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2$$

$$\frac{\partial \log L}{\partial \mu} = -\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu) \cdot 2 \cdot (-1)$$

$$= \frac{1}{\sigma^2} \left[ \left( \sum_{i=1}^n x_i \right) - n\mu \right] = 0 \quad (1)$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n (x_i - \mu)^2 = 0$$

— (2)

$$(1) \Rightarrow \hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

$$(2) \Rightarrow \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

The MLEs of VaR and ES are

$$\widehat{\text{VaR}}_p(X) = \bar{x} + \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2} \Phi^{-1}(p)$$

and

$$\widehat{\text{ES}}_p(X) = \bar{x} + \frac{1}{p} \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2} \int_0^p \Phi^{-1}(t) dt$$

Q4

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right) = p$$

$$\Rightarrow \frac{\log x - \mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow \log x = \mu + \sigma \Phi^{-1}(p)$$

$$\Rightarrow x = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$\Rightarrow \text{VaR}_p(x) = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$ES_p(x) = \frac{1}{p} \int_0^p F^{-1}(t) dt$$

$$= \frac{e^{\mu}}{p} \int_0^p e^{\sigma \Phi^{-1}(t)} dt$$

$$L(\mu, \sigma) = \prod_{i=1}^n \left[ \frac{1}{\sqrt{2\pi} \sigma x_i} e^{-\frac{(\log x_i - \mu)^2}{2\sigma^2}} \right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n \prod_{i=1}^n x_i} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (\log x_i - \mu)^2}$$

$$\log L(\mu, \sigma) = -\frac{n}{2} \log(2\pi) - n \log \sigma$$

$$- \sum_{i=1}^n \log x_i - \frac{1}{2\sigma^2} \sum_{i=1}^n (\log x_i - \mu)^2$$

$$\begin{aligned} \frac{\partial \log L}{\partial \mu} &= -\frac{1}{2\sigma^2} \sum_{i=1}^n (\log x_i - \mu) \cdot 2 \cdot (-1) \\ &= \frac{1}{\sigma^2} \left( \sum_{i=1}^n \log x_i - n\mu \right) = 0 \quad \text{--- (1)} \end{aligned}$$

$$\begin{aligned} \frac{\partial \log L}{\partial \sigma} &= -\frac{2}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n (\log x_i - \mu)^2 = 0 \\ &\quad \text{--- (2)} \end{aligned}$$

$$(1) \Rightarrow \hat{\mu} = \frac{1}{n} \sum_{i=1}^n \log x_i$$

$$(2) \Rightarrow \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (\log x_i - \hat{\mu})^2$$

The MLEs of VaR and ES are

$$\widehat{\text{VaR}}_p(x) = e^{\frac{1}{n} \sum_{i=1}^n \log x_i} + \sqrt{\frac{1}{n} \sum_{i=1}^n (\log x_i - \hat{\mu})^2} \Phi^{-1}(p)$$

and

$$\widehat{\text{ES}}_p(x) = \frac{e^{\frac{1}{n} \sum_{i=1}^n \log x_i}}{p} \int_0^p \sqrt{\frac{1}{n} \sum_{i=1}^n (\log x_i - \hat{\mu})^2} \Phi^{-1}(t) dt$$



**MATH3/4/68181: Extreme values and financial risk**  
**Semester 1**  
**Problem sheet 10**

1. If  $x_1, x_2, \dots, x_n$  is a random sample from  $\text{Exp}(\lambda)$  find the maximum likelihood estimates of  $\text{VaR}_p(X)$  and  $\text{ES}_p(X)$ .
2. If  $x_1, x_2, \dots, x_n$  is a random sample from the power function distribution with pdf  $f(x) = ax^{a-1}$ ,  $0 < x < 1$ ,  $a > 0$  find the maximum likelihood estimates of  $\text{VaR}_p(X)$  and  $\text{ES}_p(X)$ .
3. If  $x_1, x_2, \dots, x_n$  is a random sample from the normal distribution  $N(\mu, \sigma^2)$  find the maximum likelihood estimates of  $\text{VaR}_p(X)$  and  $\text{ES}_p(X)$ .
4. If  $x_1, x_2, \dots, x_n$  is a random sample from the log-normal distribution  $LN(\mu, \sigma^2)$  find the maximum likelihood estimates of  $\text{VaR}_p(X)$  and  $\text{ES}_p(X)$ .
5. If  $x_1, x_2, \dots, x_n$  is a random sample from a distribution with pdf  $f(x) = \theta_2 x^{\theta_2 - 1} \theta_1^{-\theta_2}$ ,  $0 < x < \theta_1$ ,  $\theta_1 > 0$ ,  $\theta_2 > 0$  find the maximum likelihood estimates of  $\text{VaR}_p(X)$  and  $\text{ES}_p(X)$ .
6. If  $x_1, x_2, \dots, x_n$  is a random sample from the uniform  $[\mu - \delta, \mu + \delta]$  distribution find the maximum likelihood estimates of  $\text{VaR}_p(X)$  and  $\text{ES}_p(X)$ .

Indicator function approach

**LECTURE**

**23 NOVEMBER**

**12:00-13:00PM**

**MATH4/68181**

Estimation                      Methods                      for                      ~~VAR~~ ES

- a) Parametric estimation methods  
Levels 3, 4, 6
- b) Non-parametric                      "                      "  
Levels 3, 4, 6
- c) Semi-parametric                      "                      "  
Levels 4, 6

c) Semi-parametric estimation methods  
for ES

i) Heavy tailed method

Let  $X_t$  = financial variable  
at time  $t$

Then

$$r_t = X_t - X_{t-1}$$

is the return at time  $t$

Assume that

$$P(r_t < -x) \sim x^{-\alpha} L(x) \quad (**)$$

behaves as

where  $\alpha > 0$  and  $L(\cdot)$  is a  
slowly varying function.

A function  $L(\cdot)$  is slowly varying  
if

$$\frac{L(tx)}{L(x)} \rightarrow 1$$

as  $t \rightarrow \infty$  for all  $x$

Suppose we have

$$\boxed{r_{(1)}} \leq r_{(2)} \leq \dots \leq \boxed{r_{(n)}}$$

smallest  largest.

Then

$$\widehat{ES}_p = \frac{1}{p} \int_0^p \exp \left[ \frac{\widehat{\alpha}_{l_{n,p}}}{nq} \right] dq \quad (*)$$

- 1

where  $l_{n,p} = \lfloor n(p + 0.05) \rfloor$

and  $\widehat{\alpha}_{l,n} = \left[ \frac{1}{l} \sum_{i=1}^l \left( \frac{r_{(i)}}{r_{(l)}} \right) \right]^{-1}$

(\*) is a semi parametric estimator of  $ES$ .

## ii) Necir et al's estimator

Suppose  $r_1, r_2, \dots, r_n$   
are observed returns.

Order them as

$$r_{(1)} \leq r_{(2)} \leq \dots \leq r_{(n)}$$

Assume (\*\*). Then

$$\widehat{ES}_p = \frac{1}{p} \int_{\frac{k}{n}}^p (\widehat{F})^{-1}(t) dt + \frac{k r_{(n-k)}}{n p (1 - \widehat{\alpha})}$$

where

$$\widehat{\alpha} = \frac{1}{k} \sum_{i=1}^k \log \frac{r_{(n-i+1)}}{r_{(n-k)}}$$

$$1 \leq k \leq n,$$

$\widehat{F}(\cdot)$  is the empirical CDF  
of the returns.

Q3, Prob Sheet 14

$$C(u_1, u_2) = u_1 u_2 e^{-\theta(\log u_1)(\log u_2)}$$

$$\begin{aligned} \text{i) } C(0, u) &= 0 \cdot u \cdot e^{-(\log 0)(\log u)} \\ &= 0 \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{ii) } C(u, 0) &= u \cdot 0 \cdot e^{-(\log u)(\log 0)} \\ &= 0 \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{iii) } C(1, u) &= 1 \cdot u \cdot e^{-\theta(\log 1)(\log u)} \\ &= u \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{iv) } C(u, 1) &= u \cdot 1 \cdot e^{-\theta(\log u)(\log 1)} \\ &= u \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{v) } \frac{\partial}{\partial u_1} C(u_1, u_2) &= u_2 e^{-\theta(\log u_1)(\log u_2)} \\ &\quad + \cancel{u_1} u_2 e^{-\theta(\log u_1)(\log u_2)} \\ &\quad \cdot \left( -\frac{\theta}{u_1} \right) \cdot (\log u_2) \end{aligned}$$

$$= \underbrace{u_2}_{\checkmark 0} e^{-\theta(\log u_1)(\log u_2)} \underbrace{\left[ 1 - \theta \log u_2 \right]}_{\checkmark 0} \checkmark$$

$$\text{vi) } \frac{\partial}{\partial u_2} C(u_1, u_2) \\ = \underbrace{u_1}_{\substack{\vee \\ 0}} \underbrace{e^{-\theta (\log u_1) (\log u_2)}}_{\substack{\vee \\ 0}} \underbrace{[1 - \theta \log u_1]}_{\substack{\vee \\ 0}} \quad \checkmark$$

Hence,  $C$  is a copula.



Q4, Sheet 14

- i)  $C(0, u) = 0 \cdot u [1 + \phi(1-0)(1-u)] = 0 \checkmark$
- ii)  $C(u, 0) = u \cdot 0 [1 + \phi(1-u)(1-0)] = 0 \checkmark$
- iii)  $C(1, u) = 1 \cdot u [1 + \phi(1-1)(1-u)] = u \checkmark$
- iv)  $C(u, 1) = u \cdot 1 [1 + \phi(1-u)(1-1)] = u \checkmark$
- v)  $\frac{\partial C(u_1, u_2)}{\partial u_1} = u_2 [1 + \phi(1-u_1)(1-u_2)]$

$$+ u_1 u_2 \cdot \phi \cdot (-1)(1-u_2)$$

$$= u_2 [1 + \phi(1-u_1)(1-u_2) - \phi u_1(1-u_2)]$$

$$= u_2 [1 + \underbrace{\phi(1-2u_1)}_{-1 \leq 1-1 \leq -1 \leq 1 \quad 0 \leq 1 \leq 1} (1-u_2)] \geq 0 \checkmark$$

$-1 \leq \leq 1$

$$vi) \frac{\partial C(u_1, u_2)}{\partial u_2} = u_1 [1 + \underbrace{\phi(1-u_1)(1-2u_2)}_{-1 \leq 1-1 \leq -1 \leq 1 \quad 0 \leq 1 \leq -1 \leq 1}] \geq 0 \checkmark$$

$-1 \leq \leq 1$

Hence,  $C$  is a copula.

$$\begin{aligned} \text{(i)} \quad C(0, u) &= 0 + u - 1 + \left[ (1-0)^{-\frac{1}{\alpha}} + (1-u)^{-\frac{1}{\alpha}} - 1 \right]^{-\alpha} \\ &= u - 1 + \left[ (1-u)^{-\frac{1}{\alpha}} \right]^{-\alpha} \\ &= u - 1 + 1 - u = 0 \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad C(u, 0) &= u + 0 - 1 + \left[ (1-u)^{-\frac{1}{\alpha}} + (1-0)^{-\frac{1}{\alpha}} - 1 \right]^{-\alpha} \\ &= u - 1 + 1 - u \\ &= 0 \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad C(1, u) &= 1 + u - 1 + \left[ \underbrace{(1-1)^{-\frac{1}{\alpha}}}_{\substack{=0 \\ +\infty}} + (1-u)^{-\frac{1}{\alpha}} - 1 \right]^{-\alpha} \\ &= u \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad C(u, 1) &= u + 1 - 1 + \left[ (1-u)^{-\frac{1}{\alpha}} + \underbrace{(1-1)^{-\frac{1}{\alpha}}}_{\substack{=0 \\ +\infty}} - 1 \right]^{-\alpha} \\ &= u \quad \checkmark \end{aligned}$$

**LECTURE**

**24 NOVEMBER**

**9:00-10:00AM**

**MATH3/4/68181**

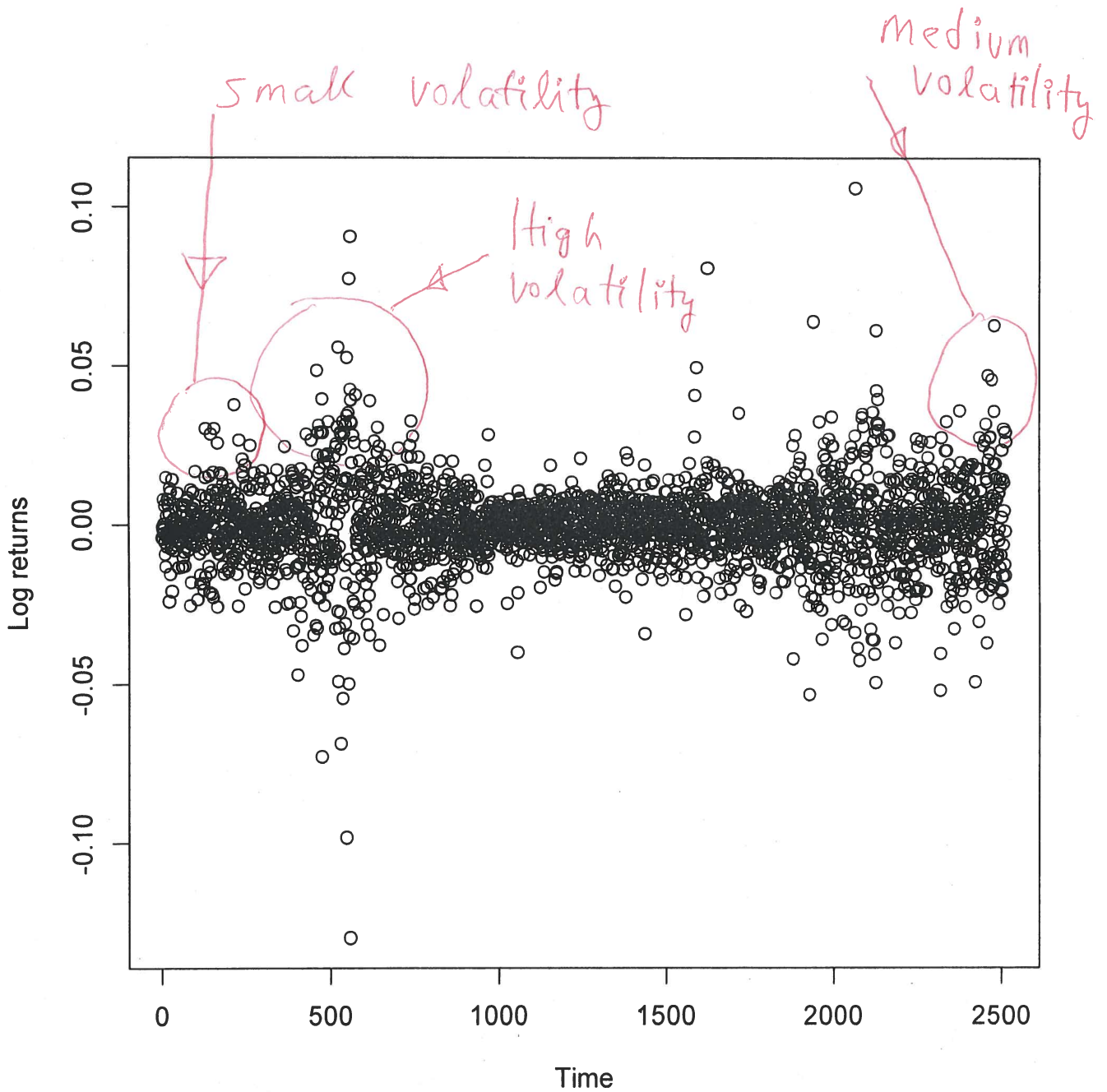
# Models for Stock

## Returns

i) Model I

ii) Model II

iii) Model III



the amount of volatility at a given time is a variable itself.

# i) Model I

Let  $X$  = stock return at a given time

$V$  = volatility at a given time

$X$  is an observable RV

$V$  is not an " "

But  $X$  depends on  $V$ .

What is the actual distribution of  $X$ ?

The PDF of  $X$  is

$$f_X(x) = \int_0^{\infty} f_{X|V}(x|v) g(v) dv$$

[ total prob rule ]

PDF of  $V$

PDF of  $X$  given  $V$

The CDF of  $X$  is

$$F_X(x) = \int_0^{\infty} F_{X|V}(x|v) g(v) dv$$

[total prob rule]

CDF of  $X$  given  $V$

PDF of  $V$

The  $n$ th moment of  $X$  is

$$\begin{aligned} E[X^n] &= E[E[X^n|V]] \\ &= \int_0^{\infty} E[X^n|V] g(v) dv \end{aligned}$$

For example,

$$E[X] = E[E[X|V]],$$

$$\begin{aligned} \text{Var}[X] &= E[E[X^2|V]] \\ &\quad - (E[E[X|V]])^2 \end{aligned}$$

Ex

$$X | V \sim N(0, \boxed{\sigma^2})$$

$\Downarrow$   
 $V$

$V$  is a RV

Let  $g(\cdot)$  denote the PDF of  $\sigma$ .

The actual PDF of  $X$  is

$$\begin{aligned} f_X(x) &= \int_0^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}}_{\text{PDF of } X|V} \underbrace{g(\sigma)}_{\text{PDF of } \sigma} d\sigma \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{\sigma} e^{-\frac{x^2}{2\sigma^2}} g(\sigma) d\sigma \end{aligned}$$

Suppose

$$g(\sigma) = \frac{2}{\sigma^3} e^{-\frac{1}{\sigma^2}}, \sigma > 0$$

Then

$$\begin{aligned} f_X(x) &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{\sigma^4} e^{-\frac{x^2}{2\sigma^2} - \frac{1}{\sigma^2}} d\sigma \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{\sigma^4} e^{-\left(\frac{x^2}{2} + 1\right) \frac{1}{\sigma^2}} d\sigma \end{aligned}$$



$$\text{Set } y = \left(\frac{x^2}{2} + 1\right) \frac{1}{\sigma^2}$$

$$\sigma^2 = \left(\frac{x^2}{2} + 1\right) \frac{1}{y}$$

$$\sigma = \sqrt{\frac{x^2}{2} + 1} \frac{1}{\sqrt{y}}$$

$$\frac{d\sigma}{dy} = -\frac{1}{2} \sqrt{\frac{x^2}{2} + 1} \frac{1}{y^{3/2}}$$

$$f_X(x) = \sqrt{\frac{2}{\pi}} \int_{\infty}^0 \frac{y^2}{\left(\frac{x^2}{2} + 1\right)^2} e^{-y} \left(-\frac{1}{2}\right) \sqrt{\frac{x^2}{2} + 1}$$

$$\cdot \frac{1}{y^{3/2}} dy$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{x^2}{2} + 1\right)^{-\frac{3}{2}}$$

$$\int_0^{\infty} y^{\frac{1}{2}} e^{-y} dy$$

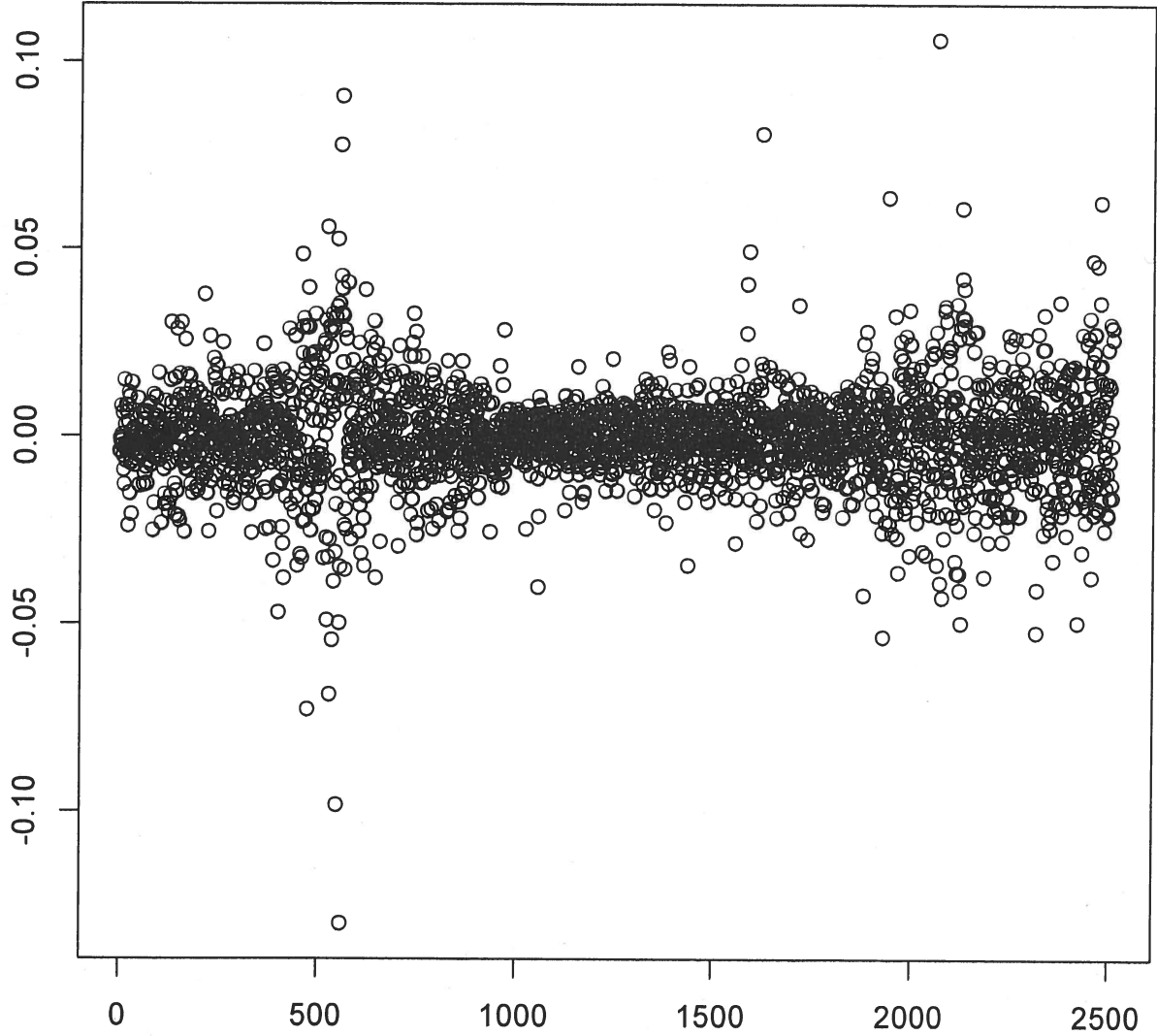
$$\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{x^2}{2} + 1\right)^{-\frac{3}{2}} \frac{\sqrt{\pi}}{2}$$

$$= \frac{1}{2\sqrt{2}} \left(\frac{x^2}{2} + 1\right)^{-\frac{3}{2}}$$

$X_t$

Log returns



Time

$t$

ii) Model II

Let  $X_t$  = stock return at  $t$

$X_0$  = stock return at time 0

Assume  $X_0$  is known (a fixed number).

$$\begin{aligned} X_t &= \boxed{X_t - X_{t-1}} = Z_t \\ &+ \boxed{X_{t-1} - X_{t-2}} = Z_{t-1} \\ &+ \boxed{X_{t-2} - X_{t-3}} = Z_{t-2} \\ &+ \dots \\ &+ \dots \\ &+ \boxed{X_1 - X_0} = Z_1 \\ &+ X_0 \end{aligned}$$

$$= \left( \sum_{i=1}^t Z_i \right) + X_0$$

$$E[X_t] = \sum_{i=1}^t E(Z_i) + X_0$$

$$E[X_t^2] = E\left\{\left[\left(\sum_{i=1}^t Z_i\right) + X_0\right]^2\right\}$$

$$= E\left[\left(\sum_{i=1}^t Z_i\right)^2\right]$$

$$+ 2X_0 E\left[\sum_{i=1}^t Z_i\right]$$

$$+ X_0^2$$

$$= \sum_{i=1}^t E(Z_i^2) + \sum_{i \neq j} E(Z_i Z_j)$$

$$+ 2X_0 \sum_{i=1}^t E(Z_i)$$

$$+ X_0^2$$

$$\text{Var}[\cancel{X}_t] = E[X_t^2] - (E[X_t])^2$$

$$= \sum_{i=1}^t E(Z_i^2) + \sum_{i \neq j} E(Z_i Z_j)$$

$$- \left[\sum_{i=1}^t E(Z_i)\right]^2.$$

Suppose  $Z_i$  are IID.

Then

$$E[X_t] = t E(Z) + X_0,$$

$$\begin{aligned} E[X_t^2] &= t E(Z^2) \\ &\quad + t(t-1) E(Z^2) \\ &\quad + 2X_0 t E(Z) \\ &\quad + X_0^2, \end{aligned}$$

$$\begin{aligned} \text{Var}[X_t] &= t E[Z^2] \\ &\quad + t(t-1) E(Z^2) \\ &\quad - t^2 (E(Z))^2 \end{aligned}$$

Ex

Suppose  $Z_i$  are independent  
 $N(\mu_i, \sigma_i^2)$ . What is  
the distribution of  $X_t$ ?

$$X_t = \left( \sum_{i=1}^t Z_i \right) + X_0$$

$$= \left( \sum_{i=1}^t N(\mu_i, \sigma_i^2) \right) + X_0$$

$$= N\left( \sum_{i=1}^t \mu_i, \sum_{i=1}^t \sigma_i^2 \right) + X_0$$

$$= N\left( X_0 + \sum_{i=1}^t \mu_i, \sum_{i=1}^t \sigma_i^2 \right).$$

$$E[X_t] = X_0 + \sum_{i=1}^t \mu_i$$

$$\text{Var}[X_t] = \sum_{i=1}^t \sigma_i^2$$

# **EXAMPLE CLASS**

**27 NOVEMBER**

**12:00-13:00PM**

**MATH3/4/68181**

Q1

$X$  = stock returns

$$X | \lambda \sim \text{Exp}(\lambda)$$

volatility

$$\lambda \sim \text{Exp}(a)$$

$$\begin{aligned} f_X(x) &= \int_0^{\infty} \underbrace{f_{X|\lambda}(x|\lambda)}_{\substack{\text{cond PDF} \\ \text{of } X \text{ given } \lambda}} \underbrace{g(\lambda)}_{\substack{\text{PDF} \\ \text{of } \lambda}} d\lambda \\ &= \int_0^{\infty} \lambda e^{-\lambda x} a e^{-a\lambda} d\lambda \\ &= a \int_0^{\infty} \lambda e^{-(x+a)\lambda} d\lambda \end{aligned}$$

$$\begin{aligned} \text{Set } y &= (x+a)\lambda \\ \lambda &= \frac{y}{x+a} \\ d\lambda &= \frac{dy}{x+a} \end{aligned}$$

$$\begin{aligned} &= a \int_0^{\infty} \frac{y}{x+a} e^{-y} \frac{dy}{x+a} \\ &= \frac{a}{(x+a)^2} \int_0^{\infty} y e^{-y} dy = \Gamma(2) \end{aligned}$$



$$f_X(x) = \frac{a}{(x+a)^2}$$

Suppose  $x_1, \dots, x_n$  are IID observations on  $X$ .

$$\begin{aligned} L(a) &= \prod_{i=1}^n \left[ \frac{a}{(x_i + a)^2} \right] \\ &= a^n \left[ \prod_{i=1}^n (x_i + a) \right]^{-2} \end{aligned}$$

$$\log L(a) = n \log a - 2 \sum_{i=1}^n \log(x_i + a)$$

$$\frac{d \log L(a)}{d a} = \frac{n}{a} - 2 \sum_{i=1}^n \frac{1}{x_i + a}$$

So  $\hat{a}$  (the MLE of  $a$ ) is the root of

$$\frac{n}{a} = 2 \sum_{i=1}^n \frac{1}{x_i + a}$$

Q2

$$X | \lambda \sim \text{Exp}(\lambda)$$

$$\lambda \sim \text{Uni}[a, b]$$

$$f_X(x) = \int_a^b f_{X|\lambda}(x|\lambda) g(\lambda) d\lambda$$

$$= \int_a^b \lambda e^{-\lambda x} \frac{1}{b-a} d\lambda$$

$$\begin{aligned} \text{Set } y &= \lambda x \\ \lambda &= \frac{y}{x} \\ d\lambda &= \frac{dy}{x} \end{aligned}$$

$$= \frac{1}{b-a} \int_{ax}^{bx} \frac{y}{x} e^{-y} \frac{dy}{x}$$

$$= \frac{1}{(b-a)x^2} \int_{ax}^{bx} y e^{-y} dy$$

Int by parts

$$= \frac{1}{(b-a)x^2} \left\{ \left[ y(e^{-y}) \right]_{ax}^{bx} + \int_{ax}^{bx} e^{-y} dy \right\}$$

$$= \frac{1}{(b-a)x^2} \left[ -bx e^{-bx} + ax e^{-ax} + e^{-ax} - e^{-bx} \right]$$

Q3

$$X | \lambda \sim \text{Exp}(\lambda)$$

$$\lambda \sim \text{PDF} = a \lambda^{a-1}, 0 < \lambda < 1$$

$$f_X(x) = \int_0^1 f_{X|\lambda}(x|\lambda) g(\lambda) d\lambda$$

$$= \int_0^1 \lambda e^{-\lambda x} a \lambda^{a-1} d\lambda$$

$$= a \int_0^1 \lambda^a e^{-\lambda x} d\lambda$$

$$\text{Set } y = \lambda x$$

$$\lambda = \frac{y}{x}$$

$$d\lambda = \frac{dy}{x}$$

$$= a \int_0^x \left(\frac{y}{x}\right)^a e^{-y} \frac{dy}{x}$$

$$= \frac{a}{x^{a+1}} \int_0^x y^a e^{-y} dy$$

$$= \frac{a}{x^{a+1}} \gamma(a+1, x)$$

$$\gamma(\beta, x) = \int_0^x t^{\beta-1} e^{-t} dt$$

Incomplete Gamma Function

Q4

$$X|\lambda \sim \text{Exp}(\lambda)$$

$$\lambda \sim \text{PDF} \quad \frac{a k^a}{\lambda^{a+1}}, \lambda > k$$

$$\begin{aligned} f_X(x) &= \int_k^\infty f_{X|\lambda}(x|\lambda) g(\lambda) d\lambda \\ &= \int_k^\infty \lambda e^{-\lambda x} \frac{a k^a}{\lambda^{a+1}} d\lambda \\ &= a k^a \int_k^\infty \lambda^{-a} e^{-\lambda x} d\lambda \end{aligned}$$

$$\begin{aligned} \text{Set } y &= \lambda x \\ \lambda &= \frac{y}{x} \\ d\lambda &= \frac{dy}{x} \end{aligned}$$

$$\begin{aligned} &= a k^a \int_{kx}^\infty \left(\frac{y}{x}\right)^{-a} e^{-y} \frac{dy}{x} \\ &= a k^a x^{a-1} \int_{kx}^\infty y^{-a} e^{-y} dy \\ &= a k^a x^{a-1} \Gamma(1-a, kx) \end{aligned}$$

$$\Gamma(\beta, x) = \int_x^\infty t^{\beta-1} e^{-t} dt$$

Complementary Incomplete Gamma Function

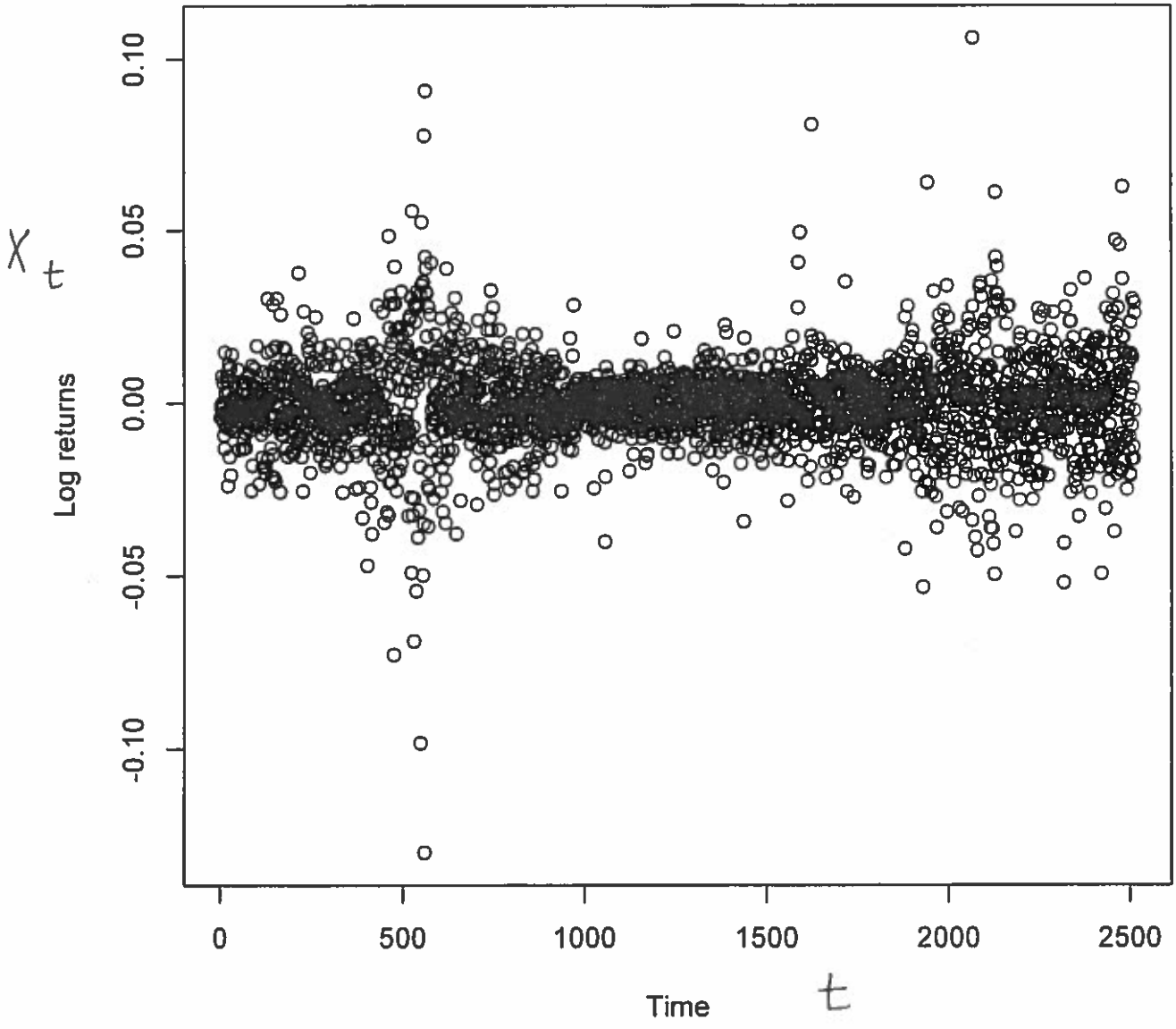
**LECTURE**

**28 NOVEMBER**

**9:00-10:00AM**

**MATH3/4/68181**

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ii) Model II

Let  $X_t$  = stock return at  $t$

$X_0$  = stock return at time 0

Assume  $X_0$  is known (a fixed number).

$$\begin{aligned} X_t &= \boxed{X_t - X_{t-1}} = Z_t \\ &+ \boxed{X_{t-1} - X_{t-2}} = Z_{t-1} \\ &+ \boxed{X_{t-2} - X_{t-3}} = Z_{t-2} \\ &+ \dots \\ &+ \dots \\ &+ \boxed{X_1 - X_0} = Z_1 \\ &+ X_0 \end{aligned}$$

$$= \left( \sum_{i=1}^t Z_i \right) + X_0$$

$$E[X_t] = \sum_{i=1}^t E(Z_i) + X_0$$

$$E[X_t^2] = E\left\{\left[\left(\sum_{i=1}^t Z_i\right) + X_0\right]^2\right\}$$

$$= E\left[\left(\sum_{i=1}^t Z_i\right)^2\right]$$

$$+ 2X_0 E\left[\sum_{i=1}^t Z_i\right]$$

$$+ X_0^2$$

$$= \sum_{i=1}^t E(Z_i^2) + \sum_{i \neq j} E(Z_i Z_j)$$

$$+ 2X_0 \sum_{i=1}^t E(Z_i)$$

$$+ X_0^2$$

$$\text{Var}[X_t] = E[X_t^2] - (E[X_t])^2$$

$$= \sum_{i=1}^t E(Z_i^2) + \sum_{i \neq j} E(Z_i Z_j)$$

$$- \left[\sum_{i=1}^t E(Z_i)\right]^2$$



$$\left( \sum_{j=1}^n a_j \right)^2$$

$$= \sum_{j=1}^n a_j^2 + \sum_{i \neq j} a_i a_j$$

Suppose  $Z_i$  are IID.

Then

$$E[X_t] = t E(Z) + X_0,$$

$$\begin{aligned} E[X_t^2] &= t E(Z^2) \\ &\quad + t(t-1) E(Z^2) \\ &\quad + 2X_0 t E(Z) \\ &\quad + X_0^2, \end{aligned}$$

$$\begin{aligned} \text{Var}[X_t] &= t E[Z^2] \\ &\quad + t(t-1) E(Z^2) \\ &\quad - t^2 (E(Z))^2 \end{aligned}$$

---

Ex

Suppose  $Z_i$  are independent  
 $N(\mu_i, \sigma_i^2)$ . What is  
the distribution of  $X_t$ ?

$$X_t = \left( \sum_{i=1}^t Z_i \right) + X_0$$

$$= \left( \sum_{i=1}^t N(\mu_i, \sigma_i^2) \right) + X_0$$

$$= N\left( \sum_{i=1}^t \mu_i, \sum_{i=1}^t \sigma_i^2 \right) + X_0$$

$$= N\left( X_0 + \sum_{i=1}^t \mu_i, \sum_{i=1}^t \sigma_i^2 \right).$$

$$E[X_t] = X_0 + \sum_{i=1}^t \mu_i$$

$$\text{Var}[X_t] = \sum_{i=1}^t \sigma_i^2$$

E<sub>x</sub> (contd)

$$P(X_t < x)$$
$$= \Phi \left( \frac{x - X_0 - \sum_{i=1}^t \mu_i}{\sqrt{\sum_{i=1}^t \sigma_i^2}} \right)$$

$\Phi(\cdot)$  is the CDF of  $N(0, 1)$

Answer questions like:

$$P(X_{5000} > x) = 0.95$$

$$\Rightarrow P(X_{5000} \leq x) = 0.05$$

$$\Rightarrow \Phi \left( \frac{x - X_0 - \sum_{i=1}^{5000} \mu_i}{\sqrt{\sum_{i=1}^{5000} \sigma_i^2}} \right) = 0.05$$

$$\Rightarrow x = X_0 + \sum_{i=1}^{5000} \mu_i + \Phi^{-1}(0.05) \sqrt{\sum_{i=1}^{5000} \sigma_i^2}$$

Ex 2 Suppose  $Z_i$  are independent gamma random variables with parameters  $(a_i, b)$ . What is the distribution of  $X_t$ ?

$$X_t = \left( \sum_{i=1}^t Z_i \right) + X_0$$

The MGF Approach

$$M_{X_t}(s) = E(e^{sX_t})$$

$$= E\left(e^{s\left(\sum_{i=1}^t Z_i + X_0\right)}\right)$$

$$= e^{sX_0} E\left(e^{s\sum_{i=1}^t Z_i}\right)$$

$$= e^{sX_0} E(e^{sZ_1}) \cdots E(e^{sZ_t})$$

Math  
= 20802

$$e^{sX_0} \left(\frac{b}{b-s}\right)^{a_1} \cdots \left(\frac{b}{b-s}\right)^{a_t}$$

$$= e^{sX_0} \left(\frac{b}{b-s}\right)^{\sum_{i=1}^t a_i}$$

= MGF of a gamma RV with paras  $(\sum_{i=1}^t a_i, b)$

$$\Rightarrow X_t = Z + X_0 \leftarrow \text{constant}$$

where  $Z$  is a gamma RV  
with parameters  $\left( \sum_{i=1}^t a_i, b \right)$

$$\begin{aligned} E(X_t) &= E(Z) + X_0 \\ &= \frac{\sum_{i=1}^t a_i}{b} + X_0 \end{aligned}$$

$$\begin{aligned} \text{Var}(X_t) &= \text{Var}(Z + X_0) \\ &= \text{Var}(Z) \\ &= \frac{\sum_{i=1}^t a_i}{b^2} \end{aligned}$$

---

$$P(X_t < x) = q$$

$$\Rightarrow P(Z + X_0 < x) = q$$

$$\Rightarrow P(Z < x - X_0) = q$$

$$\Rightarrow \frac{\gamma\left(\sum_{i=1}^t a_i, b(x - X_0)\right)}{\Gamma\left(\sum_{i=1}^t a_i\right)} = q$$

where  $\gamma(a, x)$  denotes the incomplete gamma function.

---

### Model III

Suppose  $X_0$  is again a fixed number.

$$X_t = \begin{pmatrix} X_t \\ X_{t-1} \end{pmatrix} = Z_t$$

$$\times \begin{pmatrix} X_{t-1} \\ X_{t-2} \end{pmatrix} = Z_{t-1}$$

$$\times \begin{pmatrix} X_{t-2} \\ X_{t-3} \end{pmatrix} = Z_{t-2}$$

⋮

$$\times \begin{pmatrix} X_1 \\ X_0 \end{pmatrix} = Z_1$$

$\times X_0$

$$= \begin{pmatrix} \prod_{i=1}^t Z_i \end{pmatrix} \times X_0$$



Suppose  $Z_i$  are indep RVs

$$E(X_t) = X_0 \cdot E\left(\prod_{i=1}^t Z_i\right)$$

$$= X_0 \prod_{i=1}^t E(Z_i)$$

$$E(X_t^2) = X_0^2 \prod_{i=1}^t E(Z_i^2)$$

⋮

$$E(X_t^n) = X_0^n \prod_{i=1}^t E(Z_i^n)$$

$$\text{Var}(X_t) = X_0^2 \left[ \prod_{i=1}^t E(Z_i^2) - \prod_{i=1}^t (E(Z_i))^2 \right]$$

Suppose  $Z_i$  are IID

$$E(X_t) = X_0 (E(Z))^t$$

$$E(X_t^2) = X_0^2 (E(Z^2))^t$$

⋮

$$E(X_t^n) = X_0^n (E(Z^n))^t$$

$$\text{Var}(X_t) = X_0^2 \left[ (E(Z^2))^t - (E(Z))^{2t} \right]$$

---

Ex

Suppose  $Z_i$  are  $LN(\mu_i, \sigma_i^2)$ .  
What is the distribution of  $X_t$ ?

$$X_t = X_0 \prod_{i=1}^t LN(\mu_i, \sigma_i^2)$$

$$\Rightarrow \log X_t = \log X_0 + \sum_{i=1}^t \log LN(\mu_i, \sigma_i^2)$$

$$\begin{aligned} \Rightarrow \log X_t &= \log X_0 + \sum_{i=1}^t N(\mu_i, \sigma_i^2) \\ &= \log X_0 + N\left(\sum_{i=1}^t \mu_i, \sum_{i=1}^t \sigma_i^2\right) \\ &= N\left(\log X_0 + \sum_{i=1}^t \mu_i, \sum_{i=1}^t \sigma_i^2\right) \end{aligned}$$

$$\Rightarrow X_t \equiv LN\left(\log X_0 + \sum_{i=1}^t \mu_i, \sum_{i=1}^t \sigma_i^2\right)$$

**EXAMPLE CLASS**

**27 NOVEMBER**

**10:00-11:00AM**

**MATH3/4/68181**

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Q1

$X =$  stock returns

$$X | \lambda \sim \text{Exp}(\lambda)$$

$$\lambda \sim \text{Exp}(a)$$

$$f_X(x) = \int_0^{\infty} f_{X|\lambda}(x|\lambda) g(\lambda) d\lambda$$

$$= \int_0^{\infty} \lambda e^{-\lambda x} a e^{-a\lambda} d\lambda$$

$$= a \int_0^{\infty} \lambda e^{-\lambda(x+a)} d\lambda$$

$$\begin{aligned} \text{Set } y &= \lambda(x+a) \\ \lambda &= y/(x+a) \\ \frac{d\lambda}{dy} &= 1/(x+a) \end{aligned}$$

$$= \frac{a}{(x+a)^2} \int_0^{\infty} y e^{-y} dy = \Gamma(2) = 1!$$

$$= \frac{a}{(x+a)^2}$$

$$\underline{Q2} \quad X | \lambda \sim \text{Exp}(\lambda)$$

$$\lambda \sim \text{Uni}[a, b]$$

$$f_X(x) = \int_a^b f_{X|\lambda}(x|\lambda) g(\lambda) d\lambda$$

$$= \int_a^b \lambda e^{-\lambda x} \cdot \frac{1}{b-a} d\lambda$$

$$= \frac{1}{b-a} \int_a^b \lambda e^{-\lambda x} d\lambda$$

Int by  
Parts

$$\frac{1}{b-a} \left\{ \left[ \lambda \frac{e^{-\lambda x}}{(-x)} \right]_a^b + \frac{1}{x} \int_a^b e^{-\lambda x} d\lambda \right\}$$

$$= \frac{1}{b-a} \left\{ \frac{b e^{-bx} - a e^{-ax}}{(-x)} + \frac{1}{x} \left[ \frac{e^{-\lambda x}}{(-x)} \right]_a^b \right\}$$

$$= \frac{1}{b-a} \left\{ \frac{b e^{-bx} - a e^{-ax}}{(-x)} + \frac{1}{x} \cdot \frac{e^{-bx} - e^{-ax}}{(-x)} \right\}.$$

Q4

$X|\lambda \sim \text{Exp}(\lambda)$

$\lambda$  has PDF  $\frac{ak^a}{\lambda^{a+1}}, \lambda > k$

$$f_X(x) = \int_k^\infty \lambda e^{-\lambda x} \cdot \frac{ak^a}{\lambda^{a+1}} d\lambda$$
$$= ak^a \int_k^\infty \frac{1}{\lambda^a} e^{-\lambda x} d\lambda$$

Set  $y = \lambda x \Rightarrow \lambda = \frac{y}{x} \Rightarrow d\lambda = \frac{dy}{x}$

$$= \frac{ak^a}{x^{a+1}} \int_{kx}^\infty \frac{1}{y^a} e^{-y} dy$$

$$\Gamma(\beta, u) = \int_u^\infty t^{\beta-1} e^{-t} dt$$

Complementary Incomp Gamma Fun

$$= \frac{ak^a}{x^{a+1}} \Gamma(1-a, kx).$$

Q3

$$X | \lambda \sim \text{Exp}(\lambda)$$

$$\lambda \text{ has PDF } a\lambda^{a-1}, 0 < \lambda < 1$$

$$f_X(x) = \int_0^1 \lambda e^{-\lambda x} \cdot a\lambda^{a-1} d\lambda$$
$$= a \int_0^1 \lambda^a e^{-\lambda x} d\lambda$$

$$\text{set } y = \lambda x$$
$$\lambda = \frac{y}{x}$$
$$d\lambda = \frac{dy}{x}$$

$$= \frac{a}{x^{a+1}} \int_0^x y^a e^{-y} dy$$

$$\gamma(\beta, u) = \int_0^u t^{\beta-1} e^{-t} dt$$

Incomplete Gamma Fun

$$= \frac{a}{x^{a+1}} \gamma(a+1, x) \circ$$



**LECTURE**

**30 NOVEMBER**

**12:00-13:00PM**

**MATH4/68181**

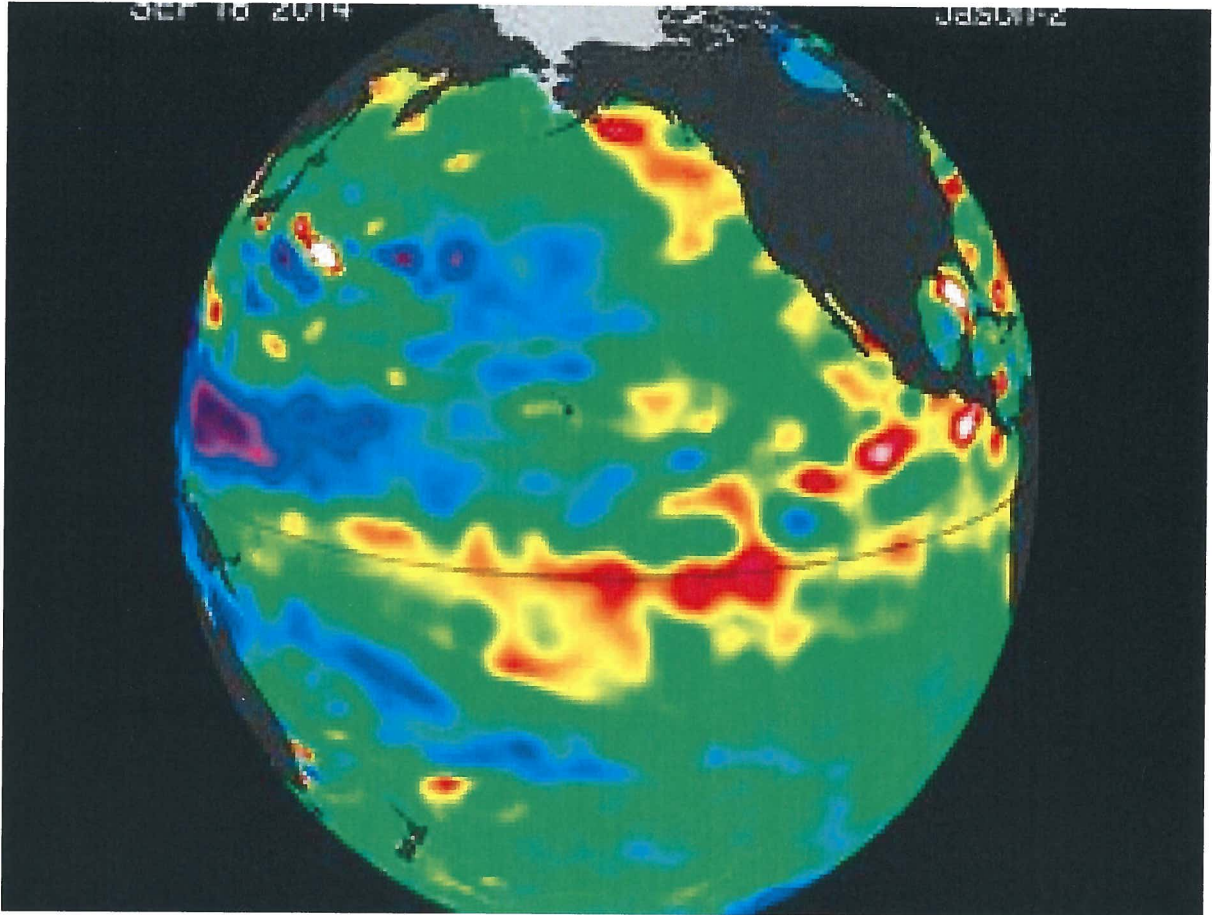
# Bivariate Extremes

There are situations where extremes of more than 1 variable are of interest.

Ex

- 1) Oil price (X)  
2) Car price (Y)
- 2) Oil price (X)  
Food price (Y)
- 3) More examples  
in pictures

# El-Nino



Caused by extreme values of sea temperature  
and air pressure

# Tornado



Caused by extreme values of humidity and  
wind speed

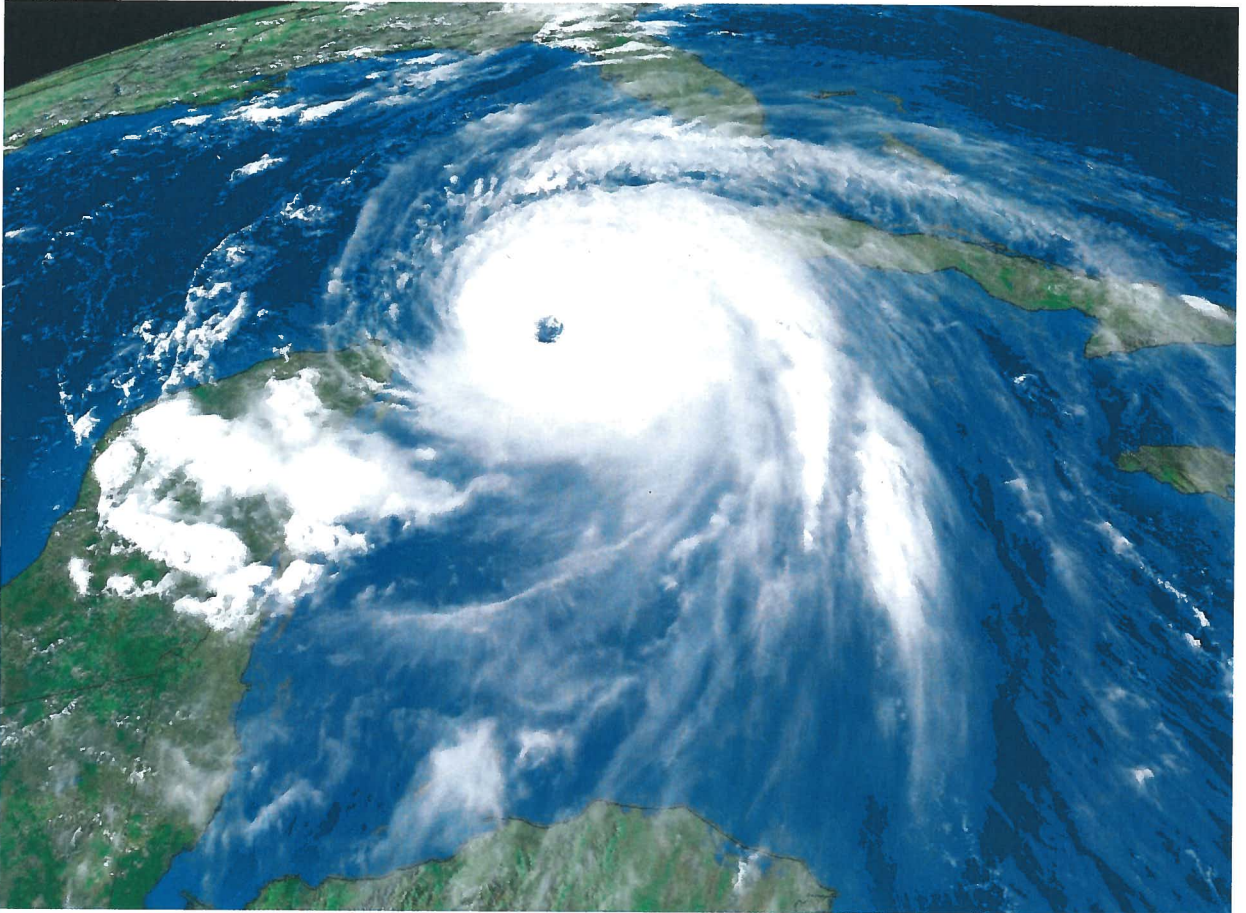
# Droughts



Caused by extreme values of rainfall and  
temperature



# Hurricanes



Caused by extreme values of sea temperature,  
rainfall and wind speed

# Floods



Caused by extreme values of rainfall and wind speed



# Forest fires



Caused by extreme values of temperature and  
wind speed



# Definition of Bivariate Extreme

## Value

Suppose  $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$  are IID with CDF  $F$ .

How to define the extreme-value?

The bivariate extreme value is

$$(M_{n1}, M_{n2})$$

$$= (\max(X_1, \dots, X_n), \max(Y_1, \dots, Y_n))$$

$(M_{n1}, M_{n2})$  may not be an observed value.

Ex

$$X_1 = -2, \quad X_2 = 0, \quad X_3 = 5$$

$$Y_1 = 2, \quad Y_2 = 5, \quad Y_3 = 0$$

$$n = 3$$

$$\max(X_1, X_2, X_3) = 5$$

$$\max(Y_1, Y_2, Y_3) = 5$$

$$\text{Biv extreme value} = (5, 5)$$

But  $(5, 5)$  is not an observation.

# Joint distribution of $(M_{n1}, M_{n2})$

$$P\left(\frac{M_{n1} - b_n}{a_n} \leq x, \frac{M_{n2} - d_n}{c_n} \leq y\right)$$

$$= P\left(M_{n1} \leq a_n x + b_n, M_{n2} \leq c_n y + d_n\right)$$

$$= P\left(\max(X_1, \dots, X_n) \leq a_n x + b_n, \max(Y_1, \dots, Y_n) \leq c_n y + d_n\right)$$

$$= P\left(X_1 \leq a_n x + b_n, \dots, X_n \leq a_n x + b_n, Y_1 \leq c_n y + d_n, \dots, Y_n \leq c_n y + d_n\right)$$

$$= P\left(X_1 \leq a_n x + b_n, Y_1 \leq c_n y + d_n, \vdots, X_n \leq a_n x + b_n, Y_n \leq c_n y + d_n\right)$$

by indep

$$= P(X_1 \leq a_n x + b_n, Y_1 \leq c_n y + d_n)$$

$$\vdots$$
$$P(X_n \leq a_n x + b_n, Y_n \leq c_n y + d_n)$$

$$= F(a_n x + b_n, c_n y + d_n)$$

⋮

$$F(a_n x + b_n, c_n y + d_n)$$

$$= [F(a_n x + b_n, c_n y + d_n)]^n$$

That is,

$$P\left(\frac{M_{n1} - b_n}{a_n} \leq x, \frac{M_{n2} - d_n}{c_n} \leq y\right)$$

$$= [F(a_n x + b_n, c_n y + d_n)]^n \quad (*)$$

What is the limit of (\*) as  $n \rightarrow \infty$ ? If the limit say  $G(x, y)$  exists then it can take an uncountably infinite different forms.

Suppose a non-degenerate  $G(x, y)$  exists. How to find the form of  $G(x, y)$ ?

- i) Find the marginal CDFs  $F_X$  and  $F_Y$  of  $F$ .
- ii) determine the max domain of attraction of  $F_X$  and  $F_Y$
- iii) if  $F_X$  and  $F_Y$  belong to Gumbel domain take

$$a_n = \gamma (F_X^{-1} (1 - \frac{1}{n}))$$

$$b_n = F_X^{-1} (1 - \frac{1}{n})$$

$$c_n = \gamma (F_Y^{-1} (1 - \frac{1}{n}))$$

$$d_n = F_Y^{-1} (1 - \frac{1}{n})$$

if  $F_X$  and  $F_Y$  belong to Frechet domain take

$$a_n = F_X^{-1} (1 - \frac{1}{n}), \quad b_n = 0$$

$$c_n = F_Y^{-1} (1 - \frac{1}{n}), \quad d_n = 0$$

if  $F_X$  and  $F_Y$  belong to Weibull domain take

$$a_n = w(F_X) - F_X^{-1}\left(1 - \frac{1}{n}\right), \quad b_n = w(F_X)$$

$$c_n = w(F_Y) - F_Y^{-1}\left(1 - \frac{1}{n}\right), \quad d_n = w(F_Y).$$

iv) Take the limit of

$$\left[ F(a_n x + b_n, c_n y + d_n) \right]^n$$

as  $n \rightarrow \infty$ .

Ex

$$F(x, y) = [1 + e^{-x} + e^{-y} + (1-a)e^{-x-y}]^{-1},$$

$x > 0, y > 0$

Find  $G$ ?

$$\begin{aligned} \text{i) } F_X(x) &= F(x, \infty) \\ &= [1 + e^{-x} + e^{-\infty} + (1-a)e^{-x-\infty}]^{-1} \\ &= [1 + e^{-x}]^{-1}, \end{aligned}$$

$$\begin{aligned} F_Y(y) &= F(\infty, y) \\ &= [1 + e^{-\infty} + e^{-y} + (1-a)e^{-\infty-y}]^{-1} \\ &= [1 + e^{-y}]^{-1}. \end{aligned}$$

$$\begin{aligned} \text{ii) } F_X(x) = 1 &\Rightarrow x = +\infty \Rightarrow w(F_X) = +\infty \\ F_Y(y) = 1 &\Rightarrow y = +\infty \Rightarrow w(F_Y) = +\infty \end{aligned}$$

$$\begin{aligned} &\lim_{t \rightarrow \infty} \frac{1 - F_X(t + x \delta(t))}{1 - F_X(t)} \\ &= \lim_{t \rightarrow \infty} \frac{1 - [1 - e^{-t - x \delta(t)}]^{-1}}{1 - [1 - e^{-t}]^{-1}} \end{aligned}$$

$$= \lim_{t \rightarrow \infty} \frac{x - [x - (-1)e^{-t - x\delta(t)}]}{x - [x - (-1)e^{-t}]}$$

$$= \lim_{t \rightarrow \infty} e^{-x\delta(t)}$$

$$= e^{-x} \quad \text{if} \quad \delta(t) \equiv 1$$

$F_X$  belongs to Gumbel domain

Similarly,  $F_Y$  " " " "

**LECTURE**

**1 DECEMBER**

**9:00-10:00AM**

**MATH3/4/68181**



# ANNOUNCEMENTS

1) the syllabus for Math 38181 will be completed today.

the syllabus for Math 4168181 will be completed on Thurs 7 Dec.

2) Revision classes

-	Mon	4	Dec	12-1	Chem G53
-	Tues	5	Dec	9-10	Rutherford
-	"	"	"	10-11	Hum Bridge
-	Fri	8	Dec	9-10	Roscoe B
-	Mon	11	Dec	12-1	Chem G53
-	Tues	12	Dec	9-10	Rutherford
-	"	"	"	10-11	Hum Bridge
-	Thurs	14	Dec	12-1	ATB G205
					[ Levels 4 & 6 only ]
-	Fri	15	Dec	9-10	Roscoe B

- 3) Check your email later today  
for what to be expected in the  
final exam in Jan 2018.
- 4) Bonus question — no bonus question  
this semester.
- 5) UEQ (Unit Evaluation Questionnaire)  
will go on-line on Mon 4 Dec.

Income

Moteling

$X$  = True income

$Z$  = Reported income

$X$  is not observable

$Z$  is observable

i) Under reporting

eg  $X = £ 50000.22$

$Z = £ 50000$

under reported if  $Z < X$

ii) Over reporting

eg  $X = £ 39999.51$

$Z = £ 40000$

Over reporting if  $Z > X$ .

What is the distribution of  $X$ ?

i) Under reporting

$$Z = XY, \quad Y \in (0, 1)$$

Assume  $Y$  has the CDF

$$F_Y(y) = y^c, \quad 0 < y < 1$$

Assume  $Z$  has the Pareto distribution with CDF

$$F_Z(z) = 1 - \left(\frac{k}{z}\right)^a, \quad z \geq k.$$

[Pareto is the first distribution used to model income data]

Assume also  $Z$  and  $Y$  are indep RVs.

Then  $X$  is Pareto distributed if and only if  $Z$  is also Pareto distributed.

**Proof** Suppose  $X$  is Pareto distributed  
with CDF

$$F_X(x) = 1 - \left(\frac{L}{x}\right)^b, \quad x \geq L.$$

Then

$$F_Z(z) = P(Z \leq z)$$

$$= P(XY \leq z)$$

$$= P\left(X \leq \frac{z}{Y}\right)$$

$$= \int_0^1 F_X\left(\frac{z}{y}\right) f_Y(y) dy$$

[total prob rule]

$$= \int_0^1 \left[1 - \left(\frac{L}{z} y\right)^b\right] c y^{c-1} dy$$

$$= c \int_0^1 y^{c-1} dy - \frac{cL^b}{z^b} \int_0^1 y^{b+c-1} dy$$

$$= 1 - \frac{cL^b}{z^b} \left[ \frac{y^{b+c}}{b+c} \right]_0^1$$

$$= 1 - \frac{cL^b}{(b+c)z^b} = 1 - \frac{\left[\frac{c}{b+c}\right]^{\frac{1}{b}} L}{z^b}^b$$

$$= 1 - \frac{K^b}{z^b}, \quad \text{Pareto CDF}$$

Suppose  $Z$  is Pareto distributed with CDF

$$F_Z(z) = 1 - \left(\frac{k}{z}\right)^a, \quad z \geq k.$$

Then  $F_X(x) = P(X \leq x)$

$$= P\left(\frac{Z}{Y} \leq x\right)$$

$$= P(Z \leq xY)$$

$$= \int_0^1 F_Z(xy) f_Y(y) dy$$

[total prob rule]

$$= \int_0^1 \left[1 - \left(\frac{k}{xy}\right)^a\right] c y^{c-1} dy$$

$$= c \int_0^1 y^{c-1} dy - \frac{ck^a}{x^a} \int_0^1 y^{c-a-1} dy$$

$$= 1 - \frac{ck^a}{x^a} \left[ \frac{y^{c-a}}{c-a} \right]_0^1$$

$$= 1 - \frac{ck^a}{x^a} \cdot \frac{1}{c-a}$$

$$= 1 - \frac{\left[\frac{ck}{(c-a)^{1/a}}\right]^a}{x^a} = 1 - \left(\frac{L}{x}\right)^a, \quad \text{Pareto CDF}$$

ii) Over reporting

$$Z = \frac{X}{Y}, \quad Y \in (0, 1)$$

Assume  $Y$  has the CDF

$$F_Y(y) = y^c, \quad 0 < y < 1.$$

Assume also  $Y$  and  $Z$  are indep RVs

Then,  $X$  is Pareto distributed if and only if  $Z$  is also Pareto distributed

Proof

Assume  $X$  is Pareto distributed

with CDF

$$F_X(x) = 1 - \left(\frac{k}{x}\right)^a, \quad x \geq k.$$

Then

$$F_Z(z) = P(Z \leq z)$$

$$= P\left(\frac{X}{Y} \leq z\right) = P(X \leq Yz)$$

$$= \int_0^1 F_X(yz) f_Y(y) dy$$

[total prob rule]

$$= \int_0^1 \left[1 - \left(\frac{k}{yz}\right)^a\right] c y^{c-1} dy$$

$$= c \int_0^1 y^{c-1} dy - \frac{c k^a}{z^a} \int_0^1 y^{a+c-1} dy$$

$$= 1 - \frac{c k^a}{z^a} \left[ \frac{y^{a+c}}{a+c} \right]_0^1$$

$$= 1 - \frac{c k^a}{z^a} \cdot \frac{1}{(a+c)}$$

$$= 1 - \frac{\left[ \frac{c^{\frac{1}{a}} k}{(a+c)^{1/a}} \right]^a}{z^a} = 1 - \left(\frac{L}{z}\right)^a, \quad \text{Pareto CDF}$$



Assume  $Z$  is Pareto distributed  
with CDF

$$F_Z(z) = 1 - \left(\frac{L}{z}\right)^b, \quad z \geq L$$

Prove that  $X$  is also. Homework

# **REVISION CLASS**

**4 DECEMBER**

**12:00-13:00PM**

**MATH3/4/68181**

# REVISION

# CLASS

Q4, 2016/17 Exam

(a)

$$\text{VaR}_p(x) = F^{-1}(p)$$

$$\text{ES}_p(x) = \frac{1}{p} \int_0^p F^{-1}(t) dt$$

(b) (i)

$$F(x) = \int_a^x \frac{3y^2}{2a^3} dy$$

$$= \frac{3}{2a^3} \left[ \frac{y^3}{3} \right]_a^x$$

$$= \frac{3}{2a^3} \left[ \frac{x^3}{3} - \frac{(-a)^3}{3} \right]$$

$$= \frac{x^3 + a^3}{2a^3}$$

(ii)

$$F(x) = p$$

$$\Rightarrow \frac{x^3 + a^3}{2a^3} = p$$

$$\Rightarrow x^3 = (2p-1)a^3$$

$$\Rightarrow x = (2p-1)^{\frac{1}{3}} a$$

$$\Rightarrow \text{VaR}_p(x) = (2p-1)^{\frac{1}{3}} a$$

$$(iii) \quad E S_p(X) = \frac{1}{p} \int_0^p (2t-1)^{\frac{1}{3}} a \, dt$$

$$= \frac{a}{p} \left[ \frac{(2t-1)^{\frac{4}{3}}}{2 \cdot (\frac{4}{3})} \right]_0^p$$

$$= \frac{3a}{8p} \left[ (2p-1)^{\frac{4}{3}} - 1 \right]$$

(c) (i)

$$L(a) = \prod_{i=1}^n \left[ \frac{3x_i^2}{2a^3} I\{-a < x_i < a\} \right]$$

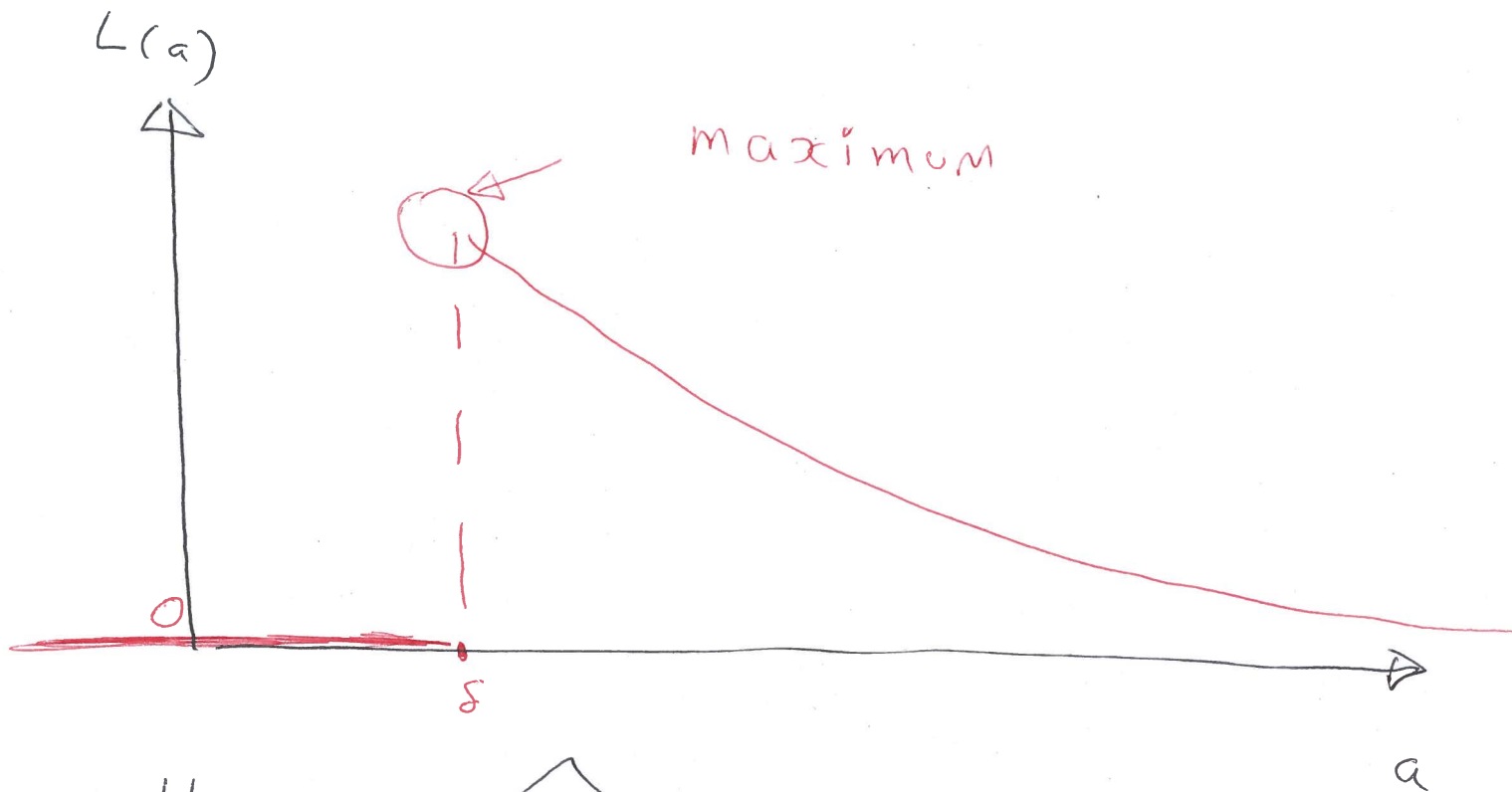
$$= \frac{3^n}{2^n a^{3n}} \left[ \prod_{i=1}^n I\{-a < x_i < a\} \right]$$

Math 20802

$$= \frac{3^n}{2^n a^{3n}} I \left\{ \max(x_1, \dots, x_n) < a \right. \\ \left. \text{and } \min(x_1, \dots, x_n) > -a \right\}$$

$$= \frac{3^n}{2^n a^{3n}} I \left\{ a > \max(x_1, \dots, x_n) \right. \\ \left. \text{and } a > -\min(x_1, \dots, x_n) \right\}$$

$$= \frac{3^n}{2^n a^{3n}} I \left\{ a > \max \left[ \max(x_1, \dots, x_n), \right. \right. \\ \left. \left. -\min(x_1, \dots, x_n) \right] \right\}$$



Hence,  $\hat{a} = \delta$ .

$$(iii) \quad \widehat{VaR}_p(x) = (2p-1)^{\frac{1}{3}} \delta$$

$$\widehat{ES}_p(x) = \frac{3\delta}{8p} \left[ (2p-1)^{\frac{4}{3}} - 1 \right]$$

c(iv)

$$Z = \max \left[ \max (X_1, \dots, X_n), \right. \\ \left. - \min (X_1, \dots, X_n) \right]$$

$$F_Z(z) = P(Z \leq z)$$

$$= P \left[ \max \left[ \max (X_1, \dots, X_n), \right. \right. \\ \left. \left. - \min (X_1, \dots, X_n) \right] \leq z \right]$$

$$= P \left[ \max (X_1, \dots, X_n) \leq z, \right. \\ \left. - \min (X_1, \dots, X_n) \leq z \right]$$

$$= P \left[ X_1 \leq z, \dots, X_n \leq z, \right. \\ \left. \min (X_1, \dots, X_n) \geq -z \right]$$

$$= P \left[ X_1 \leq z, \dots, X_n \leq z, \right. \\ \left. X_1 \geq -z, \dots, X_n \geq -z \right]$$

$$= P \left[ -z \leq X_1 \leq z, \dots, -z \leq X_n \leq z \right]$$

indep

$$= P[-z \leq X_1 \leq z] \dots P[-z \leq X_n \leq z]$$

$$= \{ P[-z \leq X \leq z] \}^n$$

$$= \{ F(z) - F(-z) \}^n$$

$$b(i) = \left\{ \frac{z^3 + a^3}{2a^3} - \frac{(-z)^3 + a^3}{2a^3} \right\}^n$$

$$= \left[ \frac{z^3}{a^3} \right]^n$$

2015 / 16 Exam, Q3 (d)

$$f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$$

$$w(F) = +\infty$$

(I) :

$$\lim_{t \rightarrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$\stackrel{\text{LH}}{=} \lim_{t \rightarrow \infty} \frac{-f(t + x\gamma(t)) \cdot (1 + x\gamma'(t))}{-f(t)}$$

$$= \lim_{t \rightarrow \infty} \frac{\cancel{\frac{1}{2}} e^{-|t + x\gamma(t)|} (1 + x\gamma'(t))}{\cancel{\frac{1}{2}} e^{-|t|}}$$

$$= \lim_{t \rightarrow \infty} \frac{e^{-(t + x\gamma(t))} (1 + x\gamma'(t))}{e^{-t}}$$

$$= \lim_{t \rightarrow \infty} e^{-x\gamma(t)} (1 + x\gamma'(t))$$

Choose  $\gamma(t) \equiv 1$   
 $\gamma'(t) = 0$

$= e^{-x} \Rightarrow F$  belongs to the Gumbel max domain.



**REVISION CLASS**

**5 DECEMBER**

**9:00-10:00AM**

**MATH3/4/68181**

# REVISION CLASS

( TUES 5 DEC 9-10 AM)

Exam 2012/2013, Q6

(a) 
$$\text{VaR}_p(X) = F^{-1}(p)$$

$$\text{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(t) dt$$

(b)

$$F(x) = \frac{x-a}{b-a}$$

$$F(x) = p$$

$$\Rightarrow \frac{x-a}{b-a} = p$$

$$\Rightarrow x = a + p(b-a)$$

$$\Rightarrow \text{VaR}_p(X) = a + p(b-a)$$

$$\text{ES}_p(X) = \frac{1}{p} \int_0^p [a + t(b-a)] dt$$

$$= \frac{1}{p} \left[ a \cdot t + \frac{t^2}{2} (b-a) \right]_0^p$$

$$= a + \frac{p}{2} (b-a)$$

(c) (i) - (iii)

$$L(a, b) = \prod_{i=1}^n \left[ \frac{1}{b-a} I \{ a \leq x_i \leq b \} \right]$$

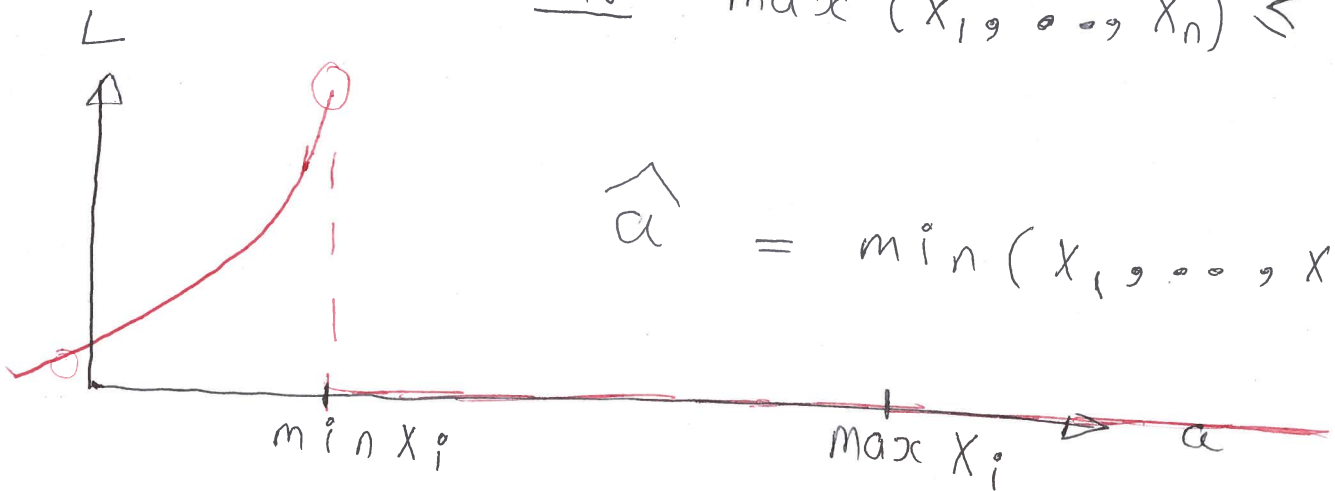
$$= \frac{1}{(b-a)^n} \prod_{i=1}^n I \{ a \leq x_i \leq b \}$$

Math 20802

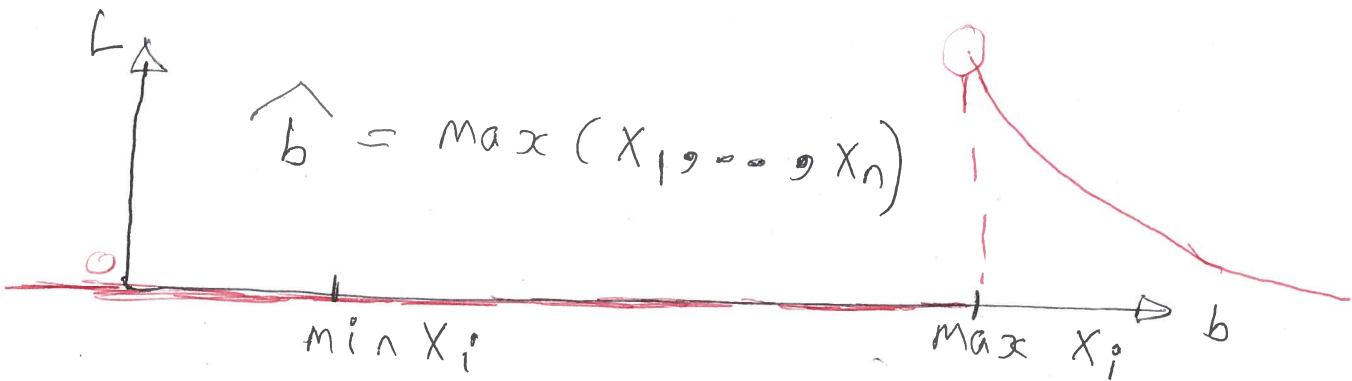
$$= \frac{1}{(b-a)^n} I \{ \min(x_1, \dots, x_n) \geq a$$

$$\text{and } \max(x_1, \dots, x_n) \leq b \}$$

$$\hat{a} = \min(x_1, \dots, x_n)$$



$$\hat{b} = \max(x_1, \dots, x_n)$$



(iv)

$$\widehat{\text{VaR}}_p(X) = \min(X_1, \dots, X_n)$$

$$+ p \left[ \max(X_1, \dots, X_n) - \min(X_1, \dots, X_n) \right]$$

$$\widehat{\text{ES}}_p(X) = \min(X_1, \dots, X_n)$$

$$+ \frac{p}{2} \left[ \max(X_1, \dots, X_n) - \min(X_1, \dots, X_n) \right]$$

$$(v) \quad E[\widehat{\text{VaR}}_p(X)] \neq \text{VaR}_p(X)$$

$$(vi) \quad E[\widehat{\text{ES}}_p(X)] \neq \text{ES}_p(X)$$

2013/14 Exam Q6

(i)  $F(x) = 1 - e^{-(1+\lambda x)^\alpha}$ ,  $x > 0$

$W(F) = +\infty$

(I)

$\lim_{t \rightarrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$

$= \lim_{t \rightarrow \infty} \frac{1 - \{1 - e^{-(1+\lambda t + \lambda x\gamma(t))^\alpha}\}}{1 - \{1 - e^{-(1+\lambda t)^\alpha}\}}$

$= \lim_{t \rightarrow \infty} e^{(1+\lambda t)^\alpha - (1+\lambda t + \lambda x\gamma(t))^\alpha}$

$= \lim_{t \rightarrow \infty} e^{(1+\lambda t)^\alpha - (1+\lambda t)^\alpha \left[1 + \frac{\lambda x\gamma(t)}{1+\lambda t}\right]^\alpha}$

$= \lim_{t \rightarrow \infty} e^{(1+\lambda t)^\alpha \left\{1 - \left[1 + \frac{\lambda x\gamma(t)}{1+\lambda t}\right]^\alpha\right\}}$

$(1+x)^\alpha \approx 1 + \alpha x$

$= \lim_{t \rightarrow \infty} e^{(1+\lambda t)^\alpha \left\{1 - \left[1 + \frac{\alpha \lambda x\gamma(t)}{1+\lambda t}\right]\right\}}$

$= \lim_{t \rightarrow \infty} e^{- (1+\lambda t)^{\alpha-1} \cdot \alpha \lambda x \gamma(t)}$

$\gamma(t) = \frac{1}{\alpha \lambda} (1+\lambda t)^{1-\alpha}$

$= e^{-x} \Rightarrow F$  belongs to Gumbel domain

2013 / 14 Exam Q6

(v)  $F(x) = 1 - q (x+1)^a$ ,  $0 < q < 1$ ,  $a > 1$   
 $x = 0, 1, 2, \dots$

$w(F) = +\infty$

$\lim_{k \rightarrow \infty} \frac{P(X=k)}{1-F(k-1)}$

$= \lim_{k \rightarrow \infty} \frac{F(k) - F(k-1)}{1-F(k-1)}$

$= \lim_{k \rightarrow \infty} \frac{1 - q (k+1)^a - [1 - q k^a]}{1 - [1 - q k^a]}$

$= \lim_{k \rightarrow \infty} \frac{q k^a - q (k+1)^a}{q k^a}$

$= \lim_{k \rightarrow \infty} 1 - q (k+1)^a - k^a$

$= \lim_{k \rightarrow \infty} 1 - q k^a \left(1 + \frac{1}{k}\right)^a - k^a$

$= \lim_{k \rightarrow \infty} 1 - q k^a \left[\left(1 + \frac{1}{k}\right)^a - 1\right]$

$$\stackrel{=} {=} \lim_{k \rightarrow \infty} 1 - q k^a \left[ \cancel{1} + \frac{a}{k} + \frac{a(a-1)}{2k^2} + \dots - \cancel{1} \right]$$

Binomial Exp

$$= \lim_{k \rightarrow \infty} 1 - q$$

$$1 - q k^{a-1} + \dots + \infty$$

$$0$$

$$= 1 \neq 0$$

Hence ETT does not hold.

**REVISION CLASS**

**5 DECEMBER**

**10:00-11:00AM**

**MATH3/4/68181**



# REVISION CLASS

(TUES 5 DEC 10-11 AM)

EXAM 2016/17 Q5

(a)

$$\bar{F}(x_1, \dots, x_k)$$

$$= \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i\right)^{-\alpha}$$

$$f(x_1, \dots, x_k) = \frac{(-1)^k \partial^k}{\partial x_1 \dots \partial x_k} \bar{F}(x_1, \dots, x_k)$$

$$\frac{\partial}{\partial x_1} \bar{F} = (-\alpha) \cdot \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i\right)^{-\alpha-1} \left(\frac{1}{\theta}\right)$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} \bar{F} = (-\alpha)(-\alpha-1) \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i\right)^{-\alpha-2} \left(\frac{1}{\theta^2}\right)$$

$$\frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} \bar{F} = (-\alpha)(-\alpha-1)(-\alpha-2) \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i\right)^{-\alpha-3} \left(\frac{1}{\theta^3}\right)$$

⋮

$$\frac{\partial^k}{\partial x_1 \dots \partial x_k} \bar{F} = (-\alpha)(-\alpha-1) \dots (-\alpha-k+1) \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i\right)^{-\alpha-k} \left(\frac{1}{\theta^k}\right)$$

$$f(x_1, \dots, x_k) = \frac{\alpha(\alpha+1) \dots (\alpha+k-1)}{\theta^k} \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i\right)^{-\alpha-k}$$

$$\begin{aligned}
 (b) \quad f_S(s) &= \int \int \dots \int_{x_1 + \dots + x_k = s} f(x_1, \dots, x_k) dx_k \dots dx_2 dx_1 \\
 &= \int \int \dots \int_{x_1 + \dots + x_k = s} \frac{\alpha(\alpha+1) \dots (\alpha+k-1)}{\theta^k} \left(1 + \frac{s}{\theta}\right)^{-\alpha-k} dx_k \dots dx_2 dx_1
 \end{aligned}$$

$$= \frac{\alpha(\alpha+1) \dots (\alpha+k-1)}{\theta^k} \left(1 + \frac{s}{\theta}\right)^{-\alpha-k}$$

$$\int \int \dots \int_{x_1 + \dots + x_k = s} 1 \cdot dx_k \dots dx_2 dx_1 = \frac{s^k}{k!}$$

$$= \frac{\alpha(\alpha+1) \dots (\alpha+k-1)}{\theta^k} \left(1 + \frac{s}{\theta}\right)^{-\alpha-k} \frac{s^k}{k!}$$

$$\begin{aligned}
 (c) \quad F_S(s) &= \int_0^s f_{S_1}(t) dt \\
 &= \frac{\alpha(\alpha+1) \dots (\alpha+k-1)}{\theta^k k!} \int_0^s \left(1 + \frac{t}{\theta}\right)^{-\alpha-k} t^k dt
 \end{aligned}$$

$$\begin{aligned}
 \text{Set } u &= 1 - \left(1 + \frac{t}{\theta}\right)^{-1} \\
 \Rightarrow 1 + \frac{t}{\theta} &= \frac{1}{1-u} \\
 \Rightarrow \frac{t}{\theta} &= \frac{u}{1-u} \\
 \Rightarrow t &= \theta \frac{u}{1-u} \\
 \Rightarrow \frac{dt}{du} &= \theta \frac{1}{(1-u)^2}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\alpha(\alpha+1) \dots (\alpha+k-1)}{\theta^k k!} \int_0^{\frac{s}{s+\theta}} (1-u)^{\alpha+k} \frac{\theta}{(1-u)^2} u^k du \\
 &= \frac{\theta \alpha(\alpha+1) \dots (\alpha+k-1)}{k!} \int_0^{\frac{s}{s+\theta}} (1-u)^{\alpha-2} u^k du
 \end{aligned}$$

$$= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} B_{\frac{s}{s+0}}(k+1, \alpha-1)$$

**LECTURE**

**7 DECEMBER**

**12:00-13:00PM**

**MATH4/68181**

# Joint distribution of $(M_{n1}, M_{n2})$

$$P\left(\frac{M_{n1} - b_n}{a_n} \leq x, \frac{M_{n2} - d_n}{c_n} \leq y\right)$$

$$= P\left(M_{n1} \leq a_n x + b_n, M_{n2} \leq c_n y + d_n\right)$$

$$= P\left(\max(X_1, \dots, X_n) \leq a_n x + b_n, \max(Y_1, \dots, Y_n) \leq c_n y + d_n\right)$$

$$= P\left(X_1 \leq a_n x + b_n, \dots, X_n \leq a_n x + b_n, Y_1 \leq c_n y + d_n, \dots, Y_n \leq c_n y + d_n\right)$$

$$= P\left(X_1 \leq a_n x + b_n, Y_1 \leq c_n y + d_n, \vdots, X_n \leq a_n x + b_n, Y_n \leq c_n y + d_n\right)$$

by indep

$$= P(X_1 \leq a_n x + b_n, Y_1 \leq c_n y + d_n)$$

$$\vdots$$
$$P(X_n \leq a_n x + b_n, Y_n \leq c_n y + d_n)$$

Suppose a non-degenerate  $G(x, y)$  exists. How to find the form of  $G(x, y)$ ?

- i) Find the marginal CDFs  $F_X$  and  $F_Y$  of  $F$ .
- ii) determine the max domain of attraction of  $F_X$  and  $F_Y$
- iii) if  $F_X$  and  $F_Y$  belong to Gumbel domain take

$$a_n = \gamma (F_X^{-1} (1 - \frac{1}{n}))$$

$$b_n = F_X^{-1} (1 - \frac{1}{n})$$

$$c_n = \gamma (F_Y^{-1} (1 - \frac{1}{n}))$$

$$d_n = F_Y^{-1} (1 - \frac{1}{n})$$

if  $F_X$  and  $F_Y$  belong to Frechet domain take

$$a_n = F_X^{-1} (1 - \frac{1}{n}), \quad b_n = 0$$

$$c_n = F_Y^{-1} (1 - \frac{1}{n}), \quad d_n = 0$$

$$= F(a_n x + b_n, c_n y + d_n)$$

⋮

$$F(a_n x + b_n, c_n y + d_n)$$

$$= [F(a_n x + b_n, c_n y + d_n)]^n$$

That is,

$$P\left(\frac{M_{n1} - b_n}{a_n} \leq x, \frac{M_{n2} - d_n}{c_n} \leq y\right)$$

$$= [F(a_n x + b_n, c_n y + d_n)]^n \quad (*)$$

What is the limit of (\*) as  $n \rightarrow \infty$ ? If the limit say  $G(x, y)$  exists then it can take an uncountably infinite different forms.

Ex

$$F(x, y) = [1 + e^{-x} + e^{-y} + (1-a)e^{-x-y}]^{-1},$$

$x > 0, y > 0$

Find  $G$ ?

i)  $F_X(x) = F(x, \infty)$

$$= [1 + e^{-x} + e^{-\infty} + (1-a)e^{-x-\infty}]^{-1}$$

$$= [1 + e^{-x}]^{-1},$$

$$F_Y(y) = F(\infty, y)$$

$$= [1 + e^{-\infty} + e^{-y} + (1-a)e^{-\infty-y}]^{-1}$$

$$= [1 + e^{-y}]^{-1}.$$

ii)  $F_X(x) = 1 \Rightarrow x = +\infty \Rightarrow w(F_X) = +\infty$

$$F_Y(y) = 1 \Rightarrow y = +\infty \Rightarrow w(F_Y) = +\infty$$

$$\lim_{t \rightarrow \infty} \frac{1 - F_X(t + x\delta(t))}{1 - F_X(t)}$$

$$= \lim_{t \rightarrow \infty} \frac{1 - [1 - e^{-t-x\delta(t)}]^{-1}}{1 - [1 - e^{-t}]^{-1}}$$



$$= \lim_{t \rightarrow \infty} \frac{\cancel{x} - [\cancel{x} - (-1) e^{-t - x\delta(t)}]}{\cancel{x} - [\cancel{x} - (-1) e^{-t}]}$$

$$= \lim_{t \rightarrow \infty} e^{-x\delta(t)}$$

$$= e^{-x} \quad \text{if} \quad \delta(t) \equiv 1$$

$F_X$  belongs to Gumbel domain

Similarly,  $F_Y$  " " " " " "

$$\text{iii) } a_n = \delta(F_X^{-1}(1 - \frac{1}{n})) = 1$$

$$b_n = F_X^{-1}(1 - \frac{1}{n})$$

$$F_X(x) = [1 + e^{-x}]^{-1} = y$$

$$\Rightarrow 1 + e^{-x} = \frac{1}{y}$$

$$\Rightarrow e^{-x} = \frac{1-y}{y}$$

$$\Rightarrow x = -\log\left(\frac{1-y}{y}\right)$$

$$= -\log\left(\frac{1 - (1 - \frac{1}{n})}{1 - \frac{1}{n}}\right)$$

$$= -\log\left(\frac{\frac{1}{n}}{1 - \frac{1}{n}}\right)$$

$$= \log(n-1)$$

Similarly,  $c_n = 1$ ,  $d_n = \log(n-1)$

$$iv) \lim F^n (a_n x + b_n, c_n y + d_n)$$

$$= \lim \left[ 1 + e^{-a_n x - b_n} + e^{-c_n y - d_n} + (1-a) e^{-a_n x - b_n - c_n y - d_n} \right]^n$$

$$= \lim_{n \rightarrow \infty} \left[ 1 + e^{-x - \log(n-1)} + e^{-y - \log(n-1)} + (1-a) e^{-x - y - 2 \log(n-1)} \right]^n$$

$$= \lim_{n \rightarrow \infty} \left[ 1 + \frac{e^{-x}}{n-1} + \frac{e^{-y}}{n-1} + (1-a) \frac{e^{-x-y}}{(n-1)^2} \right]^n$$

$$= \lim_{n \rightarrow \infty} \left[ 1 + \frac{e^{-x}}{n-1} + \frac{e^{-y}}{n-1} \right]^n$$

$$= \lim_{n \rightarrow \infty} \left[ 1 + \frac{e^{-x}}{n} \cdot \frac{n}{n-1} + \frac{e^{-y}}{n} \cdot \frac{n}{n-1} \right]^n$$

$$= \lim_{n \rightarrow \infty} \left[ 1 + \frac{e^{-x} + e^{-y}}{n} \right]^n$$

$$= e^{-[e^{-x} + e^{-y}]} = G(x, y).$$

$\left(1 + \frac{z}{n}\right)^n \rightarrow e^z$   
as  $n \rightarrow \infty$

i) Suppose  $F_X$  and  $F_Y$  belong to Gumbel max domain. In this case,

$$G(x, y) = e^{-\left(e^{-x} + e^{-y}\right) k(y-x)}$$

where  $k(\cdot)$  satisfies

$$\lim_{t \rightarrow \infty} k(t) = 1$$

$$\lim_{t \rightarrow -\infty} k(t) = 1$$

$$\frac{d}{dt} \left[ (1 + e^{-t}) k(t) \right] \leq 0 \quad \forall t$$

$$\frac{d}{dt} \left[ (1 + e^t) k(t) \right] \geq 0 \quad \forall t$$

$$\left(1 + e^{-t}\right) \frac{d^2 k(t)}{dt^2} + (1 - e^{-t}) \frac{dk(t)}{dt} \geq 0$$

$\forall t.$

ii) Suppose  $F_x$  and  $F_y$  belong to Gumbel max domains. In this case,

$$G(x, y) = e^{-\int_0^1 \min[f_1(s)e^{-x}, f_2(s)e^{-y}] ds}$$

where  $f_1(s) \geq 0$  and  $f_2(s) \geq 0$  are integrable functions such that

$$\int_0^1 f_1(t) dt = 1$$

and

$$\int_0^1 f_2(t) dt = 1.$$

iii) Suppose  $F_X$  and  $F_Y$  belong to the Fréchet domain of attraction.

In this case,

$$G(x, y) = e^{-\left(\frac{1}{x} + \frac{1}{y}\right) A\left(\frac{x}{x+y}\right)}$$

where  $A(\cdot)$  is a positive function on  $[0, 1]$  such that

$$A(0) = A(1) = 1$$

$$\max(w, 1-w) \leq A(w) \leq 1 \quad \forall w$$

$A(\cdot)$  is convex

iv) Suppose

$$G_X(x) = F_X(x) = 1 - e^{-x}$$

$$G_Y(y) = F_Y(y) = 1 - e^{-y}.$$

Then

$$\bar{G}(x, y) = e^{-(x+y)} A\left(\frac{y}{x+y}\right)$$

where  $A(\cdot)$  is a positive function on  $[0, 1]$  such that

$$A(0) = A(1) = 1$$

$$\max(w, 1-w) \leq A(w) \leq 1 \quad \forall w$$

$A(\cdot)$  is convex

$$1 - G(x, y) = e^{-x} + e^{-y} - \bar{G}(x, y)$$

$$\Rightarrow \bar{G}(x, y) = e^{-x} + e^{-y} - 1 + G(x, y)$$

$$\Rightarrow G(x, y) = 1 - e^{-x} - e^{-y} + \bar{G}(x, y)$$

v) Suppose  $F_X$  and  $F_Y$  belong to the Weibull max domain.

No form for  $G$  is known in this case.

**REVISION CLASS**

**8 DECEMBER**

**9:00-10:00AM**

**MATH3/4/68181**



REVISION

CLASS

(FRI 8 DEC 9-10 AM)

Exam 2014 (2015) Q6 (b)

$$F(x) = 1 - (1 - x^b)^a, \quad 0 < x < 1$$

$$F(x) = 1$$

$$\Rightarrow 1 - (1 - x^b)^a = 1$$

$$\Rightarrow (1 - x^b)^a = 0$$

$$\Rightarrow 1 - x^b = 0$$

$$\Rightarrow x^b = 1 \Rightarrow x = 1$$

$$\Rightarrow w(F) = 1$$

$$\text{(III)} : \lim_{t \rightarrow 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)}$$

$$= \lim_{t \rightarrow 0} \frac{1 - [1 - (1 - (1 - tx)^b)^a]}{1 - [1 - (1 - t)^b]^a}$$

$$= \lim_{t \rightarrow 0} \frac{(1 - (1 - tx)^b)^a}{(1 - (1 - t)^b)^a}$$

$$= \lim_{t \rightarrow 0} \frac{(1 - (1 - b \cdot tx))^a}{(1 - (1 - b \cdot t))^a}$$

$$= \lim_{t \rightarrow 0} \frac{(btx)^a}{(bt)^a} = x^a \Rightarrow$$

F belongs to Weibull max domain

Exam 2014/15 Q6 (e)

$$F(x) = e^{-e^{-x}}, \quad -\infty < x < \infty$$

$$F(x) = 1$$

$$\Rightarrow e^{-e^{-x}} = 1$$

$$\Rightarrow -e^{-x} = 0$$

$$\Rightarrow e^{-x} = 0$$

$$\Rightarrow -x = \log 0 = -\infty$$

$$\Rightarrow x = +\infty \Rightarrow w(F) = +\infty$$

$$\begin{aligned} (I): \quad & \lim_{t \rightarrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} \\ &= \lim_{t \rightarrow \infty} \frac{1 - e^{-e^{-t - x\gamma(t)}}}{1 - e^{-e^{-t}}} \\ &= \lim_{t \rightarrow \infty} \frac{\cancel{x} - [\cancel{x} - e^{-t - x\gamma(t)}]}{\cancel{x} - [\cancel{x} - e^{-t}]} \\ &= \lim_{t \rightarrow \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}} \\ &= \lim_{t \rightarrow \infty} e^{-x\gamma(t)} = e^{-x} \end{aligned}$$

if  $\gamma(t) \equiv 1$

$F$  belongs to the Gumbel max domain

Exam 2014/15

Q8 (c)

$$F(y) = 1 - \left(\frac{k}{y}\right)^{a\alpha} = p$$

$$\Rightarrow \left(\frac{k}{y}\right)^{a\alpha} = 1 - p$$

$$\Rightarrow y = k (1 - p)^{-\frac{1}{a\alpha}}$$

$$E S_p(Y) = \frac{k}{p} \int_0^p (1 - t)^{-\frac{1}{a\alpha}} dt$$

$$= \frac{k}{p} \cdot \frac{1}{(-1)\left(1 - \frac{1}{a\alpha}\right)} \left[ (1 - t)^{1 - \frac{1}{a\alpha}} \right]_0^p$$

$$= \frac{k}{p} \frac{1}{\left(\frac{1}{a\alpha} - 1\right)} \left[ (1 - p)^{1 - \frac{1}{a\alpha}} - 1 \right]$$

(b) (i)

$$\boxed{x > 0}$$

$$F(x) = \int_{-\infty}^x f(y) dy$$

$$= \int_{-\infty}^0 f(y) dy + \int_0^x f(y) dy$$

$$= \int_{-\infty}^0 \frac{1}{2\lambda} e^{-\frac{|y|}{\lambda}} dy + \int_0^x \frac{1}{2\lambda} e^{-\frac{|y|}{\lambda}} dy$$

$$= \frac{1}{2\lambda} \int_{-\infty}^0 e^{-\frac{y}{\lambda}} dy + \frac{1}{2\lambda} \int_0^x e^{-\frac{y}{\lambda}} dy$$

$$= \frac{1}{2\lambda} \left[ \frac{e^{-\frac{y}{\lambda}}}{-\frac{1}{\lambda}} \right]_{-\infty}^0 + \frac{1}{2\lambda} \left[ \frac{e^{-\frac{y}{\lambda}}}{(-\frac{1}{\lambda})} \right]_0^x$$

$$= \frac{1}{2} (1 - 0) + \frac{1}{2} \left( -e^{-\frac{x}{\lambda}} + 1 \right)$$

$$= 1 - \frac{1}{2} e^{-\frac{x}{\lambda}}$$

$$\boxed{x \leq 0}$$

$$F(x) = \int_{-\infty}^x f(y) dy$$

$$= \int_{-\infty}^x \frac{1}{2\lambda} e^{-\frac{|y|}{\lambda}} dy$$

$$= \frac{1}{2\lambda} \int_{-\infty}^x e^{+\frac{y}{\lambda}} dy$$

$$= \frac{1}{2\lambda} \left[ \frac{e^{\frac{y}{\lambda}}}{\frac{1}{\lambda}} \right]_{-\infty}^x$$

$$= \frac{1}{2} \left[ e^{\frac{x}{\lambda}} - 0 \right]$$

$$= \frac{1}{2} e^{\frac{x}{\lambda}}$$

$$F(x) = \begin{cases} 1 - \frac{1}{2} e^{-\frac{x}{\lambda}} & x > 0 \\ \frac{1}{2} e^{\frac{x}{\lambda}} & x \leq 0 \end{cases}$$

$F(x) > \frac{1}{2}$  for  $x > 0$   
 $F(x) \leq \frac{1}{2}$  for  $x \leq 0$

$$F(0) = \frac{1}{2} e^{\frac{0}{\lambda}} = \frac{1}{2}$$

b (ii)

$$F(x) = p$$

$$p > \frac{1}{2}$$

$$1 - \frac{1}{2} e^{-\frac{x}{\lambda}} = p$$

$$\Rightarrow \frac{1}{2} e^{-\frac{x}{\lambda}} = 1 - p$$

$$\Rightarrow e^{-\frac{x}{\lambda}} = 2(1 - p)$$

$$\Rightarrow x = -\lambda \log [2(1 - p)]$$

$$p \leq \frac{1}{2}$$

$$\frac{1}{2} e^{\frac{x}{\lambda}} = p$$

$$\Rightarrow e^{\frac{x}{\lambda}} = 2p$$

$$\Rightarrow x = \lambda \log [2p]$$

$$\text{VaR}_p(x) = \begin{cases} -\lambda \log [2(1 - p)] & p > \frac{1}{2} \\ \lambda \log [2p] & p \leq \frac{1}{2} \end{cases}$$

b (iii)

$$ES_p(X) = \frac{1}{p} \int_0^p \text{VaR}_t(X) dt$$

$$\boxed{p > \frac{1}{2}}$$

$$\begin{aligned} ES_p(X) &= \frac{1}{p} \left( \int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^p \right) \text{VaR}_t(X) dt \\ &= \frac{1}{p} \left[ \int_0^{\frac{1}{2}} \lambda \log(2t) dt \right. \\ &\quad \left. + \int_{\frac{1}{2}}^p (-\lambda) \log[2(1-t)] dt \right] \\ &= \frac{1}{p} \left[ \lambda \int_0^{\frac{1}{2}} [\log 2 + \log t] dt \right. \\ &\quad \left. - \lambda \int_{\frac{1}{2}}^p [\log 2 + \log(1-t)] dt \right] \\ &= \frac{1}{2} \left[ \frac{\lambda}{2} \log 2 + \lambda \int_0^{\frac{1}{2}} \log t dt \right. \\ &\quad \left. - \lambda \left( p - \frac{1}{2} \right) \log 2 \right. \\ &\quad \left. - \lambda \int_{\frac{1}{2}}^p \log(1-t) dt \right] \end{aligned}$$



$$= \frac{1}{2} \left[ \frac{\lambda}{2} \log 2 + \left[ \log t \cdot t \right]_0^{\frac{1}{2}} \right.$$

int  
by  
part

$$- \frac{\lambda}{2}$$

$$- \lambda \left( p - \frac{1}{2} \right) \cdot \log 2$$

$$- \lambda \left[ \log (1-t) \cdot t \right]_{\frac{1}{2}}^p$$

$$- \lambda \int_{\frac{1}{2}}^p \frac{t}{1-t} dt$$

==

**REVISION CLASS**

**11 DECEMBER**

**12:00-13:00PM**

**MATH3/4/68181**

REVISION

CLASS

(MON 11 DEC 12-1 PM)

(a)

$$f(x_1, \dots, x_k)$$

$$= (-1)^k \frac{\partial^k}{\partial x_1 \dots \partial x_k} \bar{F}(x_1, \dots, x_k) \quad (*)$$

$$\frac{\partial}{\partial x_1} \bar{F} = (-\alpha) \cdot \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i\right)^{-\alpha-1} \cdot \frac{1}{\theta}$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} \bar{F} = (-\alpha)(-\alpha-1) \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i\right)^{-\alpha-2} \left(\frac{1}{\theta}\right)^2$$

$$\frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} \bar{F} = (-\alpha)(-\alpha-1)(-\alpha-2) \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i\right)^{-\alpha-3} \cdot \left(\frac{1}{\theta}\right)^3$$

$$\frac{\partial^k}{\partial x_1 \dots \partial x_k} \bar{F} = (-\alpha)(-\alpha-1) \dots (-\alpha-k+1) \cdot \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i\right)^{-\alpha-k} \cdot \left(\frac{1}{\theta}\right)^k$$

By (\*),  $f(x_1, \dots, x_k)$

$$= \alpha(\alpha+1) \dots (\alpha+k-1) \left(1 + \frac{1}{\theta} \sum_{i=1}^k x_i\right)^{-\alpha-k} \left(\frac{1}{\theta}\right)^k$$

(b)

$$f_S(s) = \int \int \dots \int_{x_1 + x_2 + \dots + x_k = s} f(x_1, x_2, \dots, x_k) dx_k \dots dx_2 dx_1$$

$$= \frac{\alpha(\alpha+1) \dots (\alpha+k-1)}{\theta^k}$$

$$\int \int \dots \int_{x_1 + \dots + x_k = s} \left( 1 + \frac{1}{\theta} \sum_{i=1}^k x_i \right)^{-\alpha-k} dx_k \dots dx_2 dx_1$$

$$= \frac{\alpha(\alpha+1) \dots (\alpha+k-1)}{\theta^k} \left( 1 + \frac{s}{\theta} \right)^{-\alpha-k}$$

$$\int \int \dots \int_{x_1 + \dots + x_k = s} 1 \cdot dx_k \dots dx_2 dx_1 = \frac{s^k}{k!}$$

$$= \frac{\alpha(\alpha+1) \dots (\alpha+k-1)}{\theta^k} \left( 1 + \frac{s}{\theta} \right)^{-\alpha-k} \frac{s^k}{k!}$$

(c)

$$F_{\mathcal{J}}(s) = \int_0^s f_{\mathcal{J}'}(t) dt$$

$$= \frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{\theta^k k!} \int_0^s \left(1 + \frac{t}{\theta}\right)^{-\alpha-k} t^k dt$$

$$\begin{aligned} \text{Set } y &= \frac{t}{\theta+t} \\ \Rightarrow y\theta + yt &= t \\ \Rightarrow t &= \frac{y\theta}{1-y} \\ \Rightarrow \frac{dt}{dy} &= \frac{\theta}{(1-y)^2} \end{aligned}$$

$$= \frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{\theta^k k!} \int_0^{\frac{s}{\theta+s}} \left(1 + \frac{y}{1-y}\right)^{-\alpha-k} \frac{y^k \theta^k}{(1-y)^k} \cdot \frac{\theta}{(1-y)^2} dy$$

$$= \frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{\theta^k k!} \theta^{k+1} \int_0^{\frac{s}{\theta+s}} y^k (1-y)^{\alpha-2} dy$$

$$= \frac{\alpha(\alpha+1)\dots(\alpha+k-1) \theta}{k!} \mathcal{B}_{\frac{s}{\theta+s}}(k+1, \alpha-1)$$

Exam 2014/15 Q5 (c)

i) Assume  $G$  belongs to Gumbel max domain, that is there exists  $\gamma(t) > 0$  such that

$$\lim_{t \rightarrow \infty} w(G) \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x} \quad (*)$$

$$\lim_{t \rightarrow \infty} w(F) \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \rightarrow \infty} \frac{w(F)}{w(G)} \frac{1 - \left\{ 1 - \left[ 1 - (G(t + x\gamma(t)))^\alpha \right]^2 \right\}^\alpha}{1 - \left\{ 1 - \left[ 1 - (G(t))^\alpha \right]^2 \right\}^\alpha}$$

$$= \lim_{t \rightarrow \infty} w(G) \frac{1 - \left\{ 1 - \alpha \cdot \left[ 1 - (G(t + x\gamma(t)))^\alpha \right]^2 \right\}}{1 - \left\{ 1 - \alpha \cdot \left[ 1 - (G(t))^\alpha \right]^2 \right\}}$$

$$(1 - z)^\alpha \approx 1 - \alpha \cdot z \text{ as } z \rightarrow 0$$

$$= \lim_{t \rightarrow \infty} w(G) \left[ \frac{1 - (G(t + x\gamma(t)))^\alpha}{1 - (G(t))^\alpha} \right]^2$$

$$= \lim_{t \rightarrow w(\alpha)} \left[ \frac{1 - \left\{ 1 - \left[ 1 - G(t + x\gamma(t)) \right] \right\}^\theta}{1 - \left\{ 1 - \left[ 1 - G(t) \right] \right\}^\theta} \right]^2$$

$$= \lim_{t \rightarrow w(\alpha)} \left[ \frac{1 - \left\{ 1 - \left[ 1 - G(t + x\gamma(t)) \right] \right\}^\theta}{1 - \left\{ 1 - \left[ 1 - G(t) \right] \right\}^\theta} \right]^2$$

$$(1 - z)^\theta \approx 1 - \theta \cdot z \text{ as } z \rightarrow 0$$

$$= \lim_{t \rightarrow w(\alpha)} \left[ \frac{1 - G(t + x\gamma(t))}{1 - G(t)} \right]^2$$

$$\stackrel{\circledast}{=} e^{-2x}$$

So  $F$  belongs to also the Gumbel max domain.



**REVISION CLASS**

**12 DECEMBER**

**9:00-10:00AM**

**MATH3/4/68181**

$$\begin{aligned}
 \overline{F}_{X, Y, Z}(x, y, z) &= 1 - F_X(x) - F_Y(y) \\
 &\quad - F_Z(z) \\
 &\quad + F_{X, Y}(x, y) \\
 &\quad + F_{X, Z}(x, z) \\
 &\quad + F_{Y, Z}(y, z) \quad (*) \\
 &\quad - F_{X, Y, Z}(x, y, z)
 \end{aligned}$$

$$\begin{aligned}
 \overline{F}_{X, Y, Z}(x, y, z) &= 1 - \overline{F}_X(x) - \overline{F}_Y(y) - \overline{F}_Z(z) \\
 &\quad + \overline{F}_{X, Y}(x, y) + \overline{F}_{X, Z}(x, z) \\
 &\quad + \overline{F}_{Y, Z}(y, z) \quad (***) \\
 &\quad - \overline{F}_{X, Y, Z}(x, y, z)
 \end{aligned}$$

REVISION

CLASS

(TUES 12 DEC

9-10 AM)

$$F_{X, Y}(x, y) = P(X \leq x, Y \leq y)$$

$$\bar{F}_{X, Y}(x, y) = P(X > x, Y > y)$$

$$\begin{aligned} \bar{F}_{X, Y}(x, y) &= 1 - F_X(x) \\ &\quad - F_Y(y) \\ &\quad + F_{X, Y}(x, y) \end{aligned}$$

$$\begin{aligned} F_{X, Y}(x, y) &= \bar{F}_{X, Y}(x, y) - 1 \\ &\quad + F_X(x) \\ &\quad + F_Y(y) \\ &= \bar{F}_{X, Y}(x, y) - 1 \\ &\quad + 1 - \bar{F}_X(x) \\ &\quad + 1 - \bar{F}_Y(y) \end{aligned}$$

$$F_{X, Y}(x, y) = 1 - \bar{F}_X(x) - \bar{F}_Y(y) + \bar{F}_{X, Y}(x, y)$$

Exam 2012/13 Q4

(a)

$$F_M(m)$$

$$= P(M \leq m)$$

$$= P(\max(X, Y, Z) \leq m)$$

$$= P(X \leq m, Y \leq m, Z \leq m)$$

$$= F_{X, Y, Z}(m, m, m)$$

(\*\*\*)

$$= 1 - \bar{F}_X(m) - \bar{F}_Y(m) - \bar{F}_Z(m)$$

$$+ \bar{F}_{X, Y}(m, m) + \bar{F}_{X, Z}(m, m)$$

$$+ \bar{F}_{Y, Z}(m, m) - \bar{F}_{X, Y, Z}(m, m, m)$$

$$= 1 - \left[1 + \frac{m}{a}\right]^{-d} - \left[1 + \frac{m}{b}\right]^{-d} - \left[1 + \frac{m}{c}\right]^{-d}$$

$$+ \left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-d} + \left[1 + \frac{m}{a} + \frac{m}{c}\right]^{-d}$$

$$+ \left[1 + \frac{m}{b} + \frac{m}{c}\right]^{-d}$$

$$- \left[1 + \frac{m}{a} + \frac{m}{b} + \frac{m}{c}\right]^{-d}$$

$$\begin{aligned}
 (b) \quad f_M(m) &= \frac{d}{dm} F_M(m) \\
 &= \frac{d}{a} \left[ 1 + \frac{m}{a} \right]^{-d-1} + \frac{d}{b} \left[ 1 + \frac{m}{b} \right]^{-d-1} \\
 &\quad + \frac{d}{c} \left[ 1 + \frac{m}{c} \right]^{-d-1} \\
 &\quad - d \left( \frac{1}{a} + \frac{1}{b} \right) \left[ 1 + \frac{m}{a} + \frac{m}{b} \right]^{-d-1} \\
 &\quad - d \left( \frac{1}{a} + \frac{1}{c} \right) \left[ 1 + \frac{m}{a} + \frac{m}{c} \right]^{-d-1} \\
 &\quad - d \left( \frac{1}{b} + \frac{1}{c} \right) \left[ 1 + \frac{m}{b} + \frac{m}{c} \right]^{-d-1} \\
 &\quad + d \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \left[ 1 + \frac{m}{a} + \frac{m}{b} + \frac{m}{c} \right]^{-d-1}
 \end{aligned}$$

Exam 2016/17 Q3(e)

$$F(x) = 1 - \left[1 - e^{-\frac{1}{x}}\right]^a, \quad x > 0$$

$$w(F) = +\infty$$

(II) :

$$\lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t \rightarrow \infty} \frac{1 - \left\{1 - \left[1 - e^{-\frac{1}{tx}}\right]^a\right\}}{1 - \left\{1 - \left[1 - e^{-\frac{1}{t}}\right]^a\right\}}$$

$$= \lim_{t \rightarrow \infty} \frac{\left[1 - e^{-\frac{1}{tx}}\right]^a}{\left[1 - e^{-\frac{1}{t}}\right]^a}$$

$e^{-y} \approx 1 - y$

$$= \lim_{t \rightarrow \infty} \frac{\left[1 - \left(1 - \frac{1}{tx}\right)\right]^a}{\left[1 - \left(1 - \frac{1}{t}\right)\right]^a}$$

$$= x^{-a}$$

$\Rightarrow F$  belongs to the Fréchet max domain

Exam 2016/17 Q3 (b)

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$k = 0, 1, \dots, n$$

$$w(F) = n$$

$$\lim_{k \rightarrow w(F)} \frac{P(X=k)}{1-F(k-1)}$$

$$= \frac{P(X=n)}{1-F(n-1)}$$

$$= \frac{P(X=n)}{1-P(X \leq n-1)}$$

$$= \frac{P(X=n)}{P(X > n-1)}$$

$$= \frac{P(X=n)}{P(X=n)} = 1 \neq 0$$

ETR does not hold.



Exam 2016/17, Q3 (c)

$$f(x) = \frac{1}{(\log b - \log a) x}, \quad a < x < b$$

$$w(F) = b$$

$$(III) : \lim_{t \rightarrow 0} \frac{1 - F(b - tx)}{1 - F(b - t)}$$

$$\stackrel{LH}{=} \lim_{t \rightarrow 0} \frac{-f(b - tx) \cdot (-x)}{-f(b - t)}$$

$$= \lim_{t \rightarrow 0} \frac{\frac{1}{(\log b - \log a)} \cdot \frac{1}{(b - tx)} \cdot x}{\frac{1}{(\log b - \log a)} \cdot \frac{1}{b - t}}$$

$$= \lim_{t \rightarrow 0} \frac{b - t}{b - tx} \cdot x$$

$$= x$$

$\Rightarrow F$  belongs to the Weibull  
max domain.

**REVISION CLASS**

**14 DECEMBER**

**12:00-13:00PM**

**MATH4/68181**

Q1

$$\bar{G}(x, y) = e^{-\frac{\theta y^2}{x+y} + \theta y - x - y}$$

(a) Show that the distribution is a bivariate extreme value distribution.

$$\bar{G}(0, y) = e^{-\theta y + \theta y - y} = e^{-y}$$

$$\bar{G}(x, 0) = e^{-x}$$

The marginals are exponentially distributed.

$$\bar{G}(x, y) = e^{-(x+y) \left[ \frac{\theta y^2}{(x+y)^2} + \frac{\theta y}{x+y} + 1 \right]}$$

$$= e^{-(x+y) A\left(\frac{y}{x+y}\right)}$$

where  $A(w) = \theta w^2 + \theta w + 1$

(i)  $A(0) = \theta \cdot 0^2 + \theta \cdot 0 + 1 = 1 \checkmark$

$A(1) = \theta \cdot 1^2 + \theta \cdot 1 + 1 = 1 \checkmark$

(ii)  $A(w) \leq 1 \quad \forall w$

$\Leftrightarrow \theta w^2 + \theta w + 1 \leq 1 \quad \forall w \checkmark$

$\Leftrightarrow \theta w(w-1) \leq 0 \quad \forall w \checkmark$

$$A(w) \geq w \quad \forall w$$

$$\Leftrightarrow \theta w^2 - \theta w + 1 \geq w \quad \forall w$$

$$\Leftrightarrow \theta w(w-1) + 1 - w \geq 0 \quad \forall w$$

$$\Leftrightarrow \underbrace{(1-w)}_{\substack{\forall \\ 0}} \underbrace{(1-\theta w)}_{\substack{\forall \\ 0 < \theta < 1}} \geq 0 \quad \forall w \quad \checkmark$$

$$A(w) \geq 1-w \quad \forall w$$

$$\Leftrightarrow \theta w^2 - \theta w + 1 \geq 1-w \quad \forall w$$

$$\Leftrightarrow \theta w^2 + \underbrace{(1-\theta)}_{\substack{\forall \\ 0}} w \geq 0 \quad \forall w \quad \checkmark$$

(iii)  $A(\cdot)$  is convex

$$A(w) = \theta w^2 - \theta w + 1$$

$$A'(w) = 2\theta w - \theta$$

$$A''(w) = 2\theta > 0$$

$\Rightarrow A(\cdot)$  is convex

Hence,  $\bar{G}_a(x, y)$  is a bivariate extreme value distribution.

Q2 (a)

$$\bar{G}(x, y) = e^{\frac{\alpha xy}{x+y} - x - y}$$

$$\left. \begin{aligned} \bar{G}(0, y) &= e^{-y} \\ \bar{G}(x, 0) &= e^{-x} \end{aligned} \right\}$$

$$\bar{G}(x, y) = e^{-(x+y)} \left[ 1 - \frac{\alpha xy}{(x+y)^2} \right]$$

$$A(w) = 1 - \alpha w(1-w)$$

$$\begin{aligned} A\left(\frac{y}{x+y}\right) &= 1 - \alpha \frac{y}{x+y} \left(1 - \frac{y}{x+y}\right) \\ &= 1 - \alpha \frac{y}{x+y} \cdot \frac{x}{x+y} \\ &= 1 - \frac{\alpha xy}{(x+y)^2} \end{aligned}$$

$$(i) \quad A(0) = 1 - \alpha \cdot 0 \cdot (1-0) = 1 \quad \checkmark$$

$$A(1) = 1 - \alpha \cdot 1 \cdot (1-1) = 1 \quad \checkmark$$

$$(ii) \quad A(w) \leq 1 \quad \forall w$$

$$1 - \underbrace{\alpha w(1-w)}_{\geq 0} \leq 1 \quad \forall w \quad \checkmark$$

$$A(w) \geq w \quad \forall w$$

$$\Leftrightarrow 1 - \alpha w (1-w) \geq w \quad \forall w$$

$$\Leftrightarrow 1-w - \alpha w (1-w) \geq 0 \quad \forall w$$

$$\Leftrightarrow \underbrace{(1-w)}_{\substack{\forall \\ 0}} \underbrace{(1-\alpha w)}_{\substack{\forall \\ 0}} \geq 0 \quad \forall w \quad \checkmark$$

$$A(w) \geq 1-w \quad \forall w$$

$$1 - \alpha w (1-w) \geq 1-w \quad \forall w$$

$$w - \alpha w (1-w) \geq 0 \quad \forall w$$

$$\underbrace{w}_{\substack{\forall \\ 0}} \underbrace{[1-\alpha(1-w)]}_{\substack{\forall \\ 0}} \geq 0 \quad \forall w \quad \checkmark$$

$$(iii) \quad A(w) = 1 - \alpha w (1-w)$$

$$\Rightarrow A'(w) = -\alpha + 2\alpha w$$

$$\Rightarrow A''(w) = 2\alpha > 0$$

$\Rightarrow A(\cdot)$  is convex

Hence,  $\bar{G}(x, y)$  is a biv extreme  
value distribution.

Q3

$$\bar{G}(x, y) = e^{-(x^a + y^a)^{\frac{1}{a}}}$$

$$\left. \begin{aligned} \bar{G}(0, y) &= e^{-y} \\ \bar{G}(x, 0) &= e^{-x} \end{aligned} \right\}$$

$$\bar{G}(x, y) = e^{-(x+y)} \cdot \frac{1}{(x+y)} \cdot (x^a + y^a)^{\frac{1}{a}}$$

$$= e^{-(x+y)} \left[ \frac{x^a + y^a}{(x+y)^a} \right]^{\frac{1}{a}}$$

$$= e^{-(x+y)} \left[ \left( \frac{x}{x+y} \right)^a + \left( \frac{y}{x+y} \right)^a \right]^{\frac{1}{a}}$$

$$\Rightarrow A(w) = \left[ (1-w)^a + w^a \right]^{\frac{1}{a}}$$

$$(i) \quad A(0) = \left[ (1-0)^a + 0^a \right]^{\frac{1}{a}} = 1 \quad \checkmark$$

$$A(1) = \left[ (1-1)^a + 1^a \right]^{\frac{1}{a}} = 1 \quad \checkmark$$

$$(ii) \quad A(w) \leq 1 \quad \forall w$$

$$(1-w)^a + w^a \leq 1 \quad \forall w$$

REVISION

CLASS

(FRI 15 DEC 9-10 AM)



Exam date : 24 Jan Wed 9:45am

Away from Manchester between

3 Jan - 11 Jan

(feel free to email me during  
this period)

Happy to meet you during the  
period the ATB will be closed.

Please email me date / time.

Exam 2015/16 Q1

(a)

$$\begin{aligned}M_{X_i}(t) &= E[e^{tX_i}] \\&= \int_0^{\infty} e^{tx} \cdot \underbrace{\lambda e^{-\lambda x}}_{\text{pdf}} dx \\&= \lambda \int_0^{\infty} e^{-(\lambda-t)x} dx \\&= \lambda \left[ \frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^{\infty} \\&= \lambda \left[ 0 - \frac{1}{-(\lambda-t)} \right] \\&= \frac{\lambda}{\lambda-t}\end{aligned}$$

$$\begin{aligned}(b) \quad M_{T|N=n}(s) &= E[e^{sT} | N=n] \\&= E[e^{s(X_1 + \dots + X_n)} | N=n] \\&= E[e^{s(X_1 + \dots + X_n)}] \\&= \underbrace{E[e^{sX_1}]}_{\frac{\lambda}{\lambda-s}} \dots \underbrace{E[e^{sX_n}]}_{\frac{\lambda}{\lambda-s}} \\&= \left(\frac{\lambda}{\lambda-s}\right)^n\end{aligned}$$

(c)  $T | N = n$  has the gamma distribution with scale parameter  $\lambda$  and shape parameter  $n$ .

$$f_{T|N=n}(t) = \frac{\lambda (\lambda x)^{n-1} e^{-\lambda x}}{\Gamma(n)}$$

$$(d) f_T(t) = \sum_{n=1}^{\infty} f_{T|N=n}(t) P(N=n)$$

$$= \sum_{n=1}^{\infty} \frac{\lambda (\lambda x)^{n-1} e^{-\lambda x}}{\Gamma(n)} \cdot \theta (1-\theta)^{n-1}$$

$$= \theta \lambda e^{-\lambda x} \sum_{n=1}^{\infty} \frac{[\lambda x (1-\theta)]^{n-1}}{(n-1)!}$$

Set  $m = n-1$

$$= \theta \lambda e^{-\lambda x} \sum_{m=0}^{\infty} \frac{[\lambda x (1-\theta)]^m}{m!} = e^{-\lambda x (1-\theta)}$$

$$= \theta \lambda e^{-\theta \lambda x}$$

$$\Rightarrow T \sim \text{Exp}(\theta \lambda)$$

Exam 2015 / 16 Q3 (a)

$$w(F) = 1$$

(III) :

$$\lim_{t \downarrow 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)}$$

$$\stackrel{\text{L'H}}{=} \lim_{t \downarrow 0} \frac{(-1) f(1 - tx) \cdot (-x)}{(-1) f(1 - t) \cdot (-1)}$$

$$= \lim_{t \downarrow 0} \frac{C (1 - tx)^{\alpha-1} \cdot (tx)^{\beta-1} \cdot x}{C (1 - t)^{\alpha-1} t^{\beta-1}}$$

$$= x^\beta$$

$\Rightarrow F$  belongs to the Weibull  
max domain.

Exam

2015/16

Q 4 (c)

(i)

$$L(a, k) = \prod_{i=1}^n \left[ a k^a x_i^{-a-1} I\{x_i > k\} \right]$$

$$= a^n k^{na} \left( \prod_{i=1}^n x_i \right)^{-a-1}$$

$$\cdot \prod_{i=1}^n I\{x_i > k\}$$

Math

=

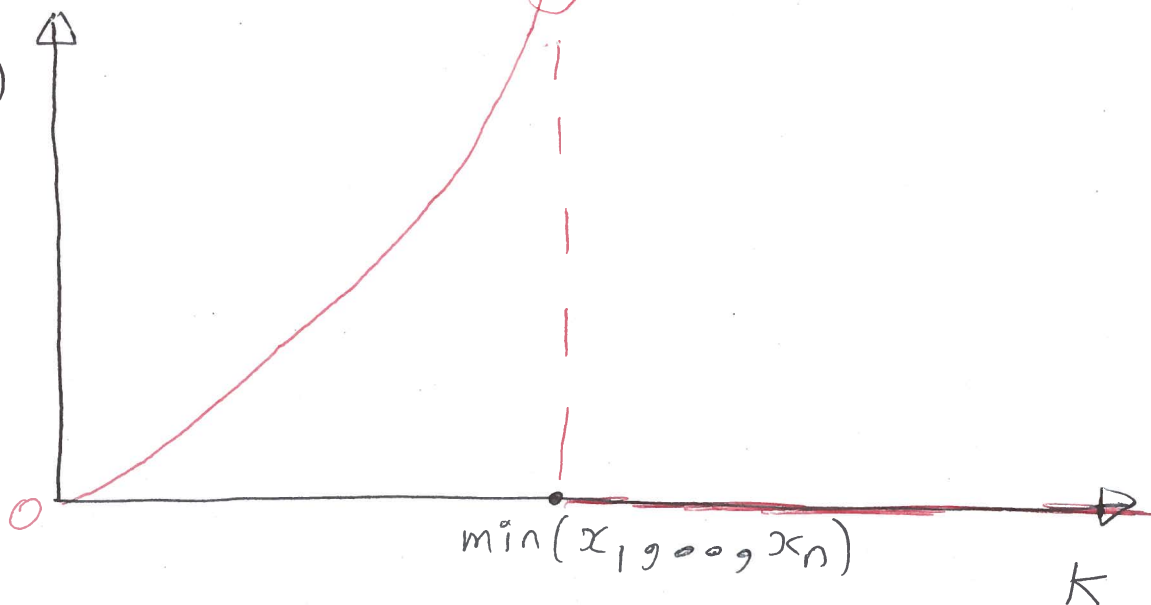
20802

$$a^n k^{na} \left( \prod_{i=1}^n x_i \right)^{-a-1}$$

$$\cdot I\{\min(x_1, \dots, x_n) > k\}$$

(ii)

$L(a, k)$



$$\Rightarrow \hat{k} = \min(x_1, \dots, x_n)$$

$$(iii) \log L(a, k) = n \log a + na \log k + (a-1) \sum_{i=1}^n \log x_i + \log I \{ \min(x_1, \dots, x_n) > k \}$$

$$\frac{d \log L}{da} = \frac{n}{a} + n \log k - \sum_{i=1}^n \log x_i = 0$$

$$\Rightarrow \hat{a} = \frac{n}{-n \log \hat{k} + \sum_{i=1}^n \log x_i}$$

(iv)

$$VaR_p(X) = k (1-p)^{-\frac{1}{a}}$$

$$VaR_0(X) = k (1-0)^{-\frac{1}{a}} = k$$

$$\widehat{VaR}_0(X) = \hat{k}$$

$$ES_p(X) = \frac{1}{p} \int_0^p VaR_t(X) dt$$

$$ES_0(X) = \lim_{p \rightarrow 0} \frac{1}{p} \int_0^p VaR_t(X) dt$$

$$\stackrel{LH}{=} \lim_{p \rightarrow 0} \frac{VaR_p(X)}{1}$$

$$= VaR_0(X) = k$$

$$\widehat{ES}_0(X) = \hat{k}$$