LECTURE

26 SEPTEMBER

9:00-10:00AM

MATH3/4/68181

Extreme Values

& Financial Risk

Lecture times

Tues	9 – 10 am	3/4/6
Fri ~~Thurs~~	9 – 10 am	3/4/6
Thurs	12 – 1	4/6

Example

Mon	12 – 1	3/4/6
Tues	10 – 11	3/4/6

Contact

Room : ATB 2.223

Email : _

Office hrs : 2–3 pm Tues

2 - 3 pm Thurs

Assessment

20% In-class test 14 Nov

80% Exam Jan 2018

2% Bonus (last 2 weeks of Semester I)

Website

http://www.maths.manchester.ac.uk/~saralees/extreme3.html

Level 3

extreme4.html

Level 4

extreme6.html

Level 6

Credits

10 — Level 3

15 — Level 4

15 — level 6

Pre-requisite

- Statistical Methods
- Statistical Inference

Books

No need

WHAT IS THE USE OF

THIS COURSE ?

Answer

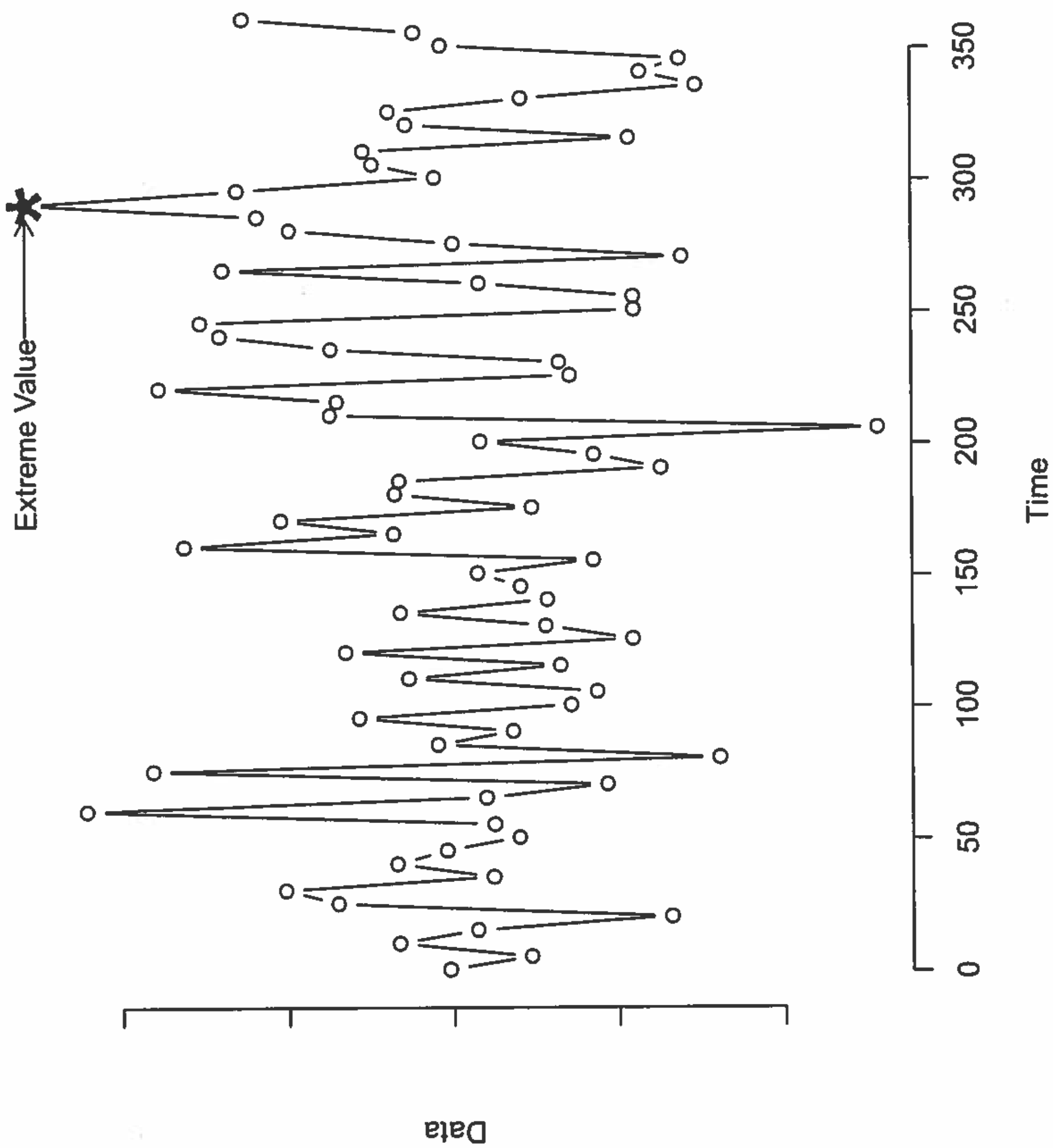
1. To minimize the chance of financial disasters

2. ~~Cat~~ Help to catch the crooks.

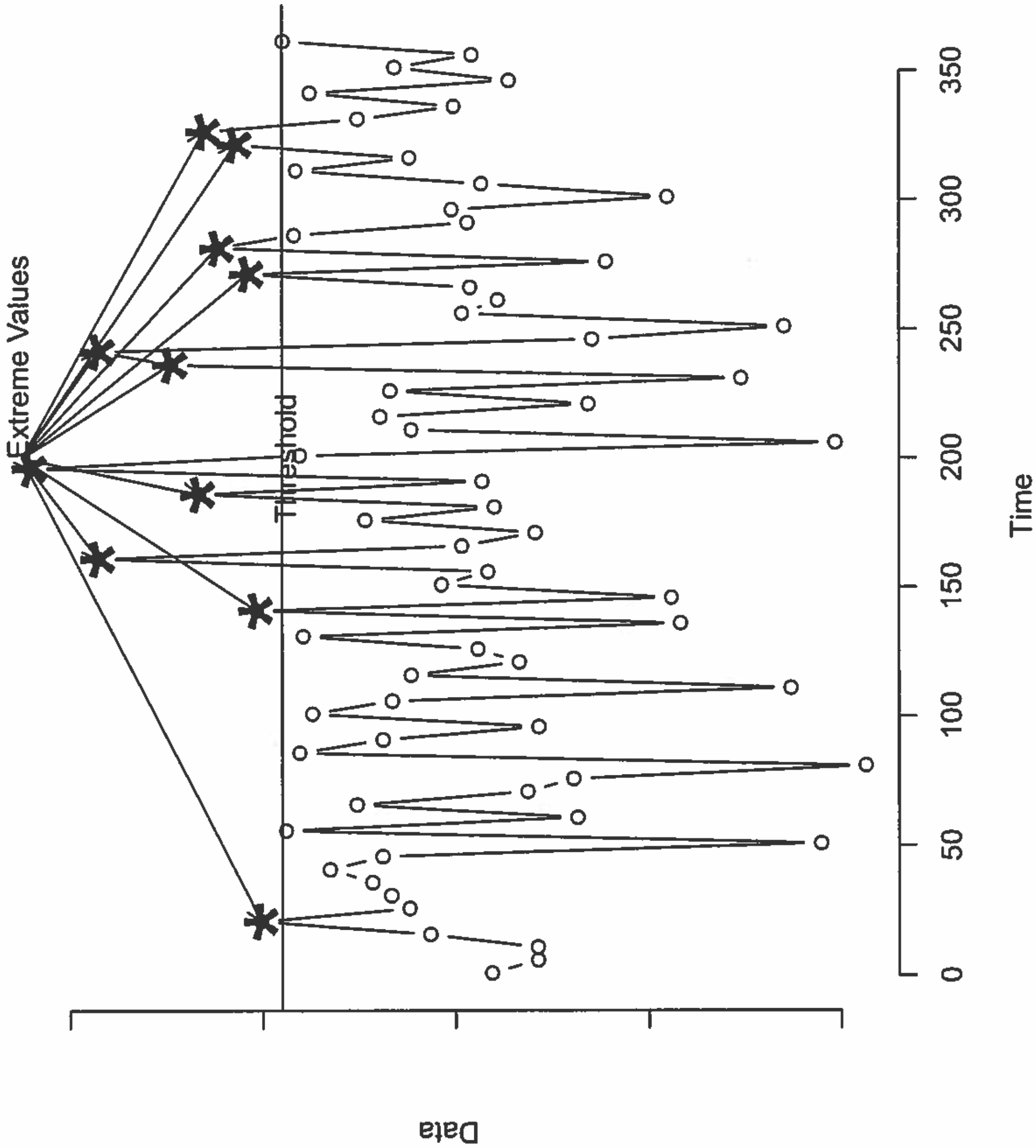
Extreme Values & Financial Risk

WHAT IS AN EXTREME VALUE?

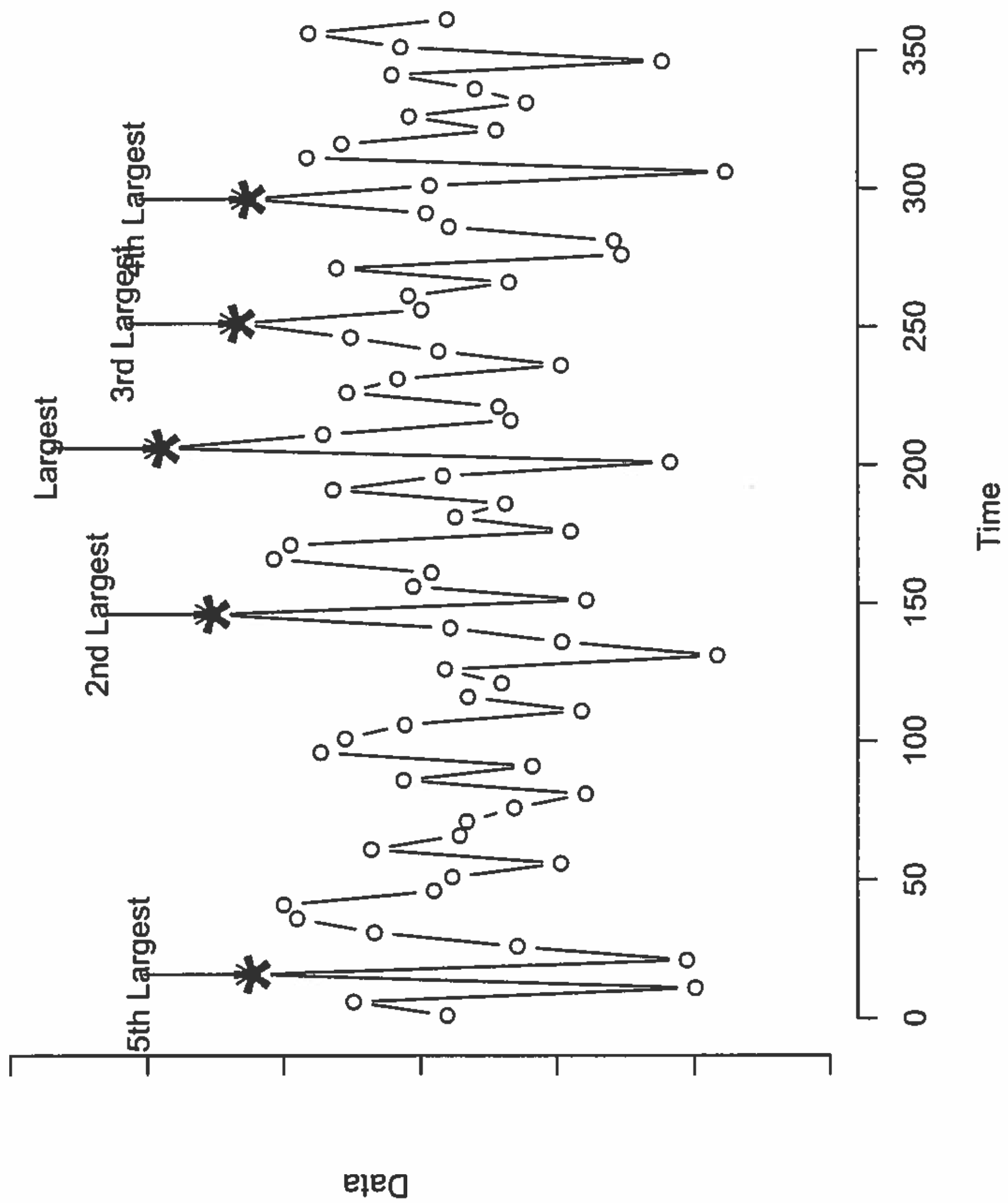
DEFINITION 1



DEFINITION 2



DEFINITION 3



Definition I

Suppose $X_1, X_2, \ldots, X_n$ are IID with CDF $F(\cdot)$.

The extreme value

$$\boxed{M_n = \max(X_1, \ldots, X_n)}$$

What is the distribution of M_n?

$$P(M_n \leqslant x)$$

$$= P(\max(X_1, \ldots, X_n) \leqslant x)$$

$$= P(X_1 \leqslant x, \ldots, X_n \leqslant x)$$

$$\overset{\text{indep}}{=} P(X_1 \leqslant x) \cdots P(X_n \leqslant x)$$

$$= F(x) \cdots F(x)$$

$$= F^n(x)$$

Usually, n is very large.

We like to know the distribution of the extrem value as $n \to \infty$.

$$\lim_{n \to \infty} P(M_n \leq x)$$

$$= \lim_{n \to \infty} F^n(x)$$

$$= \begin{cases} 0 & \text{if } F(x) < 1 \\ 1 & \text{if } F(x) = 1 \end{cases}$$

"Not very helpful" for modelling real data.

Suppose $X_1, X_2, \ldots, X_n$ are IID with mean μ and variance σ^2.

$$\lim_{n \to \infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = \mu$$

SLLN

(Strong Law of Large Numbers)

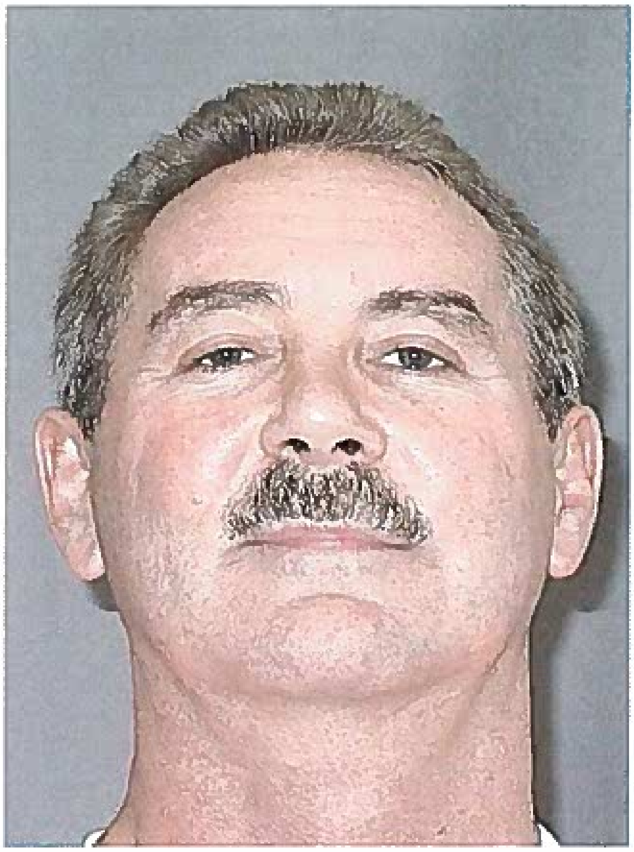
$$\lim_{n \to \infty} P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq y \right) = \Phi(y)$$

Φ is the CDF of $N(0,1)$

CLT

(Central Limit Thm)

$$P\left(\frac{M_n - b_n}{a_n} \leq x\right)$$



Allen Stanford

Stanford is serving a 110 year prison sentence, having been convicted of charges of fraud. Stanford was the chairman of the now defunct Standord Financial Group of companies.

Jordan Belfort

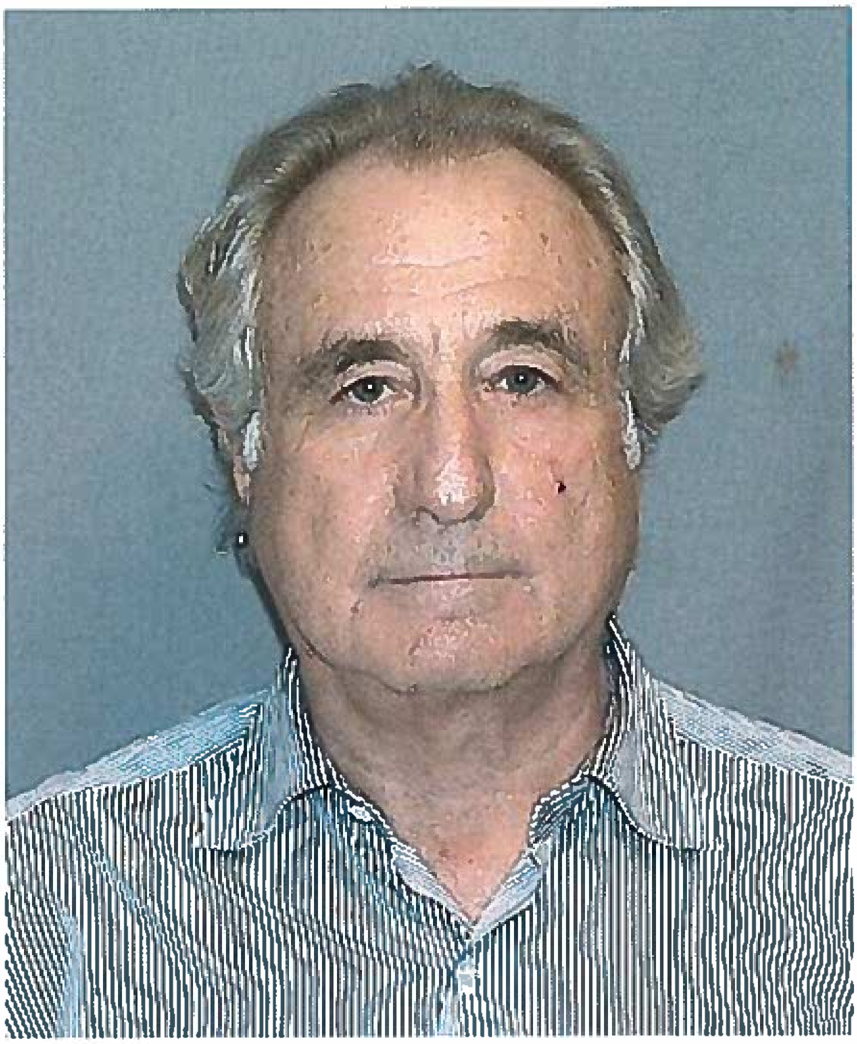
Belfort founded the brokeage firm Stratton Oakmont, and went on to defraud investors through sales of stock while employing more than 1,000 people. The firm was shut in 1998. Belfort was charged with money laundering and fraud.

Bruno Iksil

Former trader for JPMorgan Chase & Co and also known as “the London Whale”. His position in the credit default swap market caused more than $6.2 billion in losses for the firm in 2012.

Albert H. Wiggin

Career spaning from head of Chase National Bank and now JP Morgan chase. He shorted his own company with 40000 shares in 1929 and profited immensely to the tune of $4 million and wasn't even found guilty of breaking any laws.



Bernard Madoff

Former stockbroker and investment adviser. His name is connected to the most well known Ponzi scheme fraud. He earned $62 billion through such fraud and is currently sentenced to a maximum of 150 years in prison.

Barry Minkow

Founded ZZZZ Best which appeared to be an immensely successful carpet cleaning and restoration company. However, it was actually a front to attract investment for a massive Ponzi scheme. It collapsed in 1987 costing investors and lenders $100 million.

Martha Stewart

Martha is an American businesswoman, writer and television personality. In 2004, Stewart was convicted relating to the ImClone insider trading affair and sentenced to prison.

Bernard Ebbers

A Canadian born buisnessman and cofounded the telecommunications company WorlCom. In 2005, he was convicted of fraud and conspiracy as a result of WorldCom's false financial reporting. He was sentenced to 25 years for his role in an $11 billion accounting fraud. It is the largest accounting scandal in US history.

Donald Johnson

Executive of stock exchange for Nasdaq's. Convicted in 2011 of using insider information to trade shares of United Therapeutics, Honda and other companies from 2006 to 2009. He was sentenced to 3 and half years in prison for a total of $755,000 of fraud.

Doug Whitman

Managing more than $100 million through his Whitman Capital hedge fund. He was convicted in August 2012 of insider trading charges involving stocks of Google, Polycom and Marvell Technology Group. The total value of his fraud accumulated to $1 million and lead to 2 years in prison.

Raj Rajaratnam

Former head of Galleon Capital. He received the longest insider trading sentence in history of 11 years for securities fraud and five counts of conspiracy. He was ordered to pay back the $53.8 million of fraud.

Jeff Skilling

Jeff was the CEO of Enron Corp. He was convicted of federal felony charges relating to Enron's financial collapse and is currently serving 14 years prison sentence.

Ivan Boesky

Boesky is a former American stock trader who is notable for his prominent role in a Wall Street insider trading scandal that occurred in the US in the mid-1980s. In 1987, Boesky was sentenced to 3 years in prison.

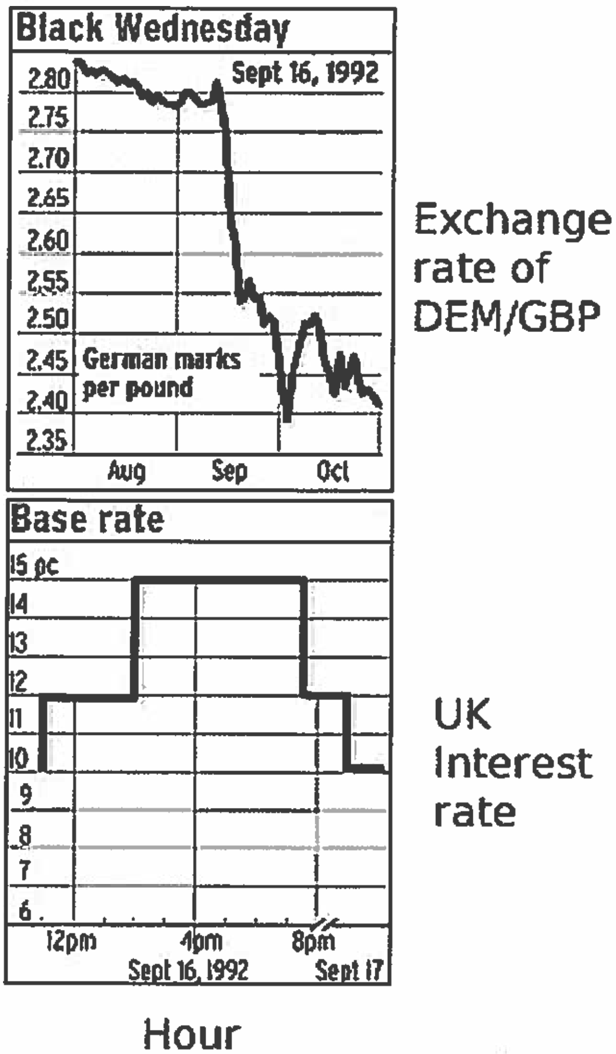


Figure 5: Black Wednesday crash of 16 September 1992. Top image shows the exchange rate of Deutsche mark to British pounds. Bottom image shows the UK interest rate on the day.

Figure 4: Asian financial crisis (Asian dollar index) in July 1997. Not fully recovered even in 2011.

Figure 3: Dot com bubble (the NASDAQ index) during 1999 and 2000. The bubble burst on 10 March 2000. The peak on that day was $5048.62. There is a recovery after 2002. Never recovered to attain the peak.

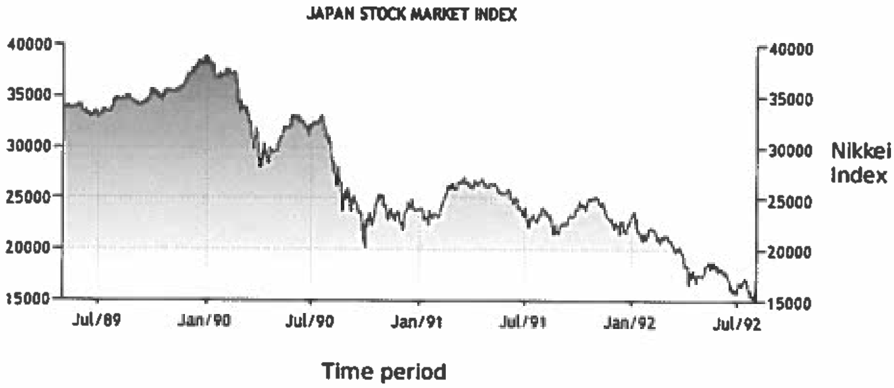


Figure 2: Japan stock price bubble near the end of 1989. A loss of $2.7 trillion in capital. A recovery happened after mid-1990.

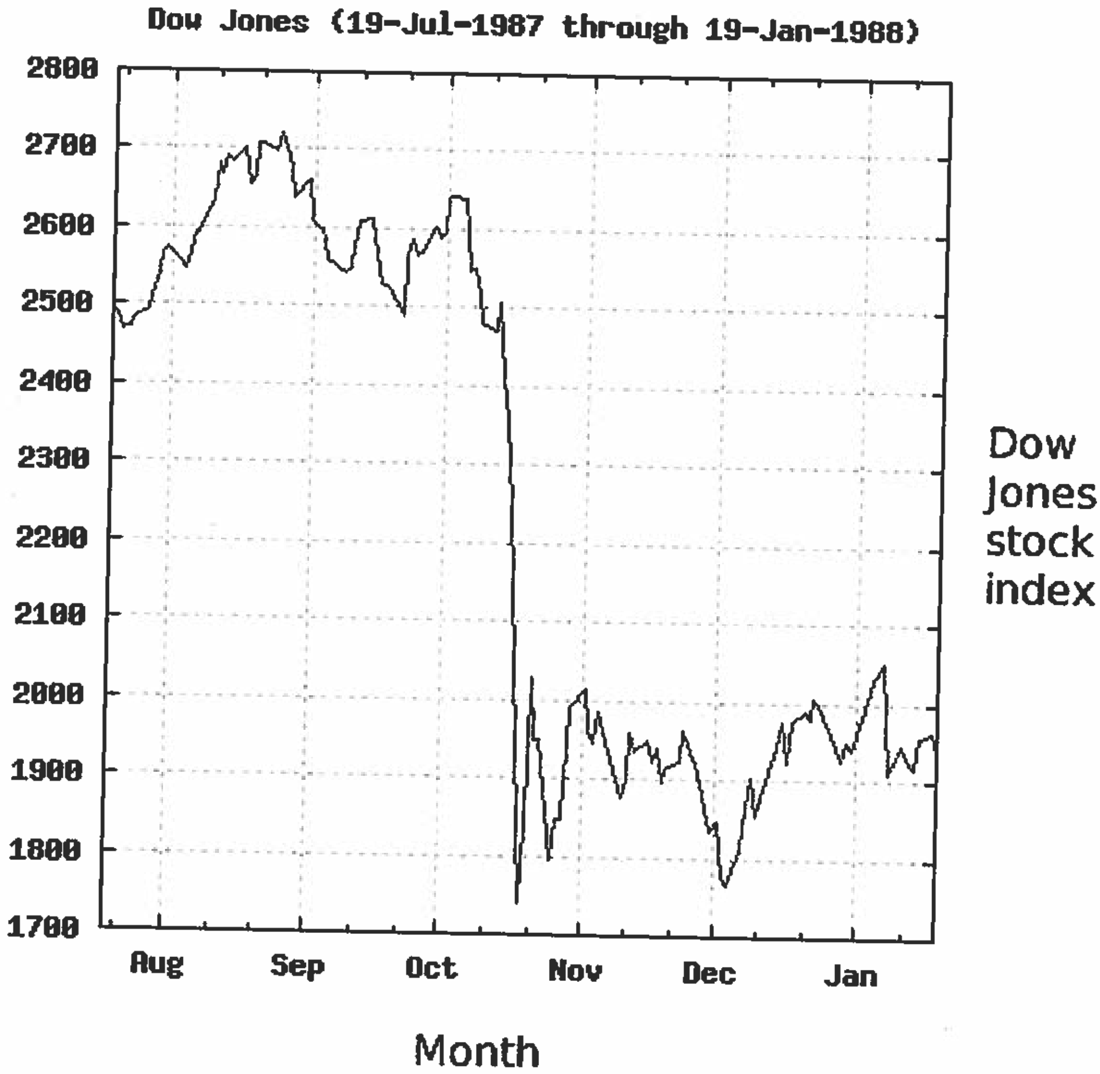


Figure 1: Black Monday crash on 19 October 1987. The Dow Jones stock index crashed down by 22.6 percent (by 508 points). Overall the stock market lost $0.5 trillion.

LECTURE

29 SEPTEMBER

9:00-10:00AM

MATH3/4/68181

Suppose $X_1, X_2, \ldots, X_n$ are IID with mean μ and variance σ^2.

$$\lim_{n \to \infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = \mu$$

SLLN

(Strong Law of Large Numbers)

$$\lim_{n \to \infty} P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq y \right) = \Phi(y)$$

Φ is the CDF of $N(0,1)$

CLT

(Central Limit Thm)

Usually, n is very large.

We like to know the distribution of the extreme value as $n \to \infty$.

$$\lim_{n \to \infty} P(M_n \leq x)$$

$$= \lim_{n \to \infty} F^n(x)$$

$$= \begin{cases} 0 & \text{if } F(x) < 1 \\ 1 & \text{if } F(x) = 1 \end{cases}$$

"Not very helpful" for modelling real data.

$$P\left(\frac{M_n-b_n}{a_n} \leqslant x\right)$$

$$= P\left(M_n \leqslant a_n x + b_n\right)$$

$$= P\left(\max(X_1, \ldots, X_n) \leqslant a_n x + b_n\right)$$

$$= P\left(X_1 \leqslant a_n x + b_n, \ldots, X_n \leqslant a_n x + b_n\right)$$

$$\overset{\text{indep}}{=} P(X_1 \leqslant a_n x + b_n) \cdots P(X_n \leqslant a_n x + b_n)$$

$$= F(a_n x + b_n) \cdots F(a_n x + b_n)$$

$$= F^n(a_n x + b_n)$$

$$\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leqslant x\right)$$

$$= \lim_{n \to \infty} F^n(a_n x + b_n) \qquad (*)$$

Extremal Types Theorem (ETT)

If (*) exists it must be of the same type as

I: $\Lambda(x) = e^{-e^{-x}}, \quad -\infty < x < \infty$

"Gumbel"

II: $\Phi_\alpha(x) = \begin{cases} e^{-x^{-\alpha}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

"Frechet"

III: $\Psi_\alpha(x) = \begin{cases} e^{-(-x)^{\alpha}} & x \leq 0 \\ 1 & x > 0 \end{cases}$

"Weibull"

The same type

Two cdfs G_1 and G_2 are of the same type if

$$G_1(x) = G_2(ax + b)$$

for all x, where $a > 0$, $b \in \mathbb{R}$.

Eg 1

$$G_1(x) = e^{-e^{-x}}$$

$$G_2(x) = e^{-e^{-(x+3)}}$$

Clearly, G_1 & G_2 are of the same type.

Eg 2

$$G_1(x) = e^{x}$$

$$G_2(x) = e^{-2x+3}$$

G_1 and G_2 are not of the same type.

Suppose $X_1, \ldots, X_n$ are IID with cdf F. Let $M_n = \max(X_1, \ldots, X_n)$. How to know which limit

$$P\left(\frac{M_n - b_n}{a_n} \leq x\right)$$ will approach to?

Conditions

I: $P\left(\frac{M_n - b_n}{a_n} < x\right) \longrightarrow \Lambda(x)$

if there exists $\gamma(t) > 0$

such that $\lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = e^{-x}$.

II: $P\left(\frac{M_n - b_n}{a_n} < x\right) \longrightarrow \Phi_\alpha(x)$

if $w(F) = \infty$, $\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}$

III: $P\left(\frac{M_n - b_n}{a_n} < x\right) \longrightarrow \Psi_\alpha(x)$

if $w(F) < \infty$, $\lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^{\alpha}$

If any only one of (I)-(III) will be satisfied

What is w(F)

$w(F)$ = "upper end point" of F

$$w(F) = \sup\{x : F(x) < 1\}$$

Easy way: set $F(x) = 1$
solve for x
the solution is $w(F)$.

eg 1

$$F(x) = 1 - e^{-x}$$

$$F(x) = 1 \Rightarrow 1 - e^{-x} = 1$$

$$\Rightarrow e^{-x} = 0 \Rightarrow x = +\infty$$

$$w(F) = +\infty$$

eg 2

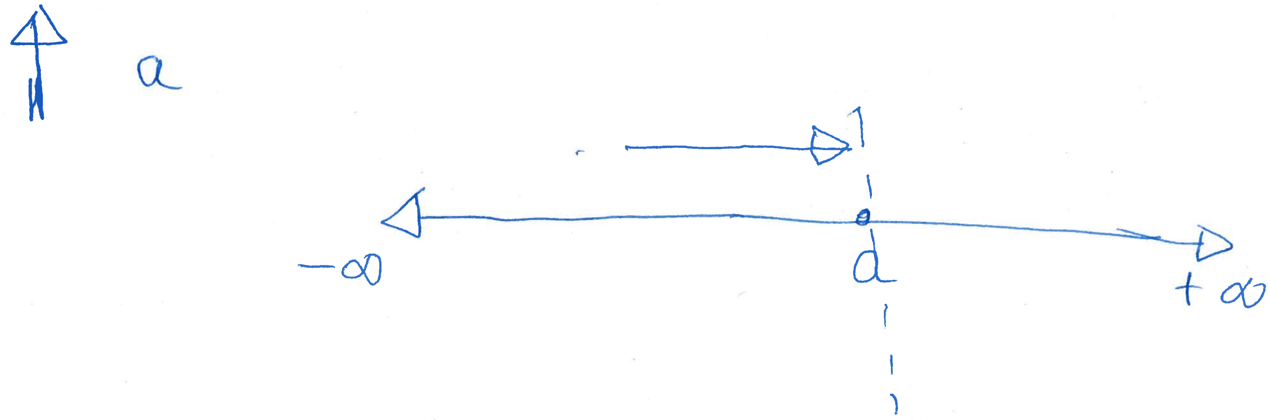
$$F(x) = 1 - \frac{1}{x}, \quad x > 1$$

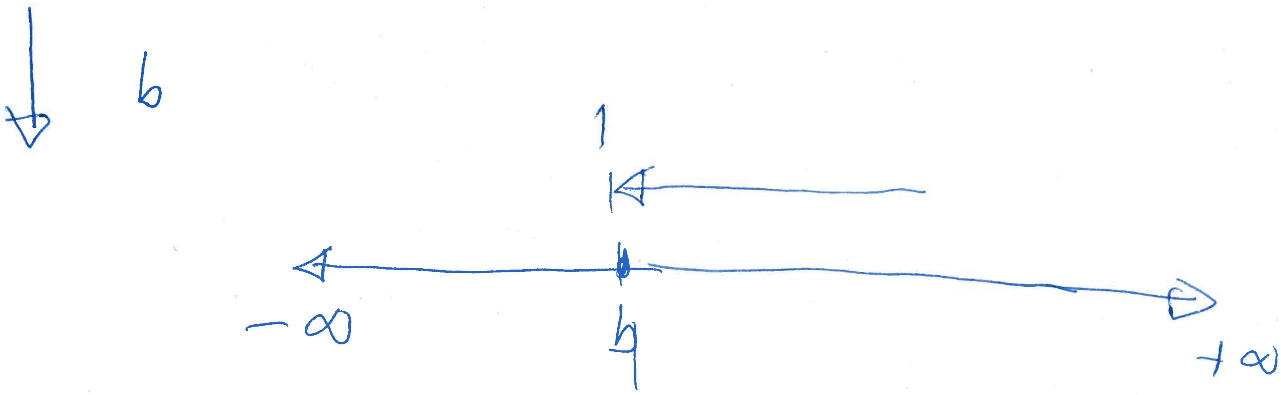
$$F(x) = 1 \Rightarrow 1 - \frac{1}{x} = 1$$

$$\Rightarrow \frac{1}{x} = 0 \Rightarrow x = +\infty$$

$$w(F) = +\infty$$

limit arrows





eg 1

$$F(x) = 1 - e^{-x}$$

$$w(F) = +\infty$$

$$\lim_{t \uparrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{1 - [1 - e^{-t - x\gamma(t)}]}{1 - [1 - e^{-t}]}$$

$$= \lim_{t \uparrow \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t \uparrow \infty} e^{-x\gamma(t)}$$

$$= e^{-x} \quad \text{if} \quad \gamma(t) \equiv 1$$

Condition (I) is satisfied.
Hence, there exist $a_n > 0$ & $b_n \in \mathbb{R}$ such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \longrightarrow e^{-e^{-x}}$$

as $n \longrightarrow \infty$.

eg 2

$$F(x) = 1 - \frac{1}{x}, \quad x > 1$$

$$\omega(F) = +\infty$$

I: $\lim_{t \uparrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$

$$= \lim_{t \uparrow \infty} \frac{1 - \left[1 - \frac{1}{t + x\gamma(t)}\right]}{1 - \left[1 - \frac{1}{t}\right]}$$

$$= \lim_{t \uparrow \infty} \frac{t}{t + x\gamma(t)}$$

$\neq e^{-x} \Rightarrow$ condition (I) not satisfied

II: $\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)}$

$$= \lim_{t \uparrow \infty} \frac{1 - \left(1 - \frac{1}{tx}\right)}{1 - \left(1 - \frac{1}{t}\right)}$$

$= \lim_{t \uparrow \infty} \frac{t}{tx} = \frac{1}{x} \Rightarrow$ condition (II) is satisfied with $\alpha = 1$.

There exists $a_n > 0$, $b_n \in \mathbb{R}$ such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \longrightarrow \begin{cases} e^{-x^{-1}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

How to choose a_n and b_n?

I :

$$a_n = \gamma\left(F^{-1}\left(1-\frac{1}{n}\right)\right)$$

$$b_n = F^{-1}\left(1-\frac{1}{n}\right)$$

II :

$$a_n = F^{-1}\left(1-\frac{1}{n}\right)$$

$$b_n = 0$$

III :

$$a_n = w(F) - F^{-1}\left(1-\frac{1}{n}\right)$$

$$b_n = w(F).$$

eg 1 (contd)

$$F(x) = 1 - e^{-x}$$

$$1 - e^{-x} = y$$

$$\Rightarrow \quad e^{-x} = 1 - y$$

$$\Rightarrow \quad x = -\log(1-y)$$

$$\Rightarrow \quad F^{-1}(x) = -\log(1-x)$$

$$\begin{cases} a_n = \gamma\left(-\log\left(1-\left(1-\frac{1}{n}\right)\right)\right) \\ b_n = -\log\left(1-\left(1-\frac{1}{n}\right)\right) \end{cases}$$

$$\Rightarrow \begin{cases} a_n = \gamma\left(-\log\left(\frac{1}{n}\right)\right) \\ b_n = -\log\left(\frac{1}{n}\right) \end{cases}$$

$$\Rightarrow \begin{cases} a_n = \gamma(\log n) = 1 \\ b_n = \log n \end{cases}$$

Hence,

$$P\left(\frac{M_n - \log n}{1} \leqslant x\right) \rightarrow e^{-e^{-x}}$$

as $n \rightarrow \infty$.

eg 2 (contd)

$$F(x) = 1 - \frac{1}{x}$$

$$1 - \frac{1}{x} = y \Rightarrow \frac{1}{x} = 1 - y$$

$$\Rightarrow x = \frac{1}{1-y}$$

$$\Rightarrow F^{-1}(x) = \frac{1}{1-x}.$$

$$a_n = F^{-1}\left(1 - \frac{1}{n}\right) = \frac{1}{1 - \left(1 - \frac{1}{n}\right)} = n$$

$$b_n = 0$$

Hence, $P\left(\frac{M_n - 0}{n} < x\right)$

$$\rightarrow \begin{cases} e^{-x^{-1}} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

EXAMPLE CLASS

2 OCTOBER

12:00-13:00PM

MATH3/4/68181

Q1

$$\Lambda(x) = e^{-e^{-x}}$$

$$\frac{d\Lambda(x)}{dx} = e^{-e^{-x}} (-1)\, e^{-x}\, (-1)$$

$$= e^{-e^{-x}} \cdot e^{-x}$$

$$\Phi_\alpha(x) = \begin{cases} e^{-x^{-\alpha}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\frac{d\Phi_\alpha(x)}{dx} = \begin{cases} e^{-x^{-\alpha}} (-1)(-\alpha)\, x^{-\alpha-1}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$= \begin{cases} \alpha\, e^{-x^{-\alpha}}\, x^{-\alpha-1}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\Psi_\alpha(x) = \begin{cases} e^{-(-x)^{\alpha}}, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

$$\frac{d\Psi_\alpha(x)}{dx} = \begin{cases} e^{-(-x)^{\alpha}} (-1)\,\alpha(-x)^{\alpha-1}(-1), & x \leq 0 \\ 0, & x > 0 \end{cases}$$

Q2

$$E(X) = \int_{-\infty}^{\infty} x \cdot \underbrace{e^{-e^{-x}} e^{-x}} dx$$

$$y = e^{-x} \Rightarrow x = -\log y$$
$$\Rightarrow \frac{dx}{dy} = -\frac{1}{y}$$

$$= \int_{\infty}^{0} (-\log y) \cdot e^{-y} y \left(-\frac{1}{y}\right) dy$$

$$= \int_{0}^{\infty} (-\log y) \cdot e^{-y} dy$$

$$= -\int_{0}^{\infty} \left(\frac{d}{da} y^a \Big|_{a=0} \right) \cdot e^{-y} dy$$

$$= -\frac{d}{da} \left(\int_{0}^{\infty} y^a e^{-y} dy \right) \Big|_{a=0}$$

$$= -\frac{d}{da} \Gamma(a+1) \Big|_{a=0}$$

$$= -\Gamma'(1)$$

$$\frac{d}{da} y^a = y^a \log y$$

$\boxed{a = 0}$: $\frac{d}{da} y^a \Big|_{a=0} = y^0 \log y$

$= \log y$

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} \, dt$$

Gamma Function

$$\frac{d^2}{da^2} y^a = \frac{d}{da}\left(y^a \log y \right)$$

$$= y^a \left(\log y \right)^2$$

$\boxed{a=0}$: $$\left. \frac{d^2}{da^2} y^a \right|_{a=0} = y^0 \left(\log y \right)^2$$

$$= \left(\log y \right)^2$$

Q3

$$\mathrm{Var}[X] = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot e^{-e^{-x}} \cdot e^{-x}\, dx$$

$y = e^{-x} \Rightarrow x = -\log y$
$\Rightarrow \frac{dx}{dy} = -\frac{1}{y}$

$$= \int_{\infty}^{0} (\log y)^2 e^{-y}\, y \left(-\frac{1}{y}\right) dy$$

$$= \int_{0}^{\infty} (\log y)^2 e^{-y}\, dy$$

$$= \int_{0}^{\infty} \left(\frac{d^2}{da^2} y^a \Big|_{a=0} \right) \cdot e^{-y}\, dy$$

$$= \frac{d^2}{da^2} \left(\int_{0}^{\infty} y^a e^{-y}\, dy \right) \Big|_{a=0}$$

$$= \frac{d^2}{da^2} \Gamma(a+1) \Big|_{a=0}$$

$$= \Gamma''(1)$$

$$\mathrm{Var}[X] = \Gamma''(1) - [\Gamma'(1)]^2.$$

Q8

$$F(x) = [1 - e^{-x}]^{\alpha}$$

$$F(x) = 1 \Rightarrow [1 - e^{-x}]^{\alpha} = 1$$

$$\Rightarrow 1 - e^{-x} = 1$$

$$\Rightarrow e^{-x} = 0 \Rightarrow x = +\infty$$

$$\Rightarrow \omega(F) = +\infty$$

I:
$$\lim_{t \uparrow \infty} \frac{1 - [1 - e^{-t - x\gamma(t)}]^{\alpha}}{1 - [1 - e^{-t}]^{\alpha}}$$

($e^{-t-x\gamma(t)} \to 0$, $e^{-t} \to 0$)

$$(1 - a)^{\alpha} \approx 1 - a\alpha \quad \text{as} \quad a \to 0$$

$$= \lim_{t \uparrow \infty} \frac{1 - [1 - \alpha e^{-t - x\gamma(t)}]}{1 - [1 - \alpha e^{-t}]}$$

$$= \lim_{t \uparrow \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t \uparrow \infty} e^{-x\gamma(t)} = e^{-x}$$

if $\gamma(t) \equiv 1$

$\Rightarrow$ Cond (I) is satisfied. There exists $a_n > 0$ & $b_n \in \mathbb{R}$ such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \to e^{-e^{-x}}.$$

Q9

$$F(x) = x, \quad 0 < x < 1$$

$$w(F) = 1$$

$$\lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)}$$

$$= \lim_{t \downarrow 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)}$$

$$= \lim_{t \downarrow 0} \frac{1 - (1 - tx)}{1 - (1 - t)} = \lim_{t \downarrow 0} \frac{tx}{t}$$

$$= x$$

cond (III) is satisfied with $\alpha = 1$.

There exists $a_n > 0$ & $b_n \in \mathbb{R}$ such that

$$P\left(\frac{M_n - b_n}{a_n} < x \right) \to \begin{cases} e^{x} & x \leq 0 \\ 1 & x > 0 \end{cases}$$

LECTURE

3 OCTOBER

9:00-10:00AM

MATH3/4/68181

More Examples on ETT

eg 1 $F(x) = e^{-\frac{1}{x}}$, $x > 0$. Find the max domain of attraction.

Sol:

$$F(x) = 1$$

$$\Rightarrow e^{-\frac{1}{x}} = 1$$

$$\Rightarrow -\frac{1}{x} = 0$$

$$\Rightarrow x = +\infty \Rightarrow w(F) = +\infty.$$

$$I: \lim_{t \to \infty} \frac{1 - e^{-\frac{1}{t + x\gamma(t)}}}{1 - e^{-\frac{1}{t}}}$$

$\frac{1}{t + x\gamma(t)} \to 0$, $\frac{1}{t} \to 0$

$e^{-y} \approx 1 - y$ as $y \to 0$

$$= \lim_{t \to \infty} \frac{1 - \left[1 - \frac{1}{t + x\gamma(t)}\right]}{1 - \left[1 - \frac{1}{t}\right]}$$

$$= \lim_{t \to \infty} \frac{t}{t + x\gamma(t)}$$

$\neq e^{-x} \Rightarrow$ Cond (I) not satisfied

II :

$$\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{1 - e^{-\frac{1}{tx}}}{1 - e^{-\frac{1}{t}}}$$

$$= \lim_{t \uparrow \infty} \frac{\cancel{1} - (\cancel{1} - \frac{1}{tx})}{\cancel{1} - (\cancel{1} - \frac{1}{t})}$$

$$= \frac{1}{x}$$

$e^{-y} \approx 1 - y$ as $y \to 0$

$\Rightarrow$ cond (II) satisfied with $\alpha = 1$

There exists $a_n > 0$ & $b_n \in \mathbb{R}$ such that

$$P\left(\frac{M_n - b_n}{a_n} < x \right) \to \begin{cases} e^{-x^{-1}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

as $n \to \infty$.

L' Hôpital's Rule

$$\lim_{t \to \infty} \frac{f_1(x)}{f_2(x)} = \lim_{t \to \infty} \frac{f_1'(x)}{f_2'(x)}$$

eg 2 Find the max domain of attraction of $F(x) = \Phi(x)$, the cdf of $N(0,1)$.

$$F(x) = 1 \Rightarrow \Phi(x) = 1$$
$$\Rightarrow x = \Phi^{-1}(1) = +\infty.$$
$$\Rightarrow w(F) = +\infty.$$

(I) : $\lim\limits_{t \uparrow \infty} \dfrac{1 - \Phi(t + x\gamma(t))}{1 - \Phi(t)}$ (numerator $\to 0$, denominator $\to 0$)

$$\overset{LH}{=} \lim_{t \uparrow \infty} \frac{-\phi(t + x\gamma(t))\,(1 + x\gamma'(t))}{-\phi(t)}$$

ϕ is the PDF of $N(0,1)$

$$= \lim_{t \uparrow \infty} \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(t + x\gamma(t))^2}{2}} (1 + x\gamma'(t))}{\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}}$$

$$= \lim_{t \uparrow \infty} e^{\frac{t^2}{2} - \frac{(t + x\gamma(t))^2}{2}} (1 + x\gamma'(t))$$

$$= \lim_{t \uparrow \infty} e^{-xt\gamma(t) - \frac{x^2\gamma^2(t)}{2}} (1 + x\gamma'(t))$$

Choose $\gamma(t) = \frac{1}{t}$, $\gamma'(t) = -\frac{1}{t^2}$

$$= \lim_{t \uparrow \infty} e^{-x - \frac{x^2}{2t^2}} \left(1 - \frac{x}{t^2}\right)$$

($\frac{x^2}{2t^2} \to 0$, $\frac{x}{t^2} \to 0$)

$$= e^{-x}$$

$\Rightarrow$ cond (I) satisfied

Hence there exist $a_n > 0$ & $b_n \in \mathbb{R}$ such that

$$P\left(\frac{M_n - b_n}{a_n} < x \right) \to e^{-e^{-x}}$$

as $n \to \infty$.

Example where ETT fails

$$F(x) = 1 - \frac{1}{\log x}, \quad x > e$$

$$F(x) = 1 \Rightarrow 1 - \frac{1}{\log x} = 1$$

$$\Rightarrow -\frac{1}{\log x} = 0 \Rightarrow x = +\infty \Rightarrow w(F) = +\infty$$

(I):

$$\lim_{t \uparrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{1 - \left[1 - \frac{1}{\log(t + x\gamma(t))}\right]}{1 - \left[1 - \frac{1}{\log t}\right]}$$

$$= \lim_{t \uparrow \infty} \frac{\log t}{\log(t + x\gamma(t))} \quad \begin{matrix} \log t \to +\infty \\ \log(t + x\gamma(t)) \to +\infty \end{matrix}$$

$$\overset{LH}{=} \lim_{t \uparrow \infty} \frac{\frac{1}{t}}{\frac{1 + x\gamma'(t)}{t + x\gamma(t)}}$$

$$= \lim_{t \uparrow \infty} \frac{t + x\gamma(t)}{t(1 + x\gamma'(t))} \neq e^{-x}$$

Cond (I) not satisfied

(II) :

$$\lim_{t \uparrow \infty} \frac{1 - \left[1 - \frac{1}{\log(tx)}\right]}{1 - \left[1 - \frac{1}{\log t}\right]}$$

$$= \lim_{t \uparrow \infty} \frac{\log t}{\log(tx)}$$

$$= \lim_{t \uparrow \infty} \frac{\log t}{\log t + \log x}$$

$$= \lim_{t \uparrow \infty} \frac{1}{1 + \frac{\log x}{\log t}} \quad \left(\frac{\log x}{\log t} \Rightarrow 0\right)$$

$= 1 \Rightarrow$ cond (II) not satisfied

(III) : clearly not satisfied since $w(F) = +\infty$

Hence ETT fails.

Ex 11

$$f(x) = K\, g(x) \left[G(x)\right]^{a-1} \left[1 - G(x)\right]^{b-1} e^{-c\,G(x)}$$

(a) Assume G belongs to Gumbel limit
& show F also " " " "

(b) Assume G belongs to Frechet limit
& show F also " " " "

(c) Assume G belongs to Weibull limit
& show F also " " " "

(a) Assume G belongs to Gumbel domain, That is, there exists $\gamma(t) > 0$ such that

$$\lim_{t \uparrow w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x} \quad (*)$$

We need to find

$$\lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$\overset{LH}{=} \lim_{t \uparrow w(F)} \frac{-f(t + x\gamma(t))\,(1 + x\gamma'(t))}{-f(t)}$$

$(1+x\gamma'(t))$

$$= \lim_{t \uparrow w(F)} \frac{k\, g(t + x\gamma(t))\,[G(t + x\gamma(t))]^{a-1}\,[1 - G(t + x\gamma(t))]^{b-1}\, e^{-cG(t+x\gamma(t))}}{k\, g(t)\,[G(t)]^{a-1}\,[1 - G(t)]^{b-1}\, e^{-cG(t)}}$$

$w(F) = w(G)$

$$= \lim_{t \uparrow w(G)} \frac{g(t + x\gamma(t))\,(1 + x\gamma'(t))}{g(t)}$$

$$\cdot \left[\frac{G(t + x\gamma(t))}{G(t)}\right]^{a-1} \quad G(t + x\gamma(t)) \to 1, \quad G(t) \to 1$$

$$\cdot \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b-1}$$

$$\cdot\ e^{c\,G(t) - c\,G(t + x\gamma(t))} \quad G(t) \to 1, \quad G(t + x\gamma(t)) \to 1$$

$$= \lim_{t \uparrow w(G)} \frac{g(t + x\gamma(t))\,(1 + x\gamma'(t))}{g(t)} \cdot \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)} \right]^{b-1}$$

$\to e^{-x}$

$$\overset{*}{=} \lim_{t \uparrow w(G)} \frac{g(t + x\gamma(t))\,(1 + x\gamma'(t))}{g(t)} \cdot \left(e^{-x}\right)^{b-1}$$

$$\overset{LH}{=} \lim_{t \uparrow w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} \cdot e^{-(b-1)x}$$

$$\overset{*}{=} e^{-x} \cdot e^{-(b-1)x} = e^{-bx},$$

same type as e^{-x}.

Hence F also belongs to the Gumbel domain.

By definition,

$$\begin{array}{ccc} G(\underline{\omega(G)}) = 1 & & \\ \wedge & & \\ G(\omega(G) + x\,\underbrace{\gamma(\omega(G))}) = 1 & & \end{array}$$

$$\begin{array}{cc} x & \gamma(\omega(G)) \\ \vee & \vee \\ 0 & 0 \end{array}$$

$$\boxed{x < y \Rightarrow G(x) \leq G(y)}$$

EXAMPLE CLASS

3 OCTOBER

10:00-11:00AM

MATH3/4/68181

Q1

$$\Lambda(x) = e^{-e^{-x}}$$

$$\frac{d\Lambda(x)}{dx} = e^{-e^{-x}} \cdot (-1) \cdot e^{-x} (-1)$$

$$= e^{-x} e^{-e^{-x}}$$

$$\Phi_\alpha(x) = \begin{cases} e^{-x^{-\alpha}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\frac{d\Phi_\alpha(x)}{dx} = \begin{cases} e^{-x^{-\alpha}} (-1)(-\alpha)\, x^{-\alpha-1}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha} & x \leq 0 \\ 1 & x > 0 \end{cases}$$

$$\frac{d\Psi_\alpha(x)}{dx} = \begin{cases} e^{-(-x)^\alpha} (-1)\, \alpha (-x)^{\alpha-1} (-1), & x \leq 0 \\ 0, & x > 0 \end{cases}$$

Q2

$$E(X) = \int_{-\infty}^{\infty} x \cdot e^{-e^{-x}} \cdot e^{-x} \, dx$$

$$y = e^{-x} \Rightarrow x = -\log y \Rightarrow \frac{dx}{dy} = -\frac{1}{y}$$

$$= \int_{\infty}^{0} (-\log y) \cdot e^{-y} \, y \left(-\frac{1}{y}\right) dy$$

$$= -\int_{0}^{\infty} \log y \; e^{-y} \, dy$$

$$= -\int_{0}^{\infty} \left(\frac{d}{da} y^{a} \Big|_{a=0} \right) e^{-y} \, dy$$

$$= -\frac{d}{da} \left(\int_{0}^{\infty} y^{a} e^{-y} \, dy \right) \Big|_{a=0}$$

$$= -\frac{d}{da} \Gamma(a+1) \Big|_{a=0} = -\Gamma'(1)$$

Q3

$$Var[X] = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot e^{-e^{-x}} e^{-x} \, dx$$

$$y = e^{-x} \Rightarrow x = -\log y \Rightarrow \frac{dx}{dy} = -\frac{1}{y}$$

$$= \int_{\infty}^{0} (\log y)^2 e^{-y} y \left(-\frac{1}{y}\right) dy$$

$$= \int_{0}^{\infty} (\log y)^2 e^{-y} dy$$

$$= \int_{0}^{\infty} \left(\frac{d^2}{da^2} y^a \Big|_{a=0} \right) e^{-y} dy$$

$$= \frac{d^2}{da^2} \left(\int_0^{\infty} y^a e^{-y} dy \right) \Big|_{a=0}$$

$$= \frac{d^2}{da^2} \Gamma(a+1) \Big|_{a=0} = \Gamma''(1)$$

$$Var[X] = \Gamma''(1) - [\Gamma'(1)]^2 .$$

$$\frac{d}{da} y^{a} = y^{a} \log y$$

$\boxed{a=0}$:
$$\left.\frac{d}{da} y^{a}\right|_{a=0} = y^{0} \log y$$
$$= \log y$$

$$\boxed{\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\, dt}$$

Gamma function

$$\frac{d^2}{da^2} y^{a} = \frac{d}{da}\left[y^{a} \log y\right] = y^{a} (\log y)^2$$

$\boxed{a=0}$:
$$\left.\frac{d^2}{da^2} y^{a}\right|_{a=0} = y^{0} (\log y)^2 = (\log y)^2$$

Q8

$$F(x) = [1 - e^{-x}]^{\alpha}$$

$$F(x) = 1 \Rightarrow [1-e^{-x}]^{\alpha} = 1$$
$$\Rightarrow 1 - e^{-x} = 1$$
$$\Rightarrow e^{-x} = 0$$
$$\Rightarrow x = +\infty \Rightarrow \omega(F) = +\infty$$

(I)

$$\lim_{t \uparrow \infty} \frac{1 - [1 - e^{-t - x\gamma(t)}]^{\alpha}}{1 - [1 - e^{-t}]^{\alpha}}$$

($e^{-t-x\gamma(t)} \to 0$, $e^{-t} \to 0$)

$$(1-a)^{\alpha} \approx 1 - a\alpha \quad \text{as} \quad a \to 0$$

$$= \lim_{t \uparrow \infty} \frac{1 - [1 - \alpha \cdot e^{-t - x\gamma(t)}]}{1 - [1 - \alpha \cdot e^{-t}]}$$

$$= \lim_{t \uparrow \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t \to \infty} e^{-x\gamma(t)} = e^{-x}$$

if $\gamma(t) \equiv 1$

Cond (I) is satisfied. Hence there exist $a_n > 0$ & $b_n \in \mathbb{R}$ such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \to e^{-e^{-x}}$$

LECTURE

6 OCTOBER

9:00-10:00AM

MATH3/4/68181

Ex 11

$$f(x) = K\, g(x) \left[G(x)\right]^{a-1} \left[1 - G(x)\right]^{b-1} e^{-c\,G(x)}$$

(a) Assume G belongs to Gumbel limit
& show F also " " " "

(b) Assume G belongs to Frechet limit
& show F also " " " "

(c) Assume G belongs to Weibull limit
& show F also " " " "

(a) Assume G belongs to Gumbel domain. That is, there exists $\gamma(t) > 0$ such that

$$\lim_{t \uparrow w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x} \quad (*)$$

We need to find

$$\lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$\overset{LH}{=} \lim_{t \uparrow w(F)} \frac{-f(t + x\gamma(t))\,(1 + x\gamma'(t))}{-f(t)}$$

$$= \lim_{t \uparrow w(F)} \frac{k\, g(t + x\gamma(t)) \left[G(t + x\gamma(t))\right]^{a-1} \left[1 - G(t + x\gamma(t))\right]^{b-1} e^{-cG(t+x\gamma(t))} \, (1 + x\gamma'(t))}{k\, g(t) \left[G(t)\right]^{a-1} \left[1 - G(t)\right]^{b-1} e^{-cG(t)}}$$

$w(F) = w(G)$

$$= \lim_{t \uparrow w(G)} \frac{g(t + x\gamma(t))\,(1 + x\gamma'(t))}{g(t)}$$

- $\left[\dfrac{G(t + x\gamma(t))}{G(t)}\right]^{a-1}$ — $G(t + x\gamma(t)) \to 1$, $G(t) \to 1$

- $\left[\dfrac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b-1}$

- $e^{cG(t) - cG(t + x\gamma(t))}$ — $G(t) \to 1$, $G(t + x\gamma(t)) \to 1$

$$= \lim_{t \uparrow w(G)} \frac{g(t + x\gamma(t))\,(1 + x\gamma'(t))}{g(t)} \cdot \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)} \right]^{b-1}$$

$\to e^{-x}$

$$\overset{(*)}{=} \lim_{t \uparrow w(G)} \frac{g(t + x\gamma(t))\,(1 + x\gamma'(t))}{g(t)} \cdot \left(e^{-x}\right)^{b-1}$$

$$\overset{LH}{=} \lim_{t \uparrow w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} \cdot e^{-(b-1)x}$$

$$\overset{(*)}{=} e^{-x} \cdot e^{-(b-1)x} = e^{-bx},$$

same type as e^{-x}.

Hence F also belongs to the Gumbel domain.

By defn of $w(G)$,

$$G(w(G)) = 1.$$

Also

$$\underbrace{G\Big(w(G) + \underbrace{x}_{\geq 0} \cdot \underbrace{\gamma(w(G))}_{\geq 0}\Big)}$$

$$\geq$$

$$G(w(G))$$

$$=$$

$$1$$

$$\Rightarrow \quad G\big(w(G) + x \cdot \gamma(w(G))\big) = 1.$$

(b) Assume G belongs to Frechet domain. that is

$$\omega(G) = \infty \quad , \quad \lim_{t \uparrow \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{-\alpha} \quad \cdots \circledast$$

We want to find

$$\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)}$$

($F(tx) \to 1$, $F(t) \to 1$)

$$\overset{LH}{=} \lim_{t \uparrow \infty} \frac{-x f(tx)}{-f(t)}$$

$$= \lim_{t \uparrow \infty} \frac{x \cdot k\, g(tx) [G(tx)]^{a-1} [1 - G(tx)]^{b-1} e^{-cG(tx)}}{k \cdot g(t) [G(t)]^{a-1} [1 - G(t)]^{b-1} e^{-cG(t)}}$$

$$= \lim_{t \uparrow \infty} \left[\frac{x\, g(tx)}{g(t)}\right] \cdot \left[\frac{G(tx)}{G(t)}\right]^{a-1} \cdot \left[\frac{1 - G(tx)}{1 - G(t)}\right]^{b-1} e^{cG(t) - cG(tx)}$$

($G(tx) \to 1$, $G(t) \to 1$, $G(t) \to 1$, $G(tx) \to 1$)

$$= \lim_{t \uparrow \infty} \left[\frac{x\, g(tx)}{g(t)}\right] \left[\frac{1 - G(tx)}{1 - G(t)}\right]^{b-1}$$

$$\overset{\circledast}{=} \lim_{t \uparrow \infty} \frac{x\, g(tx)}{g(t)} \cdot \left(x^{-\alpha}\right)^{b-1}$$

$$\overset{LH}{=} \lim_{t \uparrow \infty} \frac{1 - G(tx)}{1 - G(t)} \cdot \left(x^{-\alpha}\right)^{b-1}$$

$$\overset{\circledast}{=} \left(x^{-\alpha}\right) \cdot \left(x^{-\alpha}\right)^{b-1} = x^{-b\alpha}$$

$x^{-b\alpha}$ is of the same type as $x^{-\alpha}$.

Hence F also belongs to the Fréchet domain.

(c) Assume G belongs to Weibull domain. That is

$$w(G) < \infty, \quad \lim_{t \downarrow 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^{\alpha} \quad \cdots (*)$$

We want to find

$$\lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)}$$

(red annotations: $F(w(F) - tx) \to 1$, $F(w(F) - t) \to 1$)

$$\stackrel{LH}{=} \lim_{t \downarrow 0} \frac{x f(w(F) - tx)}{f(w(F) - t)}$$

$$\stackrel{w(F) = w(G)}{=} \lim_{t \downarrow 0} \frac{x f(w(G) - tx)}{f(w(G) - t)}$$

$$= \lim_{t \downarrow 0} \frac{x \cdot \cancel{k}\, g(w(G) - tx) [G(w(G) - tx)]^{a-1} [1 - G(w(G) - tx)]^{b-1}}{\cancel{k} \cdot g(w(G) - t) [G(w(G) - t)]^{a-1} [1 - G(w(G) - t)]^{b-1}} \cdot \frac{e^{-c\, G(w(G) - tx)}}{e^{-c\, G(w(G) - t)}}$$

(red annotations: $G(w(G) - tx) \to 1$, $G(w(G) - t) \to 1$)

$$= \lim_{t \downarrow 0} \frac{x\, g(w(G) - tx)}{g(w(G) - t)} \cdot \left[\frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)}\right]^{b-1}$$

$$= \lim_{t \downarrow 0} \frac{x\, g(w(G) - tx)}{g(w(G) - t)} \cdot (x^{\alpha})^{b-1}$$

$$\overset{LH}{=} \lim_{t \downarrow 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} \cdot (x^{\alpha})^{b-1}$$

$$\overset{(*)}{=} x^{\alpha} \cdot (x^{\alpha})^{b-1} = x^{b\alpha},$$

the same type as x^{α}.

Hence F also belongs to the Weibull domain.

F belongs to the same domain of attraction as G.

The ETT

For a practitioner, it is difficult to deal with the 3 types. Can the 3 limits be combined into one form?

The GEV distribution.

(Generalized Extreme Value)

It has the CDF

$$G(x) = e^{-\left(1 + \xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$-\infty < \mu < +\infty$ location parameter

$\sigma > 0$ scale ″

$-\infty < \xi < +\infty$ shape ″

$$1 + \xi \frac{x-\mu}{\sigma} > 0$$

Special cases of GEV

a) $\boxed{\xi = 0}$

$$\lim_{\xi \to 0} G(x) = \lim_{\xi \to 0} e^{-\left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= \lim_{\xi \to 0} e^{-\left[\left[1+\frac{\frac{x-\mu}{\sigma}}{\frac{1}{\xi}}\right]^{\frac{1}{\xi}}\right]^{-1}}$$

[set $m = \frac{1}{\xi}$]

$$= \lim_{m \to \infty} e^{-\left[\left[1+\frac{\frac{x-\mu}{\sigma}}{m}\right]^{m}\right]^{-1}}$$

$$\left[\lim_{m \to \infty} \left(1+\frac{z}{m}\right)^{m} = e^{z}\right]$$

$$= e^{-\left(e^{\frac{x-\mu}{\sigma}}\right)^{-1}}$$

$$= e^{-e^{-\frac{x-\mu}{\sigma}}},$$

the <u>same type</u> as $\Lambda(x)$

b) $\boxed{\xi > 0}$

$$G(x) = e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= e^{-\left(\frac{\xi x}{\sigma} + 1 - \frac{\xi\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

($\frac{\xi}{\sigma} = a$, $1 - \frac{\xi\mu}{\sigma} = b$)

$$= e^{-(ax+b)^{-\frac{1}{\xi}}}$$

the <u>same</u> <u>type</u> as $\boxed{e^{-x^{-\frac{1}{\xi}}}}$ $= \Phi_{\frac{1}{\xi}}(x)$

$a > 0$

$b \in \mathbb{R}$

c) $\boxed{\xi < 0}$

$$G(x) = e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= e^{-\left(\frac{\xi x}{\sigma} + 1 - \frac{\xi\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= e^{-\left(-\left(-\frac{\xi x}{\sigma} + \frac{\xi\mu}{\sigma} - 1\right)\right)^{-\frac{1}{\xi}}}$$

$-\frac{\xi x}{\sigma} = a$, $\frac{\xi\mu}{\sigma} - 1 = b$

$$= e^{-(-(ax+b))^{-\frac{1}{\xi}}} \quad , \; a > 0, \; b \in \mathbb{R}$$

the same type as

$$e^{-(-x)^{-\frac{1}{\xi}}} = \Psi_{-\frac{1}{\xi}}(x)$$

PDF

$$g(x) = \frac{1}{\sigma}\left(1 + \xi\,\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \cdot e^{-\left(1+\xi\cdot\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

quantile

Set $G(x) = p$

$$\Rightarrow e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} = p$$

$$\Rightarrow \left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} = -\log p$$

$$\Rightarrow 1+\xi\frac{x-\mu}{\sigma} = (-\log p)^{-\xi}$$

$$\Rightarrow x = \mu + \frac{\sigma}{\xi}\left[(-\log p)^{-\xi} - 1\right],$$

pth quantile

Likelihood

Data: $x_1, x_2, \ldots, x_n$

$$L(\mu, \sigma, \xi) = \prod_{i=1}^{n} g(x_i)$$

$$= \frac{1}{\sigma^n} \left[\prod_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right) \right]^{-\frac{1}{\xi} - 1}$$

$$\cdot e^{-\sum_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$\log L = -n \log \sigma$$

$$- \left(\frac{1}{\xi} + 1\right) \sum_{i=1}^{n} \log \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)$$

$$- \sum_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi}}$$

MLEs

The MLEs of μ, σ & ξ are the solutions of

$$\frac{\partial \log L}{\partial \mu} = 0$$

$$\frac{\partial \log L}{\partial \sigma} = 0$$

$$\frac{\partial \log L}{\partial \xi} = 0$$

MLE equations for the GEV distribution

The MLEs of μ, σ and ξ are the simultaneous solutions of

$$
\begin{aligned}
\frac{\partial \log L}{\partial \mu} &= \frac{1+\xi}{\sigma} \sum_{i=1}^{n} \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-1} \\
&\quad - \frac{1}{\sigma} \sum_{i=1}^{n} \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \log L}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{1+\xi}{\sigma^2} \sum_{i=1}^{n} (x_i-\mu) \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-1} \\
&\quad - \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i-\mu) \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \log L}{\partial \xi} &= \frac{1}{\xi^2} \sum_{i=1}^{n} \log \left(1+\xi \frac{x_i-\mu}{\sigma}\right) \\
&\quad - \frac{1+\xi}{\xi\sigma} \sum_{i=1}^{n} (x_i-\mu) \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-1} \\
&\quad - \frac{1}{\xi^2} \sum_{i=1}^{n} \log \left(1+\xi \frac{x_i-\mu}{\sigma}\right) \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}} \\
&\quad + \frac{1}{\xi\sigma} \sum_{i=1}^{n} (x_i-\mu) \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\
&= 0.
\end{aligned}
$$

fgev in R that can return the MLEs.

EXAMPLE CLASS

9 OCTOBER

12:00-13:00PM

MATH3/4/68181

Q1

$$L(\sigma) = \prod_{i=1}^{n} \left[\frac{1}{\sigma} e^{-\frac{x_i}{\sigma}} e^{-e^{-\frac{x_i}{\sigma}}} \right]$$

$$= \frac{1}{\sigma^n} e^{-\frac{1}{\sigma}\sum_{i=1}^{n} x_i} \, e^{-\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}}$$

$$\log L(\sigma) = -n \log \sigma - \frac{1}{\sigma}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}$$

$$\frac{d \log L}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}} \cdot \frac{(-x_i)(-1)}{\sigma^2}$$

$$= -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{1}{\sigma^2}\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}} \cdot x_i$$

The MLE of σ is the root of

$$-\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{1}{\sigma^2}\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}} \cdot x_i = 0$$

Solve this using a numerical routine.

Q2

$$L(\lambda, \sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{\lambda} x_i^{-\lambda-1} e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}} \right]$$

$$= \lambda^n \sigma^{n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{-\lambda-1} e^{-\sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda}}$$

$$\log L(\lambda, \sigma) = n \log \lambda + n\lambda \log \sigma - (\lambda+1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n \log \sigma - \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda} \log \left(\frac{\sigma}{x_i}\right) = 0 \quad \text{— (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} - \sum_{i=1}^{n} \frac{\lambda \sigma^{\lambda-1}}{x_i^{\lambda}} = 0 \quad \text{— (2)}$$

(2) $\Rightarrow$ $\frac{n}{\sigma^{\lambda}} = \sum_{i=1}^{n} x_i^{-\lambda}$

$$\Rightarrow \sigma = \left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \right]^{\frac{1}{\lambda}} \quad \text{—— (3)}$$

Sub (3) into (1) gives

$$\frac{n}{\lambda} + \frac{n}{\lambda} \log \left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \right] - \sum_{i=1}^{n} \log x_i - \left[\frac{n}{\sum x_i^{-\lambda}} \right] \sum_{i=1}^{n} x_i^{-\lambda} \log \left(\frac{\sigma}{x_i}\right) = 0$$

$$\Rightarrow \frac{n}{\lambda} + \frac{n}{\lambda} \log\left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}}\right] - \sum_{i=1}^{n} \log x_i$$

$$- \frac{1}{\lambda}\left(\log\left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}}\right]\right)\left(\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}}\right) \cdot \left(\sum_{i=1}^{n} x_i^{-\lambda}\right)$$

$$+ \frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \sum_{i=1}^{n} x_i^{-\lambda} \log x_i = \underline{0} \quad (4)$$

$\hat{\lambda}$ is the root of (4)

Sub into eq (3) to get $\hat{\sigma}$.

Q3

$$L(\lambda, \sigma) = \prod_{i=1}^{n} \left[\frac{\lambda}{\sigma^{\lambda}} x_i^{\lambda-1} e^{-\left(\frac{x_i}{\sigma}\right)^{\lambda}} \right]$$

$$= \frac{\lambda^n}{\sigma^{n\lambda}} \left(\prod_{i=1}^{n} x_i \right)^{\lambda-1} e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}}$$

$$\log L(\lambda, \sigma) = n \log \lambda - n\lambda \log \sigma + (\lambda - 1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - n \log \sigma + \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda} \log\left(\frac{x_i}{\sigma}\right) = 0 \quad \text{— (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n\lambda}{\sigma} + \lambda \sum_{i=1}^{n} \frac{x_i^{\lambda}}{\sigma^{\lambda+1}} = 0 \quad \text{— (2)}$$

(2) $\Rightarrow$ $$\sigma = \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right]^{\frac{1}{\lambda}} \quad \text{— (3)}$$

Sub (3) into (1)

$$\frac{n}{\lambda} - \frac{n}{\lambda} \log \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right] + \sum_{i=1}^{n} \log x_i - \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right]^{-1} \sum_{i=1}^{n} x_i^{\lambda} \log x_i$$

$$+ \frac{1}{\lambda} \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right]^{-1} \log \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right] \sum_{i=1}^{n} x_i^{\lambda} = 0 \quad \text{— (4)}$$

$\hat{\lambda}$ is the root of (4).

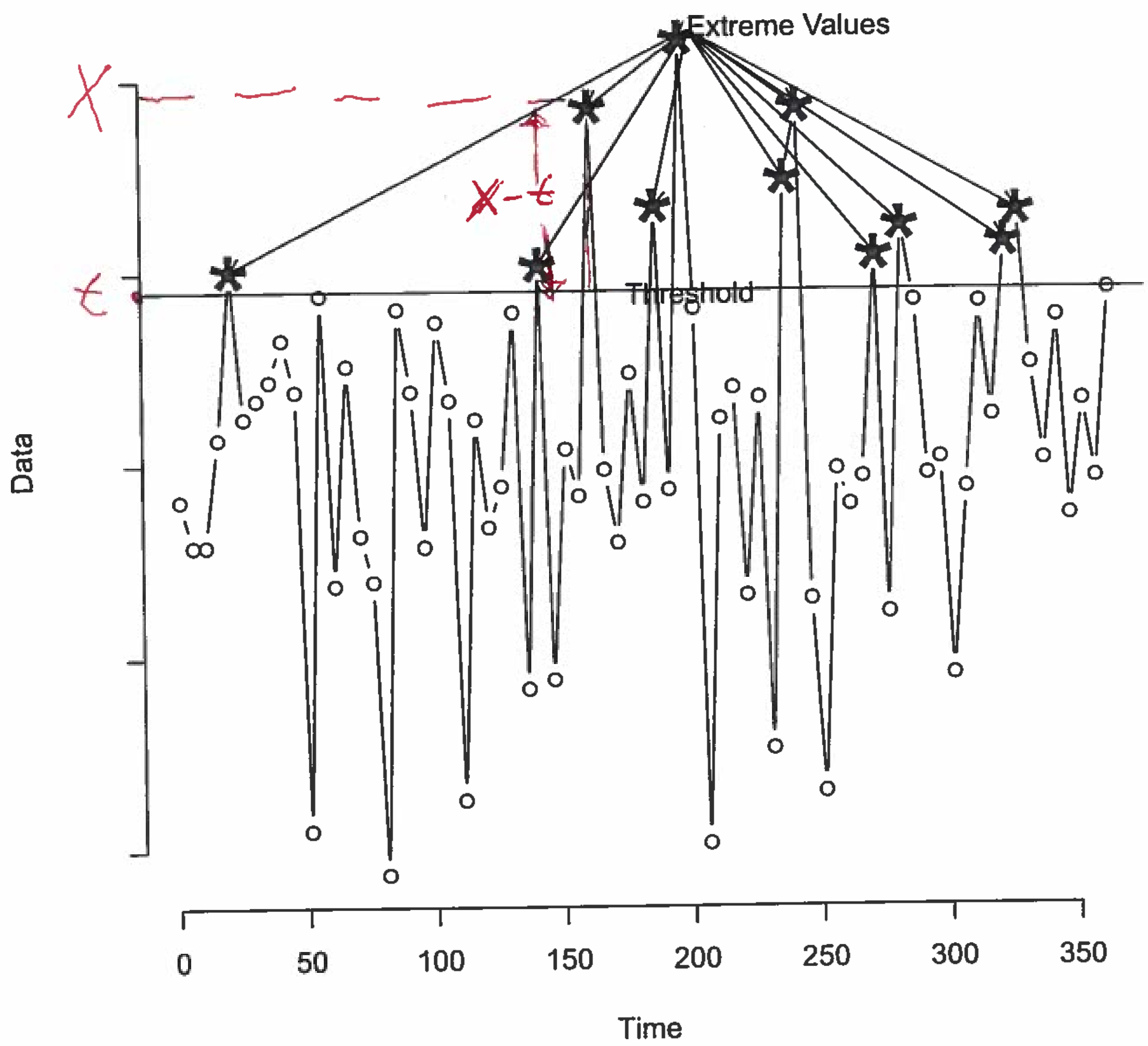
Sub into (3) to get $\hat{\sigma}$.

LECTURE

10 OCTOBER

9:00-10:00AM

MATH3/4/68181



DEFINITION 2

Excess = $X - t$

$$P(\underbrace{X - t}_{\text{Excess}} > x \mid X > t)$$

$$= \frac{P(X - t > x,\ X > t)}{P(X > t)}$$

$$= \frac{P(X > t + x,\ X > t)}{P(X > t)}$$

$$= \frac{P(X > t + x)}{P(X > t)} = \frac{1 - P(X \leq t + x)}{1 - P(X \leq t)}$$

$$= \frac{1 - F(t + x)}{1 - F(t)}$$

F cdf of X

$\downarrow$ as $t \to \omega(F)$

$$\left(1 + \xi \frac{x}{\sigma}\right)^{-\frac{1}{\xi}}$$

due to James Pickand (1975).

If t is large enough,

$$\frac{1-F(t+x)}{1-F(t)} \approx \left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$\Rightarrow 1-F(t+x) \approx \underbrace{[1-F(t)]}_{=p}\left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$\Rightarrow F(\underbrace{t+x}_{=y}) \approx 1-p\left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$\Rightarrow \boxed{F(y) \approx 1-p\left(1+\xi\frac{y-t}{\sigma}\right)^{-\frac{1}{\xi}}}$$

<u>Generalized</u> <u>Pareto</u> <u>distribution</u>

Special case

$\boxed{\xi = 0}$

$$F(y) = 1 - p \lim_{\xi \to 0} \left(1 + \xi \frac{y-t}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$= 1 - p \cdot \lim_{\xi \to 0} \left\{ \left[1 + \frac{\frac{y-t}{\sigma}}{\frac{1}{\xi}} \right]^{\frac{1}{\xi}} \right\}^{-1}$$

$$\boxed{m = \frac{1}{\xi}}$$

$$= 1 - p \lim_{m \to \infty} \left(\left(1 + \frac{(y-t)/\sigma}{m}\right)^{m} \right)^{-1}$$

$$= 1 - p \cdot \left(e^{\frac{y-t}{\sigma}}\right)^{-1}$$

$$\left(1 + \frac{z}{m}\right)^{m} \to e^{z}$$

$$\boxed{= 1 - p\, e^{\frac{-y+t}{\sigma}}}$$

Exponential CDF

PDF

$$f(y) = \frac{1}{\sigma}\left(1 + \xi \cdot \frac{y-t}{\sigma}\right)^{-\frac{1}{\xi}-1}$$

$\sigma > 0$ scale parameter

$-\infty < \xi < +\infty$ shape "

$$1 + \xi \frac{y-t}{\sigma} > 0$$

Quantile

$$F(y) = q \,, \quad 0 < q < 1$$

$$\Rightarrow 1 - p\left(1 + \xi \frac{y-t}{\sigma}\right)^{-\frac{1}{\xi}} = q$$

$$\Rightarrow \left(1 + \xi \frac{y-t}{\sigma}\right)^{-\frac{1}{\xi}} = \frac{1-q}{p}$$

$$\Rightarrow 1 + \xi \frac{y-t}{\sigma} = \left(\frac{1-q}{p}\right)^{-\xi}$$

$$\Rightarrow y = t + \frac{\sigma}{\xi}\left[\left(\frac{1-q}{p}\right)^{-\xi} - 1\right]$$

q th quantile

Estimation

Data: $x_1, x_2, \ldots, x_n$

$$L(\sigma, \xi) = \prod_{i=1}^{n} \frac{p}{\sigma}\left(1+\xi\,\frac{x_i-t}{\sigma}\right)^{-\frac{1}{\xi}-1}$$

$$= \frac{p^n}{\sigma^n}\left[\prod_{i=1}^{n}\left(1+\xi\,\frac{x_i-t}{\sigma}\right)\right]^{-\frac{1}{\xi}-1}$$

$$\log L(\sigma, \xi) = n\log p - n\log\sigma$$

$$-\left(1+\frac{1}{\xi}\right)\sum_{i=1}^{n}\log\left(1+\xi\,\frac{x_i-t}{\sigma}\right)$$

The MLEs of σ and ξ will be the solutions of

$$\frac{\partial \log L}{\partial \sigma} = 0$$

$$\frac{\partial \log L}{\partial \xi} = 0$$

MLE equations for the GP distribution

The MLEs of σ and ξ are the simultaneous solutions of

$$\begin{aligned}\frac{\partial \log L}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{1+\xi}{\sigma^2}\sum_{i=1}^{n}(x_i - t)\left(1+\xi\frac{x_i - t}{\sigma}\right)^{-1} \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \xi} &= \frac{1}{\xi^2}\sum_{i=1}^{n}\log\left(1+\xi\frac{x_i - t}{\sigma}\right) \\ &\quad -\frac{1+\xi}{\xi\sigma}\sum_{i=1}^{n}(x_i - t)\left(1+\xi\frac{x_i - t}{\sigma}\right)^{-1} \\ &= 0.\end{aligned}$$

fpot in R solves these to give you $\hat{\sigma}$ and $\hat{\xi}$.

Definition 1	}	Level	3/4/6
" 2			
Definition 3	}	Level	4/6

Conditions

I: $P\left(\frac{M_n - b_n}{a_n} < x\right) \longrightarrow \Lambda(x)$

if there exists $\gamma(t) > 0$

such that $\lim\limits_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = e^{-x}$.

II: $P\left(\frac{M_n - b_n}{a_n} < x\right) \longrightarrow \Phi_\alpha(x)$

if $w(F) = \infty$, $\lim\limits_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}$

III: $P\left(\frac{M_n - b_n}{a_n} < x\right) \longrightarrow \Psi_\alpha(x)$

if $w(F) < \infty$, $\lim\limits_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^{\alpha}$

If any only one of (I) - (III) will be satisfied

Is there a way to find out that none of (I) – (III) will hold? Yes.

Discrete Case

None of the conditions will be satisfied (ETT will fail) if

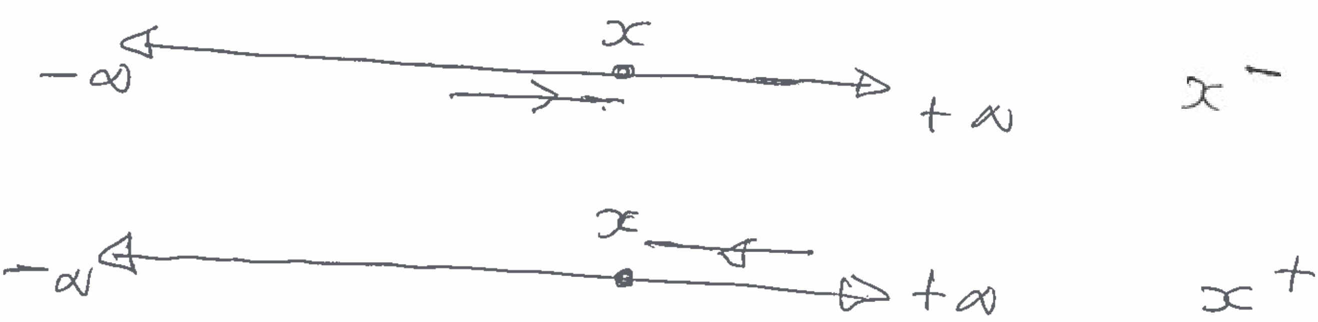
PMF → $P(X=k)$, CDF → F

$$\frac{P(X=k)}{1-F(k-1)} \not\to 0 \quad \text{as } k \to w(F).$$

Continuous Case

PDF → f, CDF → F

$$\frac{f(x)}{1-F(x^-)} \not\to 0 \quad \text{as } x \to w(F)$$



eg 1

$$P(X=k) = (1-p)^{k-1} p, \quad k = 1, 2, \ldots$$

Geometric

$$\frac{P(X=k)}{1-F(k-1)}$$

$$= \frac{(1-p)^{k-1} p}{1 - \sum_{j=1}^{k-1} (1-p)^{j-1} p}$$

$$= \frac{(1-p)^{k-1} p}{1 - [1 - (1-p)^{k-1}]}$$

$$= \frac{\cancel{(1-p)^{k-1}} p}{\cancel{(1-p)^{k-1}}} = p \neq 0$$

$\Rightarrow$ ETT will not hold.

eg 2

$$P(X=k) = \frac{1}{N}, \quad k = 1, 2, \ldots, N$$

Discrete Uniform

$$\lim_{k \to w(F)} \frac{P(X=k)}{1-F(k-1)}$$

$$= \lim_{k \to N} \frac{P(X=k)}{1-F(k-1)}$$

$$= \frac{P(X=N)}{1-F(N-1)}$$

$$= \frac{P(X=N)}{1-P(X \leq N-1)}$$

$$= \frac{P(X=N)}{P(X > N-1)}$$

$$= \frac{P(X=N)}{P(X=N)} = 1 \neq 0$$

ETT fails to hold.

EXAMPLE CLASS

10 OCTOBER

10:00-11:00AM

MATH3/4/68181

Q1

$$L(\sigma) = \prod_{i=1}^{n} \left[\frac{1}{\sigma} e^{-\frac{x_i}{\sigma}} e^{-e^{-\frac{x_i}{\sigma}}} \right]$$

$$= \frac{1}{\sigma^n} e^{-\frac{1}{\sigma}\sum_{i=1}^{n} x_i} e^{-\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}}$$

$$\log L(\sigma) = -n \log \sigma - \frac{1}{\sigma}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}$$

$$\frac{d \log L}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{1}{\sigma^2}\sum x_i e^{-\frac{x_i}{\sigma}}$$

The MLE $\hat{\sigma}$ is the root of

$$-n\sigma + \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}} = 0.$$

$\hat{\sigma}$ can be computed numerically.

Q2

$$L(\lambda, \sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{\lambda} x_i^{-\lambda-1} e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}} \right]$$

$$= \lambda^n \sigma^{n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{-\lambda-1} e^{-\sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda}}$$

$$\log L = n \log \lambda + n\lambda \log \sigma - (\lambda+1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n \log \sigma - \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda} \log \left(\frac{\sigma}{x_i}\right) = 0 \quad (1)$$

$$\left[\frac{d z^a}{d a} = z^a \log z \right]$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} - \lambda \sum_{i=1}^{n} \frac{\sigma^{\lambda-1}}{x_i^{\lambda}} = 0 \quad (2)$$

$$(2) \Rightarrow \frac{n}{\sigma^{\lambda}} = \sum_{i=1}^{n} x_i^{-\lambda}$$

$$\Rightarrow \sigma = \left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \right]^{\frac{1}{\lambda}} \quad (3)$$

Sub (3) into (1) gives

$$\frac{n}{\lambda} + \frac{n}{\lambda} \log\left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}}\right] - \sum_{i=1}^{n} \log x_i$$

$$- \left[\frac{n}{\sum x_i^{-\lambda}}\right] \cdot \frac{1}{\lambda} \cdot \log\left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}}\right] \cdot \sum_{i=1}^{n} x_i^{-\lambda}$$

$$+ \left[\frac{n}{\sum x_i^{-\lambda}}\right] \cdot \left[\sum_{i=1}^{n} x_i^{-\lambda} \log x_i\right] = 0 \quad \text{—— (4)}$$

$\hat{\lambda}$ is the root of (4)

Sub $\hat{\lambda}$ into (3) to obtain $\hat{\sigma}$

Q3

$$L(\lambda, \sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{-\lambda} x_i^{\lambda - 1} e^{-\left(\frac{x_i}{\sigma}\right)^{\lambda}} \right]$$

$$= \lambda^n \sigma^{-n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{\lambda - 1} e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}}$$

$$\log L(\lambda, \sigma) = n \log \lambda - n\lambda \log \sigma + (\lambda - 1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - n \log \sigma + \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda} \log \left(\frac{x_i}{\sigma}\right) = 0 \quad \text{— (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n\lambda}{\sigma} + \lambda \sum_{i=1}^{n} \frac{x_i^{\lambda}}{\sigma^{\lambda+1}} = 0 \quad \text{— (2)}$$

$$(2) \Rightarrow \quad \sigma = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right)^{\frac{1}{\lambda}} \quad \text{— (3)}$$

Sub (3) into (1) to get

$$\frac{n}{\lambda} - \frac{n}{\lambda} \log \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right) - \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right)^{-1} \cdot \left(\sum_{i=1}^{n} x_i^{\lambda} \log x_i \right)$$

$$+ \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right)^{-1} \cdot \frac{1}{\lambda} \cdot \log \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right) \sum_{i=1}^{n} x_i^{\lambda}$$

$$+ \sum_{i=1}^{n} \log x_i = 0 \quad \text{— (4)}$$

$\hat{\lambda}$ is the root of (4). Sub into (3) to get

Q4

$$L(\lambda) = \prod_{i=1}^{n} (1-\lambda x_i)^{\frac{1}{\lambda}-1}$$

$$\log L(\lambda) = \left(\frac{1}{\lambda}-1\right) \sum_{i=1}^{n} \log(1-\lambda x_i)$$

$$\frac{d \log L}{d\lambda} = -\frac{1}{\lambda^2} \sum_{i=1}^{n} \log(1-\lambda x_i)$$

$$+ \left(\frac{1}{\lambda}-1\right) \sum_{i=1}^{n} \frac{(-x_i)}{1-\lambda x_i} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \log(1-\lambda x_i) = -\lambda(1-\lambda) \sum_{i=1}^{n} \frac{x_i}{1-\lambda x_i}$$

$\hat{\lambda}$ is the root of this equation. Can be solved numerically.

LECTURE

13 OCTOBER

9:00-10:00AM

MATH3/4/68181

Is there a way to find out that none of (I) – (III) will hold? Yes.

Discrete Case

None of the conditions will be satisfied (ETT will fail) if

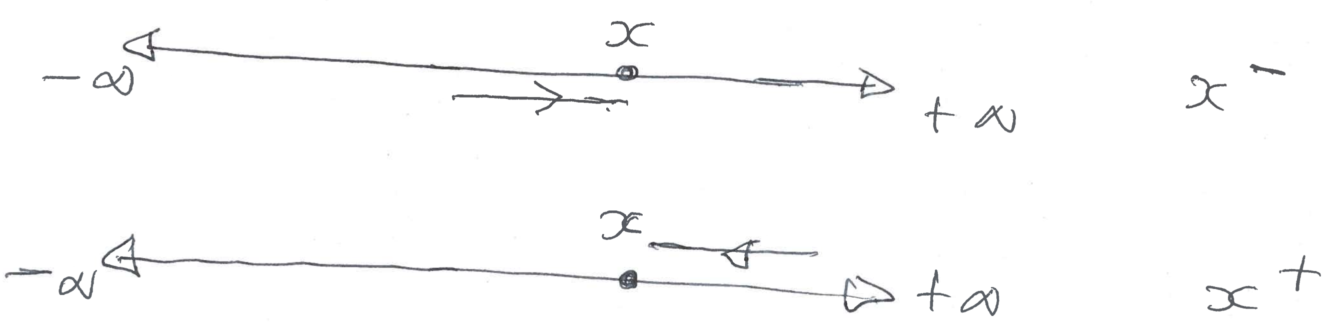
PMF → $P(X = k)$, CDF → F

$$\frac{P(X = k)}{1 - F(k-1)} \not\to 0 \quad \text{as } k \to w(F).$$

Continuous Case

PDF → f, CDF → F

$$\frac{f(x)}{1 - F(x^-)} \not\to 0 \quad \text{as } x \to w(F)$$



eg 3

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

Binomial

$$\omega(F) = n$$

$$\lim_{k \to n} \frac{P(X=k)}{1-F(k-1)} = \frac{P(X=n)}{1-F(n-1)}$$

$$= \frac{P(X=n)}{1-P(X \leqslant n-1)} = \frac{P(X=n)}{P(X > n-1)}$$

$$= \frac{P(X=n)}{P(X=n)} = 1 \neq 0$$

$\Rightarrow$ ETT fails

eg 4

$$P(X=k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k=0,1,\ldots$$

Poisson

$$w(F) = +\infty.$$

$$\lim_{k\to\infty} \frac{P(X=k)}{1-F(k-1)} = \lim_{k\to\infty} \frac{P(X=k)}{1-P(X\leq k-1)}$$

$$= \lim_{k\to\infty} \frac{P(X=k)}{P(X\geq k)} = \lim_{k\to\infty} \frac{P(X=k)}{\sum_{j=k}^{\infty} P(X=j)}$$

$$= \lim_{k\to\infty} \frac{P(X=k)}{P(X=k) + \sum_{j=k+1}^{\infty} P(X=j)}$$

$$= \lim_{k\to\infty} \frac{\frac{e^{-\lambda}\lambda^k}{k!}}{\frac{e^{-\lambda}\lambda^k}{k!} + \sum_{j=k+1}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!}}$$

$$= \lim_{k\to\infty} \frac{\frac{\lambda^k}{k!}}{\frac{\lambda^k}{k!} + \sum_{j=k+1}^{\infty} \frac{\lambda^j}{j!}}$$

$$= \lim_{k\to\infty} \frac{1}{1 + \sum_{j=k+1}^{\infty} \frac{\lambda^{j-k}\, k!}{j!}}$$

$$= \lim_{k \to \infty} \frac{1}{1 + \sum_{j=k+1}^{\infty} \frac{\lambda^{j-k}}{(k+1)(k+2)\cdots j}}$$

$$(k+1)(k+2)\cdots j \geq k^{j-k}, \quad k+1 \geq k, \; k+2 \geq k, \; j \geq k$$

$$\geq \lim_{k \to \infty} \frac{1}{1 + \sum_{j=k+1}^{\infty} \left(\frac{\lambda}{k}\right)^{j-k}}$$

$$[\, m = j - k \,]$$

$$= \lim_{k \to \infty} \frac{1}{1 + \sum_{m=1}^{\infty} \left(\frac{\lambda}{k}\right)^{m}}$$

$$= \lim_{k \to \infty} \frac{1}{1 + \frac{\lambda/k}{1 - \frac{\lambda}{k}}} \qquad \lambda/k \to 0, \quad \frac{\lambda}{k} \to 0$$

$$\sum_{m=1}^{\infty} a^m = \frac{a}{1-a}$$

$$= 1$$

$$\Rightarrow \lim_{k \to \infty} \frac{P(X=k)}{1 - F(k-1)} \geq 1$$

$$\Rightarrow \lim_{k \to \infty} \frac{P(X=k)}{1 - F(k-1)} \neq 0$$

$\Rightarrow$ ETT fails

Portfolio Theory

Portfolio is a collection of investments.

Suppose there are n investments.

Let X_1 = loss in investment 1

X_2 = " " " 2

$\vdots$

X_n = " " " n

The six cases

1) $X_1, \ldots, X_n$ are IID RVs and n is fixed

2) $X_1, \ldots, X_n$ are IID RVs and n is a RV

3) $X_1, \ldots, X_n$ are INID RVs (independent but not identical) and n is a fixed

4) $X_1, \ldots, X_n$ are INID RVs and n is a RV

5) $X_1, \ldots, X_n$ are dependent RVs and n is fixed

6) $X_1, \ldots, X_n$ are dependent RVs and n is a RV.

Variables of interest

1) Total portfolio loss

$$S = X_1 + X_2 + \dots + X_n$$

2) Maximum portfolio loss

$$U = \max(X_1, \dots, X_n)$$

3) Minimum portfolio loss

$$V = \min(X_1, \dots, X_n)$$

Case 1 The CDF of S' is

$$F_{S'}(s) = \int\int\cdots\int_{\{x_1+\cdots+x_n \le s\}} f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(s-x_1-\cdots-x_{n-1}) \, dx_{n-1} \cdots dx_2 \, dx_1$$

The PDF of S' is

$$f_{S'}(s) = \int\int\cdots\int_{\{x_1+\cdots+x_n = s\}} f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(s-x_1-\cdots-x_{n-1}) \, dx_{n-1} \cdots dx_2 \, dx_1$$

The m^{th} moment of S' is

$$E(S'^m) = E\left[(X_1+\cdots+X_n)^m\right]$$

$$= \sum_{\{m_1+\cdots+m_n = m\}} \frac{m!}{m_1! \cdots m_n!} E(X_1^{m_1}) \cdots E(X_n^{m_n})$$

The CDF of U is

$$P(U \le u) = P(\max(X_1, \dots, X_n) \le u)$$

$$= P(X_1 \le u, \dots, X_n \le u)$$

$$\overset{\text{indep}}{=} P(X_1 \le u) \cdots P(X_n \le u)$$

$$= F_{X_1}(u) \cdots F_{X_n}(u)$$

$$= [F_X(u)]^n$$

The PDF of U is

$$f_U(u) = n [F_X(u)]^{n-1} f_X(u)$$

The mth moment of U is

$$E(U^m) = n \int_{-\infty}^{+\infty} u^m [F_X(u)]^{n-1} f_X(u) \, du$$

The CDF of $\bar{V}$ is

$$F_{\bar{V}}(v) = P(\bar{V} \leq v)$$

$$= 1 - [1 - F_X(v)]^n$$

The PDF of $\bar{V}$ is

$$f_{\bar{V}}(v) = n[1 - F_X(v)]^{n-1} f_X(v)$$

The mth moment $\bar{V}$ is

$$E(\bar{V}^m) = n \int_{-\infty}^{\infty} v^m [1 - F_X(v)]^{n-1} f_X(v) dv$$

EXAMPLE CLASS

16 OCTOBER

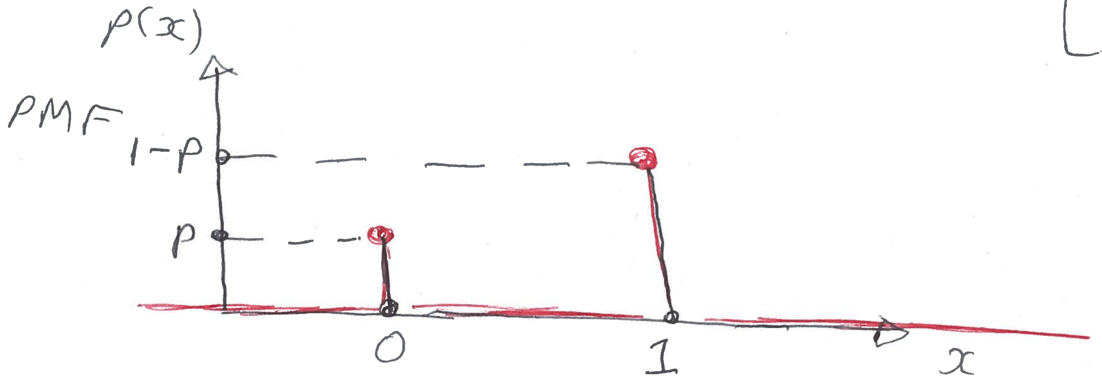
12:00-13:00PM

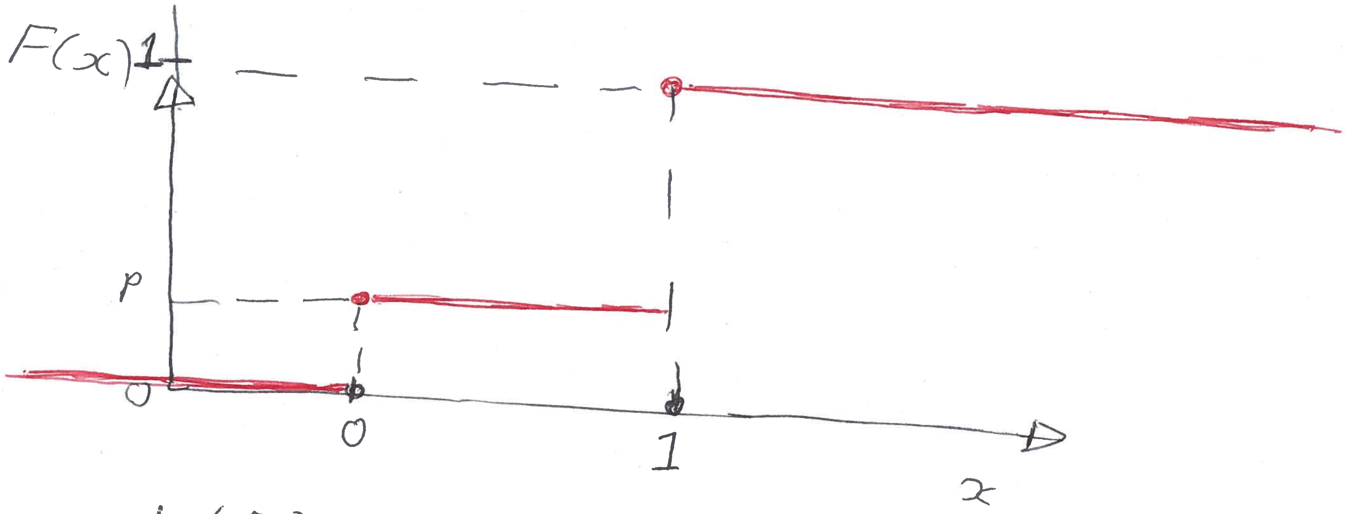
MATH3/4/68181

Q1

Bernoulli (p)

$$X = \begin{cases} 0 & \text{wp } p \\ 1 & \text{wp } 1-p \end{cases}$$





$w(F) = 1$

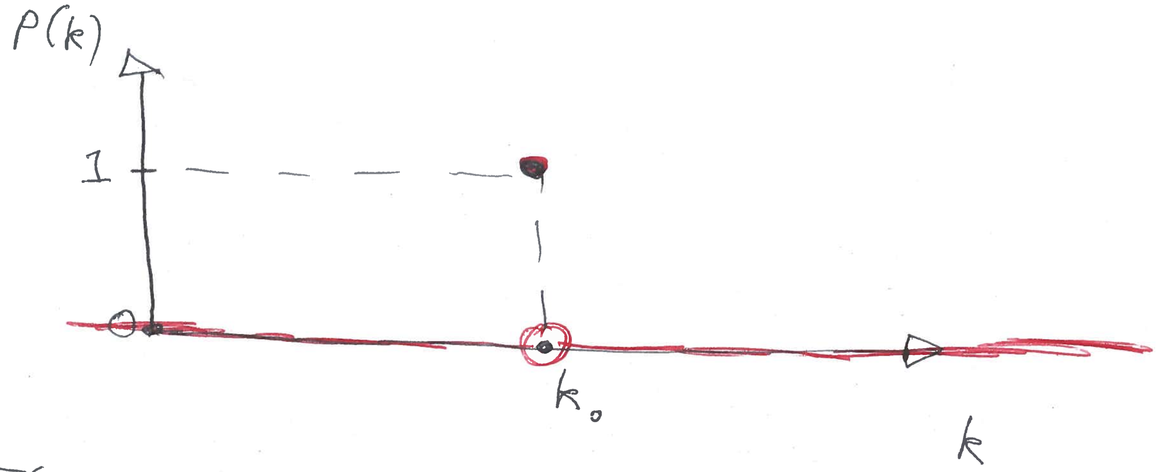
$$\lim_{k \to w(F)} \frac{P(X=k)}{1-F(k-1)} = \frac{P(X=1)}{1-F(0)}$$

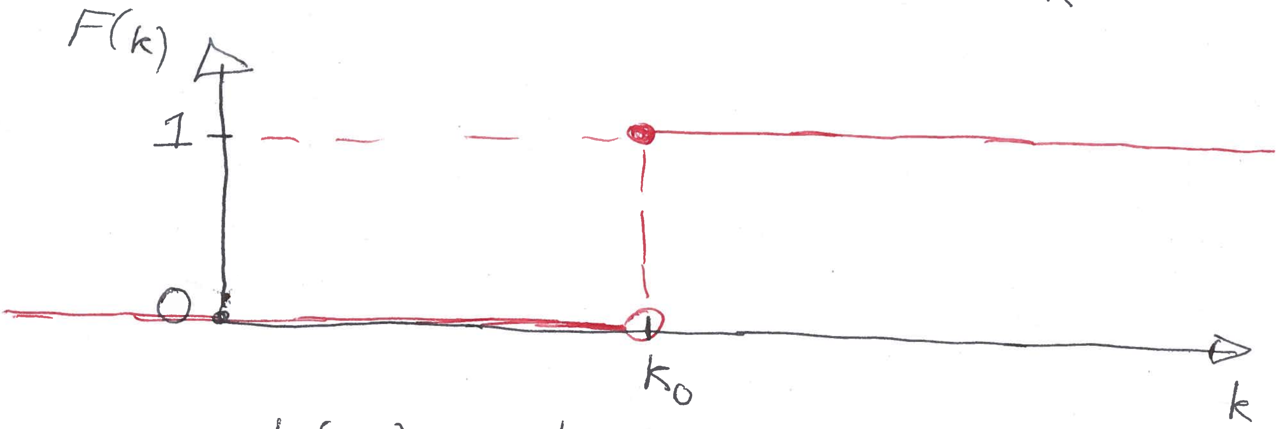
$$= \frac{1-p}{1-p}$$

$$= 1 \neq 0$$

ETT fails.

Q2





$$w(F) = k_0$$

$$\lim_{k \to w(F)} \frac{P(X=k)}{1-F(k-1)} = \frac{P(X=k_0)}{1-F(k_0-1)}$$

$$= \frac{1}{1-0} = 1 \neq 0$$

ETT fails

Q4

$$P(k) = \frac{k^{-s}}{\zeta(s)}, \quad k \geq 1$$

$$w(F) = \infty.$$

$$\lim_{k \to \infty} \frac{P(X=k)}{1-F(k-1)} = \lim_{k \to \infty} \frac{P(X=k)}{1-P(X \leq k-1)}$$

$$= \lim_{k \to \infty} \frac{P(X=k)}{P(X \geq k)} = \lim_{k \to \infty} \frac{P(X=k)}{\sum_{j=k}^{\infty} P(X=j)}$$

$$= \lim_{k \to \infty} \frac{\frac{k^{-s}}{\zeta(s)}}{\sum_{j=k}^{\infty} \frac{j^{-s}}{\zeta(s)}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{\sum_{j=k}^{\infty} j^{-s}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{\int_k^{\infty} x^{-s}\, dx}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{\left[\frac{x^{1-s}}{1-s}\right]_k^{\infty}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{0 - \frac{k^{1-s}}{1-s}} = 0, \quad s > 1$$

$\Rightarrow$ ETT will hold.

$$\frac{d}{dx} \log_2 x = \frac{1}{x \cdot \log 2}$$

Q5

$$P(k) = -\log_2\left[1 - \frac{1}{(k+1)^2}\right], \quad k \geq 1$$

$$w(F) = +\infty$$

$$\lim_{k\to\infty} \frac{-\log_2\left[1 - \frac{1}{(k+1)^2}\right]}{1 - \left\{1 - \log_2\left[\frac{k+1}{k}\right]\right\}}$$

$$= \lim_{k\to\infty} - \frac{\log_2\left[\frac{k^2+2k}{(k+1)^2}\right]}{\log_2\left[\frac{k+1}{k}\right]}$$

$$= \lim_{k\to\infty} - \frac{\log_2(k^2+2k) - 2\log_2(k+1)}{\log_2(k+1) - \log_2 k}$$

$$\overset{LH}{=} \lim_{k\to\infty} - \frac{\frac{2k+2}{k^2+2k}\cdot\frac{1}{\log 2} - \frac{2}{k+1}\cdot\frac{1}{\log 2}}{\frac{1}{k+1}\cdot\frac{1}{\log 2} - \frac{1}{k}\cdot\frac{1}{\log 2}}$$

$$= \lim_{k\to\infty} - \frac{\frac{2(k+1)}{k(k+2)} - \frac{2}{k+1}}{\frac{1}{k(k+1)}}$$

$$= \lim_{k\to\infty} - \frac{2(k+1)^2}{k+2} + 2k$$

$$= \lim_{k\to\infty} -2\left[\frac{(k+1)^2 - k(k+2)}{(k+2)}\right]$$

$= 0 \Rightarrow$ ETT will hold.

LECTURE

17 OCTOBER

9:00-10:00AM

MATH3/4/68181

Portfolio Theory

Portfolio is a collection of investments.

Suppose there are n investments.

Let X_1 = loss in investment 1

X_2 = " " " 2

⋮

X_n = " " " n

The six cases

1) $X_1, \ldots, X_n$ are IID RVs and n is fixed

2) $X_1, \ldots, X_n$ are IID RVs and n is a RV

3) $X_1, \ldots, X_n$ are INID RVs (independent but not identical) and n is a fixed

4) $X_1, \ldots, X_n$ are INID RVs and n is a RV

5) $X_1, \ldots, X_n$ are dependent RVs and n is fixed

6) $X_1, \ldots, X_n$ are dependent RVs and n is a RV.

Variables of interest

1) Total portfolio loss

$$S = X_1 + X_2 + \dots + X_n$$

2) Maximum portfolio loss

$$U = \max(X_1, \dots, X_n)$$

3) Minimum portfolio loss

$$V = \min(X_1, \dots, X_n)$$

Case 1 The CDF of S' is

$$F_{S'}(s) = \int\int\cdots\int_{\{x_1+\cdots+x_n \le s\}} f_{X_1}(x_1)\, f_{X_2}(x_2) \cdots f_{X_n}(s-x_1-\cdots-x_{n-1})\, dx_{n-1}\cdots dx_2\, dx_1$$

The PDF of S' is

$$f_{S'}(s) = \int\int\cdots\int_{\{x_1+\cdots+x_n = s\}} f_{X_1}(x_1)\, f_{X_2}(x_2) \cdots f_{X_n}(s-x_1-\cdots-x_{n-1})\, dx_{n-1}\cdots dx_2\, dx_1$$

The m^{th} moment of S' is

$$E(S'^m) = E\left[(X_1+\cdots+X_n)^m\right]$$

$$= \sum_{\substack{A \\ \{m_1+\cdots+m_n=m\}}} \frac{m!}{m_1!\cdots m_n!} E(X_1^{m_1})\cdots E(X_n^{m_n})$$

The CDF of U is

$$P(U \le u) = P(\max(X_1, \dots, X_n) \le u)$$

$$= P(X_1 \le u, \dots, X_n \le u)$$

$$\overset{\text{indep}}{=} P(X_1 \le u) \cdots P(X_n \le u)$$

$$= F_{X_1}(u) \cdots F_{X_n}(u)$$

$$= [F_X(u)]^n$$

The PDF of U is

$$f_U(u) = n [F_X(u)]^{n-1} f_X(u)$$

The mth moment of U is

$$E(U^m) = n \int_{-\infty}^{+\infty} u^m [F_X(u)]^{n-1} f_X(u)\, du$$

The CDF of $\bar{V}$ is

$$F_{\bar{V}}(v) = P(\bar{V} \leq v)$$

$$= 1 - [1 - F_X(v)]^n$$

The PDF of $\bar{V}$ is

$$f_{\bar{V}}(v) = n[1 - F_X(v)]^{n-1} f_X(v)$$

The mth moment $\bar{V}$ is

$$E(\bar{V}^m) = n \int_{-\infty}^{\infty} v^m [1 - F_X(v)]^{n-1} f_X(v) dv$$

Case 2 The CDF of S' is

$$P(S' \leqslant s)$$

$$\overset{\text{total prob thm}}{=} \sum_{n=1}^{\infty} \underbrace{P(S' \leqslant s \mid N=n)}\, P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[\int\int \cdots \int_{\{x_1 + \cdots + x_n \leqslant s\}} f_{X_1}(x_1)\, f_{X_2}(x_2) \cdots f_{X_n}(s - x_1 - \cdots - x_{n-1})\, dx_{n-1} \cdots dx_2\, dx_1 \right] \cdot P(N=n)$$

The PDF of S' is

$$f_{S'}(s) = \sum_{n=1}^{\infty} \left[\int\int \cdots \int_{\{x_1 + \cdots + x_n = s\}} f_{X_1}(x_1)\, f_{X_2}(x_2) \cdots f_{X_n}(s - x_1 - \cdots - x_{n-1})\, dx_{n-1} \cdots dx_2\, dx_1 \right] \cdot P(N=n)$$

The m th moment of S is

$$E(S^m)$$

$$= E\left[E\left(S^m \mid N\right)\right]$$

$$= E\left[E\left((X_1 + \cdots + X_n)^m \mid N = n\right)\right]$$

$$= \sum_{n=1}^{\infty}\left[\sum_{\{m_1 + \cdots + m_n = m\}} \frac{m!}{m_1! \cdots m_n!} E(X_1^{m_1}) \cdots E(X_n^{m_n})\right] \cdot P(N = n)$$

The CDF of U is

$$P(U \le u)$$

$$= \sum_{n=1}^{\infty} \underbrace{P(U < u \mid N=n)} P(N=n)$$

$$= \sum_{n=1}^{\infty} [F_X(u)]^n P(N=n)$$

The PDF of U is

$$f_U(u) = \sum_{n=1}^{\infty} n [F_X(u)]^{n-1} f_X(u) \cdot P(N=n)$$

The mth moment of U is

$$E(U^m) = \sum_{n=1}^{\infty} n \cdot P(N=n) \cdot \int_{-\infty}^{\infty} u^m [F_X(u)]^{n-1} f_X(u)\, du$$

The CDF of V is

$$P(V \leq v)$$

$$= \sum_{n=1}^{\infty} \underbrace{P(V \leq v \mid N=n)} P(N=n)$$

$$= \sum_{n=1}^{\infty} \{1-[1-F_X(v)]^n\} \cdot P(N=n)$$

$$= \sum_{n=1}^{\infty} P(N=n) - \sum_{n=1}^{\infty} [1-F_X(v)]^n P(N=n)$$

$$= 1 - \sum_{n=1}^{\infty} [1-F_X(v)]^n P(N=n)$$

The PDF of V is

$$f_V(v) = \sum_{n=1}^{\infty} n[1-F_X(v)]^{n-1} f_X(v) \cdot P(N=n)$$

The mth moment of V is

$$E(V^m) = \sum_{n=1}^{\infty} n \cdot P(N=n) \cdot \int_{-\infty}^{\infty} v^m [1-F_X(v)]^{n-1} f_X(v)\,dv$$

Ex1 Suppose $X_1, \ldots, X_N$ are IID representing losses on N investments. Suppose too N is Poisson (λ), indep of $X_1, X_2, \ldots$.

The CDF of U is

$$P(U \leq u) = \sum_{n=1}^{\infty} [F_X(u)]^n \frac{e^{-\lambda} \lambda^n}{n!}$$

$$= \left[\sum_{n=1}^{\infty} \frac{[F_X(u) \lambda]^n}{n!} \right] e^{-\lambda}$$

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (*)$$

$$= \left[e^{\lambda F_X(u)} - 1 \right] e^{-\lambda}$$

$$f_U(u) = \lambda e^{\lambda F_X(u) - \lambda} f_X(u)$$

The CDF of $\bar{V}$ is

$$P(\bar{V} \leq v) = 1 - \sum_{n=1}^{\infty} [1 - F_X(v)]^n \frac{e^{-\lambda} \lambda^n}{n!}$$

$$= 1 - e^{-\lambda} \left[e^{\lambda[1 - F_X(v)]} - 1 \right]$$

by (*)

$$= 1 - e^{-\lambda F_X(v)} - e^{-\lambda}$$

The PDF of $\bar{V}$ is

$$f_{\bar{V}}(v) = \lambda e^{-\lambda F_X(v)} f_X(v).$$

Ex 2

Same as example I but N is Geometric (p).

The CDF of U is

$$P(U \leqslant u) = \sum_{n=1}^{\infty} [F_X(u)]^n \, p(1-p)^{n-1}$$

$$= p F_X(u) \sum_{n=1}^{\infty} [F_X(u)]^{n-1} (1-p)^{n-1}$$

$[k = n-1]$

$$= p F_X(u) \sum_{k=0}^{\infty} [F_X(u)]^k (1-p)^k$$

$$\boxed{\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}} \quad (*)(*)$$

$$= \frac{p F_X(u)}{1-(1-p)F_X(u)}$$

The PDF of U is

$$f_U(u) = \{[1-(1-p)F_X(u)]\, p f_X(u) + p F_X(u)(1-p) f_X(u)\} / [1-(1-p)F_X(u)]^2$$

$$= \frac{p f_X(u)}{[1-(1-p)F_X(u)]^2}$$

The CDF of V is

$$F_V(v) = 1 - \sum_{n=1}^{\infty} [1 - F_X(v)]^n \, p(1-p)^{n-1}$$

$$= 1 - \frac{p\,[1 - F_X(v)]}{1 - (1-p)[1 - F_X(v)]}$$

by ⊛⊛

$$= \frac{F_X(v)}{1 - (1-p)[1 - F_X(v)]}$$

The PDF of V is

$$f_V(v) = \Big[\{1 - (1-p)[1 - F_X(v)]\} f_X(v) - F_X(v)\,(1-p)\,f_X(v) \Big] \Big/ \{1 - (1-p)[1 - F_X(v)]\}^2$$

$$= \frac{p\, f_X(v)}{\{1 - (1-p)[1 - F_X(v)]\}^2}$$

EXAMPLE CLASS

17 OCTOBER

10:00-11:00AM

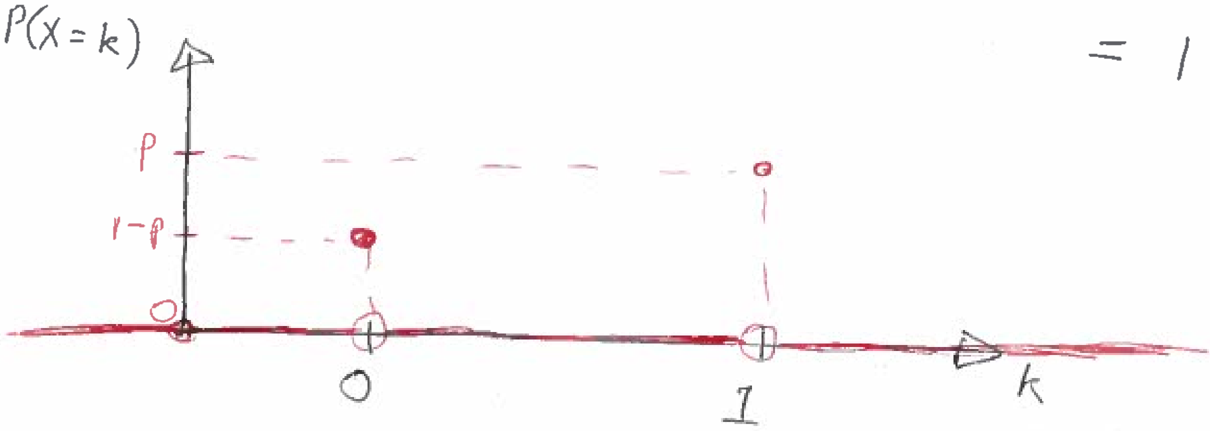
MATH3/4/68181

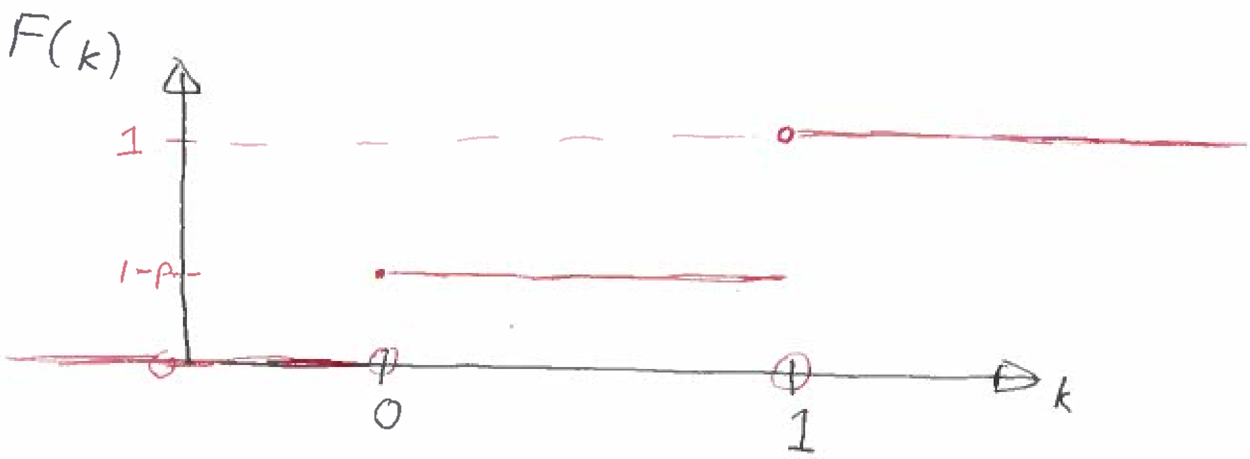
Q1

Bernoulli (p)

$$X = \begin{cases} 0 & \text{with prob } 1-p \\ 1 & \text{" " } p \end{cases}$$

$$\lim_{k \to w(F)} \frac{P(X=k)}{1-F(k-1)} = \frac{P(X=1)}{1-F(0)} = \frac{p}{1-(1-p)} = 1 \neq 0$$

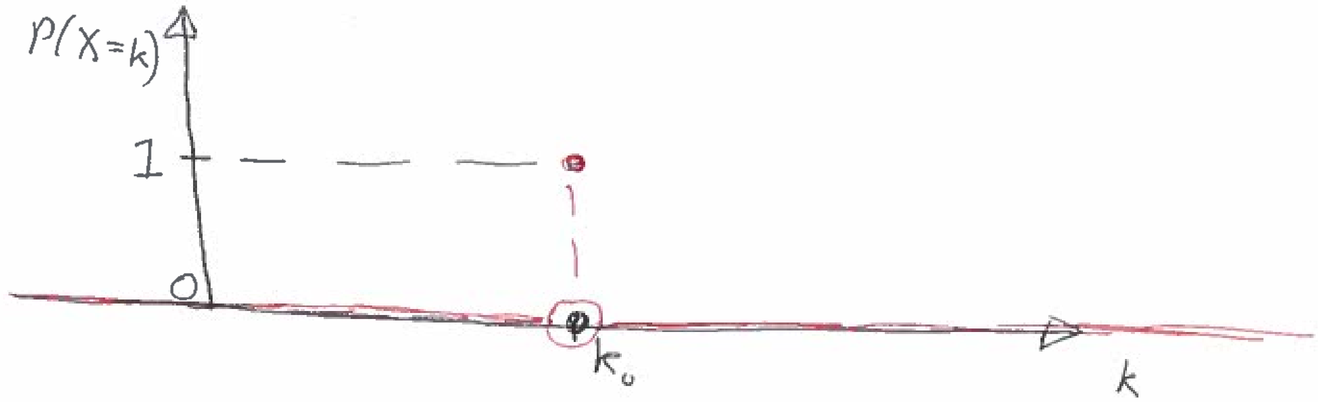


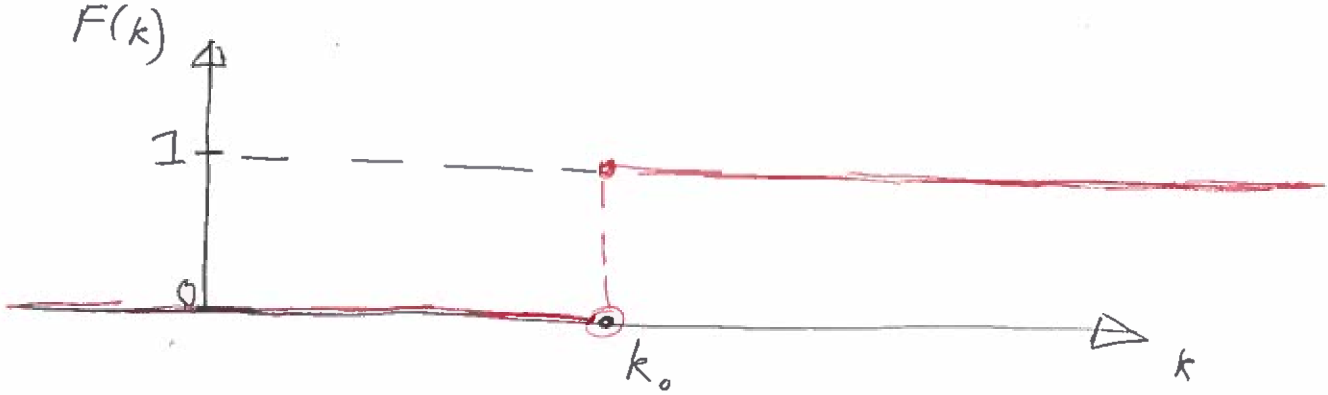


$w(F) = 1$

ETT fails to hold.

Q2





$$\lim_{k \to w(F)} \frac{P(X=k)}{1-F(k-1)} = \frac{P(X=k_0)}{1-F(k_0-1)}$$

$$= \frac{1}{1-0} = 1 \neq 0$$

ETT fails to hold.

Q4

$$p(k) = \frac{k^{-s}}{\zeta(s)}, \quad k \geq 1$$

$$w(F) = +\infty$$

$$\lim_{k \to \infty} \frac{P(X = k)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{\frac{k^{-s}}{\zeta(s)}}{1 - P(X \leq k-1)}$$

$$= \lim_{k \to \infty} \frac{\frac{k^{-s}}{\zeta(s)}}{P(X \geq k)}$$

$$= \lim_{k \to \infty} \frac{\frac{k^{-s}}{\zeta(s)}}{\sum_{j=k}^{\infty} P(X = j)}$$

$$= \lim_{k \to \infty} \frac{\frac{k^{-s}}{\zeta(s)}}{\sum_{j=k}^{\infty} \frac{j^{-s}}{\zeta(s)}} = \lim_{k \to \infty} \frac{k^{-s}}{\sum_{j=k}^{\infty} j^{-s}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{\int_k^{\infty} x^{-s}\, dx} = \lim_{k \to \infty} \frac{k^{-s}}{\left[\frac{x^{1-s}}{1-s}\right]_k^{\infty}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{0 - \frac{k^{1-s}}{1-s}} \quad \text{if } s > 1$$

$$= 0 \Rightarrow \text{ETT will hold.}$$

Q5

$$\lim_{k\to\infty} \frac{-\log_2\left[1-\frac{1}{(k+1)^2}\right]}{1-\left[1-\log_2\left[\frac{k+1}{k}\right]\right]}$$

$$= \lim_{k\to\infty} - \frac{\log_2\left[\frac{k^2+2k}{(k+1)^2}\right]}{\log_2\left[\frac{k+1}{k}\right]}$$

($\frac{k^2+2k}{(k+1)^2} \to 1$, $\frac{k+1}{k} \to 1$)

$$= -\lim_{k\to\infty} \frac{\log_2(k^2+2k) - 2\log_2(k+1)}{\log_2(k+1) - \log_2 k}$$

$$\overset{LH}{=} -\lim_{k\to\infty} \frac{\frac{2k+2}{k^2+2k}\cdot\frac{1}{\log 2} - \frac{2}{k+1}\cdot\frac{1}{\log 2}}{\frac{1}{k+1}\cdot\frac{1}{\log 2} - \frac{1}{k}\cdot\frac{1}{\log 2}}$$

$$= -\lim_{k\to\infty} \frac{\frac{2(k+1)}{k(k+2)} - \frac{2}{k+1}}{\frac{1}{k(k+1)}}$$

$$= -\lim_{k\to\infty} \left[\frac{2(k+1)^2}{k+2} - 2k\right]$$

$$= -\lim_{k\to\infty} 2\cdot\left[\frac{(k+1)^2 - k(k+2)}{k+2}\right]$$

$$= -\lim_{k\to\infty} \frac{2}{k+2} = 0$$

ETT will hold.

$$\frac{d}{dx} \cdot \log_2 x = \frac{1}{x} \cdot \frac{1}{\log 2}$$

LECTURE

20 OCTOBER

9:00-10:00AM

MATH3/4/68181

Portfolio Theory

Portfolio is a collection of investments.

Suppose there are n investments.

Let X_1 = loss in investment 1

X_2 = " " " 2

$\vdots$

X_n = " " " n

The six cases

1) $X_1, \ldots, X_n$ are IID RVs and n is fixed

2) $X_1, \ldots, X_n$ are IID RVs and n is a RV

3) $X_1, \ldots, X_n$ are INID RVs (independent but not identical) and n is a fixed

4) $X_1, \ldots, X_n$ are INID RVs and n is a RV

5) $X_1, \ldots, X_n$ are dependent RVs and n is fixed

6) $X_1, \ldots, X_n$ are dependent RVs and n is a RV.

Variables of interest

1) Total portfolio loss

$$S = X_1 + X_2 + \dots + X_n$$

2) Maximum portfolio loss

$$U = \max(X_1, \dots, X_n)$$

3) Minimum portfolio loss

$$V = \min(X_1, \dots, X_n)$$

Ex 3

Suppose $X_1, X_2, \ldots, X_n$ are IID $\text{Exp}(\lambda)$ RVs. Suppose n is <u>fixed</u>. Derive

(i) the CDF of S

(ii) " PDF of S

(iii) " CDF of U

(iv) " PDF of U

(v) " CDF of V

(vi) " PDF of V

Ex 3 (solutions)

$$S = X_1 + \dots + X_n$$

The MGF of S' is
(moment generating function)

$$M_{S'}(s) = E\left[e^{tS'}\right]$$

$$= E\left[e^{t(X_1 + \dots + X_n)}\right]$$

$$= E\left[e^{tX_1} \dots e^{tX_n}\right]$$

indep

$$= E\left[e^{tX_1}\right] \dots E\left[e^{tX_n}\right]$$

$$= \left(\int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} dx\right) \dots \left(\int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} dx\right)$$

$$= \left(\lambda \int_0^\infty e^{-(\lambda - t)x} dx\right)^n$$

$$= \left(\lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)}\right]_0^\infty\right)^n$$

$$= \left(\lambda \left(0 - \frac{1}{-(\lambda - t)}\right)\right)^n$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^n,$$ the MGF of a Gamma RV with parameters n & λ.

$\Rightarrow$ S is a gamma RV with parameters n & λ.

$$\Rightarrow \quad f_S(s) = \frac{\lambda^n s^{n-1} e^{-\lambda s}}{\Gamma(n)}$$

[please look at Math 20802 notes]

The CDF of U is

$$F_U(u) = [F_X(u)]^n$$

$$= [1 - e^{-\lambda u}]^n$$

The PDF of U is

$$f_U(u) = n\lambda e^{-\lambda u} [1 - e^{-\lambda u}]^{n-1}$$

The CDF of V is

$$F_V(v) = 1 - [1 - F_X(v)]^n$$

$$= 1 - [e^{-\lambda v}]^n$$

$$= 1 - e^{-n\lambda v}$$

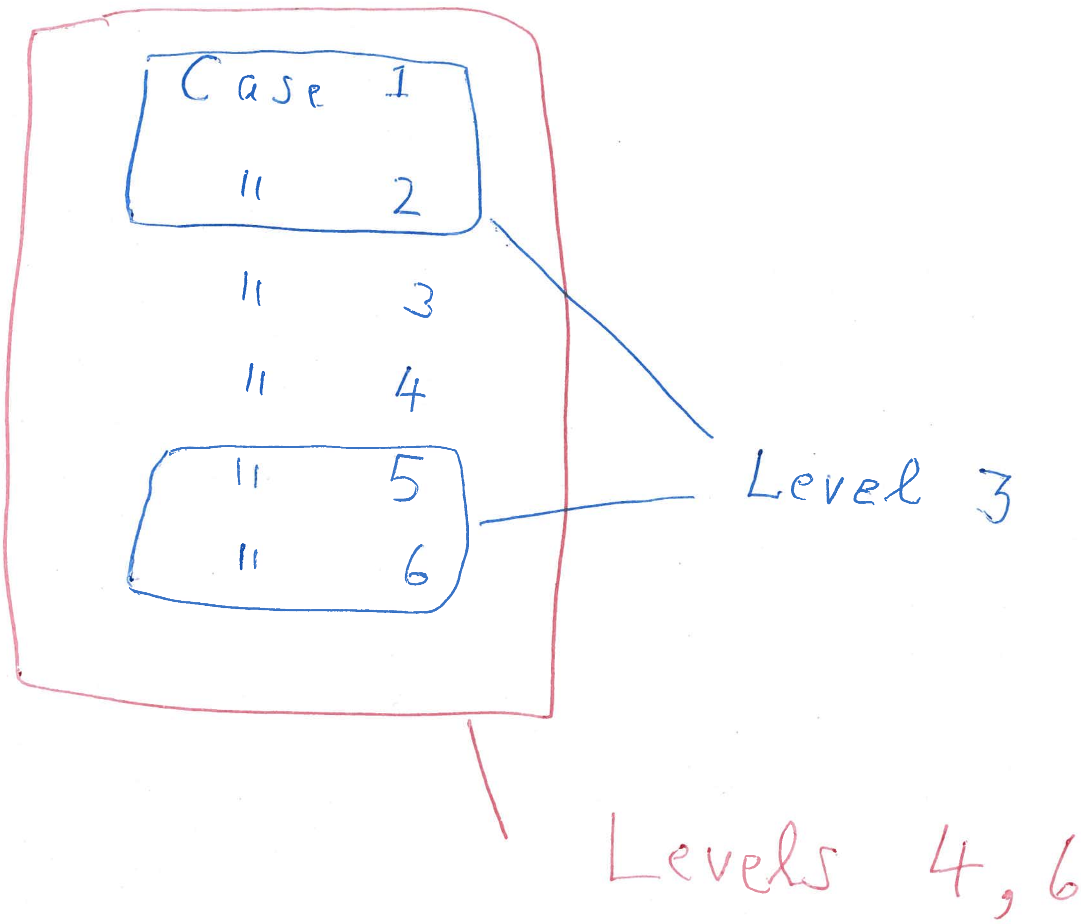
The PDF of V is

$$f_V(v) = n\lambda e^{-n\lambda v},$$

the PDF of $Exp(n\lambda)$.

$\Rightarrow$ V is an $Exp(n\lambda)$.

PORTFOLIO THEORY



Terminology

Suppose (X, Y) is a random vector.

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

— Joint CDF of (X, Y)

$$\overline{F}_{X,Y}(x,y) = P(X > x, Y > y)$$

— Joint survival function of (X, Y)

$$\left.\begin{aligned} F_X(x) &= F_{X,Y}(x, \infty) \\ F_X(x) &= 1 - \overline{F}_{X,Y}(x, -\infty) \end{aligned}\right\} \text{CDF of } X.$$

$$\left.\begin{aligned} F_Y(y) &= F_{X,Y}(\infty, y) \\ F_Y(y) &= 1 - \overline{F}_{X,Y}(-\infty, y) \end{aligned}\right\} \text{CDF of } Y$$

Relation between joint CDF & joint SF

$$F_{X,Y}(x,y) = 1 - \left\{ [1 - F_X(x)] + [1 - F_Y(y)] - \bar{F}_{X,Y}(x,y) \right\}$$

$$= -1 + F_X(x) + F_Y(y) + \bar{F}_{X,Y}(x,y)$$

$$\bar{F}_{X,Y}(x,y)$$

$$= 1 - F_X(x) - F_Y(y) + F_{X,Y}(x,y)$$

Suppose (X, Y, Z) is a random vector.

$$F_{X,Y,Z}(x, y, z) = P(X \leq x, Y \leq y, Z \leq z)$$

Joint CDF of (X, Y, Z)

$$\bar{F}_{X,Y,Z}(x, y, z) = P(X > x, Y > y, Z > z)$$

Joint SF of (X, Y, Z)

$$\bar{F}_{X,Y,Z}(x, y, z) = 1 - F_X(x) - F_Y(y) - F_Z(z) + F_{X,Y}(x, y) + F_{X,Z}(x, z) + F_{Y,Z}(y, z) - F_{X,Y,Z}(x, y, z)$$

Suppose $(X_1, \ldots, X_n)$ is a random vector.

$$F_{X_1, \ldots, X_n}(x_1, \ldots, x_n)$$
$$= P(X_1 < x_1, \ldots, X_n < x_n)$$

Joint CDF of $(X_1, \ldots, X_n)$

$$\bar{F}_{X_1, \ldots, X_n}(x_1, \ldots, x_n)$$
$$= P(X_1 > x_1, \ldots, X_n > x_n)$$

Joint SF of $(X_1, \ldots, X_n)$

$$\bar{F}_{X_1, \ldots, X_n}(x_1, \ldots, x_n)$$
$$= 1 - \sum_{i=1}^{n} F_{X_i}(x_i)$$
$$+ \sum_{i<j}\sum F_{X_i, X_j}(x_i, x_j)$$
$$- \sum_{i<j<k}\sum\sum F_{X_i, X_j, X_k}(x_i, x_j, x_k)$$
$$+ \cdots + (-1)^n F_{X_1, \ldots, X_n}(x_1, \ldots, x_n)$$

Case 5 $X_1, \ldots, X_n$ are dependent losses. n is fixed.

The CDF of S' is

$$P(S' \leq s) = \int\int \cdots \int_{x_1 + \cdots + x_n \leq s} f_{X_1, \ldots, X_n}(y_1, \ldots, y_n) \, dy_1 \cdots dy_n$$

Joint PDF of $(X_1, \ldots, X_n)$

The PDF of S' is

~~$P(S'$~~ $$f_{S'}(s) = \int\int \cdots \int_{x_1 + \cdots + x_n = s} f_{X_1, \ldots, X_n}(y_1, \ldots, y_n) \, dy_1 \cdots dy_n$$

The mth moment of S' is

$$E(S'^m) = \sum_{m_1 + \cdots + m_n = m} \frac{m!}{m_1! \cdots m_n!} E(X_1^{m_1} \cdots X_n^{m_n})$$

The CDF of U is

$$F_U(u) = P(U \leq u)$$

$$= P(\max(X_1, \ldots, X_n) \leq u)$$

$$= P(X_1 \leq u, \ldots, X_n \leq u)$$

$$= F_{X_1, \ldots, X_n}(u, \ldots, u)$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_{X_1, \ldots, X_n}(u, \ldots, u)$$

The mth moment of U is

$$E(U^m) = \int_{-\infty}^{\infty} u^m \frac{d}{du} F_{X_1, \ldots, X_n}(u, \ldots, u)\, du$$

The CDF of $\bar{V}$ is

$$P(\bar{V} \leq v)$$
$$= 1 - P(\bar{V} > v)$$
$$= 1 - P(\min(X_1, \ldots, X_n) > v)$$
$$= 1 - P(X_1 > v, \ldots, X_n > v)$$
$$= 1 - \bar{F}_{X_1, \ldots, X_n}(v, \ldots, v)$$

The PDF of $\bar{V}$ is

$$f_{\bar{V}}(v) = -\frac{d}{dv} \bar{F}_{X_1, \ldots, X_n}(v, \ldots, v)$$

The mth moment of $\bar{V}$ is

$$E(\bar{V}^m) = -\int_{-\infty}^{\infty} v^m \frac{d}{dv} \bar{F}_{X_1, \ldots, X_n}(v, \ldots, v) dv$$

EXAMPLE CLASS

23 OCTOBER

12:00-13:00PM

MATH3/4/68181

[1]

$$P(X \leq x)$$

$$= P(\max(X_1, \ldots, X_\alpha) \leq x)$$

$$= P(X_1 \leq x, \ldots, X_\alpha \leq x)$$

$$\overset{\text{indep}}{=} P(X_1 \leq x) \cdots P(X_\alpha \leq x)$$

$$= [1 - e^{-\lambda x}] \cdots [1 - e^{-\lambda x}]$$

$$= [1 - e^{-\lambda x}]^\alpha$$

[2] $$f_X(x) = \alpha [1 - e^{-\lambda x}]^{\alpha - 1} \lambda e^{-\lambda x}$$

3

$$E(X^n) = \alpha\lambda \int_0^\infty x^n e^{-\lambda x} \left[1 - e^{-\lambda x}\right]^{\alpha-1} dx$$

Set $y = e^{-\lambda x}$

$$x = -\frac{1}{\lambda} \log y$$

$$\frac{dx}{dy} = -\frac{1}{\lambda y}$$

$$= \alpha \lambda \int_1^0 \left(-\frac{1}{\lambda} \log y\right)^n y \left[1-y\right]^{\alpha-1} \left(-\frac{dy}{\lambda y}\right)$$

$$= \alpha \int_0^1 \left(-\frac{1}{\lambda} \log y\right)^n \left[1-y\right]^{\alpha-1} dy$$

$$= \alpha \left(-\frac{1}{\lambda}\right)^n \int_0^1 (\log y)^n \left[1-y\right]^{\alpha-1} dy$$

$$\overset{(*)}{=} \alpha \left(-\frac{1}{\lambda}\right)^n \int_0^1 \left(\frac{d^n}{db^n} y^b \Big|_{b=0}\right) (1-y)^{\alpha-1} dy$$

$$= \alpha \left(-\frac{1}{\lambda}\right)^n \frac{d^n}{db^n} \left(\int_0^1 y^b (1-y)^{\alpha-1} dy\right) \Big|_{b=0}$$

$$= \alpha \left(-\frac{1}{\lambda}\right)^n \frac{d^n}{db^n} B(b+1, \alpha) \Big|_{b=0}$$

$$\frac{d}{db} y^b = y^b \log y$$

$$\frac{d^2}{db^2} y^b = y^b (\log y)^2$$

$$\vdots$$

$$\left.\frac{d^n}{db^n} y^b\right|_{b=0} = \left. y^b (\log y)^n \right|_{b=0}$$

$$\left.\frac{d^n}{db^n} y^b\right|_{b=0} = (\log y)^n \quad (*)$$

Beta Function

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} \, dt$$

4

$$E(X) = -\frac{\alpha}{\lambda} \frac{d}{db} B(b+1, \alpha) \Big|_{b=0}$$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$= -\frac{\alpha}{\lambda} \frac{d}{db} \frac{\Gamma(b+1)\Gamma(\alpha)}{\Gamma(b+1+\alpha)} \Big|_{b=0}$$

$$= -\frac{\alpha\Gamma(\alpha)}{\lambda} \left[\Gamma(b+1+\alpha)\Gamma'(b+1) - \Gamma(b+1)\Gamma'(b+1+\alpha)\right] / \Gamma^2(b+1+\alpha) \Big|_{b=0}$$

$$= -\frac{\alpha\Gamma(\alpha)}{\lambda} \frac{\Gamma(1+\alpha)\Gamma'(1) - \Gamma'(1+\alpha)}{\Gamma^2(1+\alpha)}$$

8

$$L(\alpha, \lambda) = \alpha \lambda e^{-\lambda x} \left[1 - e^{-\lambda x}\right]^{\alpha - 1}$$

$$\log L = \log(\alpha \lambda) - \lambda x + (\alpha - 1) \log\left[1 - e^{-\lambda x}\right]$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{1}{\alpha} + \log\left[1 - e^{-\lambda x}\right] = 0 \quad - (1)$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{1}{\lambda} - x + (\alpha - 1) \frac{x e^{-\lambda x}}{1 - e^{-\lambda x}} = 0 \quad - (2)$$

$$(1) \Rightarrow 1 - e^{-\lambda x} = e^{-1/\alpha}$$

$$\Rightarrow e^{-\lambda x} = 1 - e^{-1/\alpha}$$

$$\Rightarrow \lambda = -\frac{1}{x} \log\left(1 - e^{-\frac{1}{\alpha}}\right) \quad - (3)$$

Sub (3) into (2)

$$-\frac{x}{\log\left(1 - e^{-\frac{1}{\alpha}}\right)} - x + \frac{(\alpha - 1)\, x \left(1 - e^{-\frac{1}{\alpha}}\right)}{e^{-\frac{1}{\alpha}}} = 0 \quad - (4)$$

$\hat{\alpha}$ is the root of (4)

Sub $\hat{\alpha}$ into (3) to get $\hat{\lambda}$.

LECTURE

24 OCTOBER

9:00-10:00AM

MATH3/4/68181

Portfolio Theory

Portfolio is a collection of investments.

Suppose there are n investments.

Let X_1 = loss in investment 1

X_2 = " " " 2

$\vdots$

X_n = " " " n

The six cases

1) $X_1, \ldots, X_n$ are IID RVs and n is fixed

2) $X_1, \ldots, X_n$ are IID RVs and n is a RV

3) $X_1, \ldots, X_n$ are INID RVs (independent but not identical) and n is a fixed

4) $X_1, \ldots, X_n$ are INID RVs and n is a RV

5) $X_1, \ldots, X_n$ are dependent RVs and n is fixed

6) $X_1, \ldots, X_n$ are dependent RVs and n is a RV.

Variables of interest

1) Total portfolio loss

$$S = X_1 + X_2 + \dots + X_n$$

2) Maximum portfolio loss

$$U = \max(X_1, \dots, X_n)$$

3) Minimum portfolio loss

$$V = \min(X_1, \dots, X_n)$$

Case 5 $X_1, \ldots, X_n$ are dependent losses. n is fixed.

The CDF of S' is

$$P(S' \leq s)$$

$$= \int\int \cdots \int_{\substack{\cancel{x_1 + \cdots + x_n} \\ y_1 + \cdots + y_n \leq s}} f_{X_1, \ldots, X_n}(y_1, \ldots, y_n) \, dy_1 \cdots dy_n$$

Joint PDF of $(X_1, \ldots, X_n)$

The PDF of S' is

~~P(S'~~ $f_{S'}(s) =$

$$\int\int \cdots \int_{\substack{\cancel{x_1 + \cdots + x_n} \\ y_1 + \cdots + y_n = s}} f_{X_1, \ldots, X_n}(y_1, \ldots, y_n) \, dy_1 \cdots dy_n$$

The mth moment of S' is

$$E(S'^m) =$$

$$\sum_{m_1 + \cdots + m_n = m} \frac{m!}{m_1! \cdots m_n!} E(X_1^{m_1} \cdots X_n^{m_n})$$

The CDF of $\bar{U}$ is

$$F_{\bar{U}}(u) = P(\bar{U} \leq u)$$

$$= P(\max(X_1, \ldots, X_n) \leq u)$$

$$= P(X_1 \leq u, \ldots, X_n \leq u)$$

$$= F_{X_1, \ldots, X_n}(u, \ldots, u)$$

The PDF of $\bar{U}$ is

$$f_{\bar{U}}(u) = \frac{d}{du} F_{X_1, \ldots, X_n}(u, \ldots, u)$$

The mth moment of $\bar{U}$ is

$$E(\bar{U}^m) = \int_{-\infty}^{\infty} u^m \frac{d}{du} F_{X_1, \ldots, X_n}(u, \ldots, u)\, du$$

The CDF of $\bar{V}$ is

$$P(\bar{V} \leq v)$$
$$= 1 - P(\bar{V} > v)$$
$$= 1 - P(\min(X_1, \ldots, X_n) > v)$$
$$= 1 - P(X_1 > v, \ldots, X_n > v)$$
$$= 1 - \bar{F}_{X_1, \ldots, X_n}(v, \ldots, v)$$

The PDF of $\bar{V}$ is

$$f_{\bar{V}}(v) = -\frac{d}{dv} \bar{F}_{X_1, \ldots, X_n}(v, \ldots, v)$$

The mth moment of $\bar{V}$ is

$$E(\bar{V}^m) = -\int_{-\infty}^{\infty} v^m \frac{d}{dv} \bar{F}_{X_1, \ldots, X_n}(v, \ldots, v) dv$$

CASE 6: $X_1, \dots, X_N$ are dep random variables and N is a random variable indep of $X_1, X_2, \dots$

The CDF of S' is

$$P(S' \leq s)$$

total prob rule

$$= \sum_{n=1}^{\infty} P(S' \leq s \mid N=n)\, P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[\int\int \cdots \int_{y_1 + \cdots + y_n \leq s} f_{X_1, \dots, X_n}(y_1, \dots, y_n)\, dy_1 \cdots dy_n \right] P(N=n)$$

The PDF of S' is

$$f_{S'}(s) = \sum_{n=1}^{\infty} \left[\int\int \cdots \int_{y_1 + \cdots + y_n = s} f_{X_1, \dots, X_n}(y_1, \dots, y_n)\, dy_1 \cdots dy_n \right] P(N=n)$$

The mth moment

$$E(S'^m) = \sum_{n=1}^{\infty} E(S'^m \mid N=n)\, P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[\sum_{m_1 + \cdots + m_n = m} \frac{m!}{m_1! \cdots m_n!} E(X_1^{m_1} \cdots X_n^{m_n}) \right] P(N=n)$$

The CDF of U is

$$F_U(u) = P(U \leq u)$$

$$= \sum_{n=1}^{\infty} \underbrace{P(U \leq u \mid N = n)} P(N = n)$$

$$= \sum_{n=1}^{\infty} F_{X_1, \ldots, X_n}(u, \ldots, u) P(N = n)$$

The PDF of U is

$$f_U(u) = \sum_{n=1}^{\infty} \frac{\partial}{\partial u} F_{X_1, \ldots, X_n}(u, \ldots, u) \cdot P(N = n)$$

The m th moment of U is

$$E(U^m) = \sum_{n=1}^{\infty} \left[\int_{-\infty}^{\infty} u^m \frac{\partial}{\partial u} F_{X_1, \ldots, X_n}(u, \ldots, u) \, du \right] \cdot P(N = n)$$

The CDF of $\bar{V}$ is

$$F_{\bar{V}}(v) = P(\bar{V} \le v)$$

$$= 1 - P(\bar{V} > v)$$

$$= 1 - \sum_{n=1}^{\infty} \underbrace{P(\bar{V} > v \mid N = n)} P(N = n)$$

$$= 1 - \sum_{n=1}^{\infty} \bar{F}_{X_1, \ldots, X_n}(v, \ldots, v) P(N = n)$$

The PDF of $\bar{V}$ is

$$f_{\bar{V}}(v) = -\sum_{n=1}^{\infty} \frac{\partial}{\partial v} \bar{F}_{X_1, \ldots, X_n}(v, \ldots, v) \cdot P(N = n)$$

The mth moment of $\bar{V}$ is

$$E(\bar{V}^m) = -\sum_{n=1}^{\infty} \left[\int_{-\infty}^{\infty} v^m \frac{\partial}{\partial v} \bar{F}_{X_1, \ldots, X_n}(v, \ldots, v) \, dv \right] \cdot P(N = n)$$

Ex 1 Suppose a portfolio has 2 assets. Their losses say X and Y have the joint survival function

$$\bar{F}_{X,Y}(x,y) = \left[1 + \frac{x}{a} + \frac{y}{b}\right]^{-c}, \quad x > 0, \; y > 0$$

Find the following

(i) cdf of $\bar{U}$

(ii) pdf " $\bar{U}$

(iii) cdf of $\bar{V}$

(iv) pdf " $\bar{V}$

(v) " " $\bar{S}$

(vi) cdf " $\bar{S}$

(i) $F_U(u) = F_{X,Y}(u,u)$

$$= 1 - [1 - F_X(u)] - [1 - F_Y(u)] + \bar{F}_{X,Y}(u,u)$$

$$= -1 + F_X(u) + F_Y(u) + \bar{F}_{X,Y}(u,u)$$

$$= -1 + [1 - \bar{F}_{X,Y}(u,0)] + [1 - \bar{F}_{X,Y}(0,u)] + \bar{F}_{X,Y}(u,u)$$

$$= 1 - \bar{F}_{X,Y}(u,0) - \bar{F}_{X,Y}(0,u) + \bar{F}_{X,Y}(u,u)$$

$$= 1 - \left[1 + \frac{u}{a}\right]^{-c} - \left[1 + \frac{u}{b}\right]^{-c} + \left[1 + \frac{u}{a} + \frac{u}{b}\right]^{-c}.$$

(ii) $$f_U(u) = \frac{c}{a}\left[1+\frac{u}{a}\right]^{-c-1} + \frac{c}{b}\left[1+\frac{u}{b}\right]^{-c-1} - c\left(\frac{1}{a}+\frac{1}{b}\right)\left[1+\frac{u}{a}+\frac{u}{b}\right]^{-c-1}$$

(iii) $$F_V(v) = 1 - \bar{F}_{X,Y}(v,v) = 1 - \left[1+\frac{v}{a}+\frac{v}{b}\right]^{-c}$$

(iv) $$f_V(v) = c\left(\frac{1}{a}+\frac{1}{b}\right)\left[1+\frac{v}{a}+\frac{v}{b}\right]^{-c-1}$$

(v)

The joint PDF of (X, Y)

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} \bar{F}_{X,Y}(x, y)$$

$$= \frac{\partial}{\partial x} \left(\frac{-c}{b}\right) \left[1 + \frac{x}{a} + \frac{y}{b}\right]^{-c-1}$$

$$= \frac{c(c+1)}{ab} \left[1 + \frac{x}{a} + \frac{y}{b}\right]^{-c-2}$$

$$f_S(s) = \int_{x+y=s} \frac{c(c+1)}{ab} \left[1 + \frac{x}{a} + \frac{y}{b}\right]^{-c-2} dx\,dy$$

$$= \int_0^s \frac{c(c+1)}{ab} \left[1 + \frac{x}{a} + \frac{s-x}{b}\right]^{-c-2} \sout{dx\,dy}\ dx$$

$$= \frac{c(c+1)}{ab} \int_0^s \left[1 + \frac{s}{b} + \left(\frac{1}{a} - \frac{1}{b}\right)x\right]^{-c-2} dx$$

$$= \frac{c(c+1)}{ab} \left\{ \frac{\left[1 + \frac{s}{b} + \left(\frac{1}{a} - \frac{1}{b}\right)x\right]^{-c-1}}{(-c-1)\left(\frac{1}{a} - \frac{1}{b}\right)} \right\}_0^s$$

$$= \frac{c(c+1)}{ab} \left\{ \frac{\left(1+\frac{s}{a}\right)^{-c-1}}{(-c-1)\left(\frac{1}{a}-\frac{1}{b}\right)} + \frac{\left(1+\frac{s}{b}\right)^{-c-1}}{(c+1)\left(\frac{1}{a}-\frac{1}{b}\right)} \right\}$$

$$= \frac{c}{b-a} \left\{ \left(1+\frac{s}{b}\right)^{-c-1} - \left(1+\frac{s}{a}\right)^{-c-1} \right\}$$

(vi) $F_{S'}(s) = \int_0^s f_{S'}(t)\, dt$

$$= \frac{c}{b-a} \left\{ \int_0^s \left(1+\frac{t}{b}\right)^{-c-1} dt - \int_0^s \left(1+\frac{t}{a}\right)^{-c-1} dt \right\}$$

$$= \frac{c}{b-a} \left\{ \left[\frac{b\left(1+\frac{t}{b}\right)^{-c}}{(-c)} \right]_0^s - \left[\frac{a\left(1+\frac{t}{a}\right)^{-c}}{(-c)} \right]_0^s \right\}$$

$$= \frac{c}{b-a} \left\{ \frac{b\left(1+\frac{s}{b}\right)^{-c}}{(-c)} + \frac{b}{c} - \frac{a\left(1+\frac{s}{a}\right)^{-c}}{(-c)} - \frac{a}{c} \right\}$$

$$= \frac{1}{b-a} \left\{ - b\left(1+\frac{s}{b}\right)^{-c} + b \right.$$
$$\left. + a\left(1+\frac{s}{a}\right)^{-c} - a \right\}$$

Financial

Risk

Measures

EXAMPLE CLASS

24 OCTOBER

10:00-11:00AM

MATH3/4/68181

1

$$F_X(x) = P(X \leqslant x)$$

$$= P[\max(X_1, \ldots, X_\alpha) \leqslant x]$$

$$= P[X_1 \leqslant x, \ldots, X_\alpha \leqslant x]$$

$$\overset{\text{indep}}{=} P(X_1 \leqslant x) \cdots P(X_\alpha \leqslant x)$$

$$= [1 - e^{-\lambda x}] \cdots [1 - e^{-\lambda x}]$$

$$= [1 - e^{-\lambda x}]^\alpha$$

2

$$f_X(x) = \alpha [1 - e^{-\lambda x}]^{\alpha - 1} \lambda e^{-\lambda x}$$

$$= \alpha \lambda e^{-\lambda x} [1 - e^{-\lambda x}]^{\alpha - 1}$$

$$\frac{d\, y^b}{db} = y^b \log y$$

$$\frac{d^2}{db^2} y^b = y^b (\log y)^2$$

$$\vdots$$

$$\left.\frac{d^n}{db^n} y^b\right|_{b=0} = \left. y^b (\log y)^n \right|_{b=0}$$

$$\left.\frac{d^n}{db^n} y^b\right|_{b=0} = (\log y)^n . \quad (*)$$

Beta Function

$$B(a,b) = \int_0^1 t^{a-1} (1-t)^{b-1} \, dt$$

3

$$E(X^n) = \alpha\lambda \int_0^\infty x^n e^{-\lambda x} \left[1-e^{-\lambda x}\right]^{\alpha-1} dx$$

Set $y = e^{-\lambda x}$

$$x = -\frac{1}{\lambda}\log y$$

$$\frac{dx}{dy} = -\frac{1}{\lambda y}$$

$$= \alpha\lambda \int_1^0 \left(-\frac{1}{\lambda}\log y\right)^n y (1-y)^{\alpha-1} \frac{dy}{(-\lambda y)}$$

$$= \alpha\left(-\frac{1}{\lambda}\right)^n \int_1^0 (\log y)^n (1-y)^{\alpha-1} (-dy)$$

$$= \alpha\left(-\frac{1}{\lambda}\right)^n \int_0^1 (\log y)^n (1-y)^{\alpha-1} dy$$

(*)

$$= \alpha\left(-\frac{1}{\lambda}\right)^n \int_0^1 \frac{d^n}{db^n} y^b \Big|_{b=0} (1-y)^{\alpha-1} dy$$

$$= \alpha\left(-\frac{1}{\lambda}\right)^n \frac{d^n}{db^n} \left. \int_0^1 y^b (1-y)^{\alpha-1} dy \right|_{b=0}$$

$$= \alpha\left(-\frac{1}{\lambda}\right)^n \frac{d^n}{db^n} B(b+1, \alpha) \Big|_{b=0}$$

4

$$E(X) = -\frac{\alpha}{\lambda}\frac{d}{db} B(b+1, \alpha)\Big|_{b=0}$$

$$B(a, b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

$$= -\frac{\alpha}{\lambda}\frac{d}{db}\frac{\Gamma(b+1)\,\Gamma(\alpha)}{\Gamma(b+1+\alpha)}\Big|_{b=0}$$

$$= -\frac{\alpha\,\Gamma(\alpha)}{\lambda}\frac{\Gamma(b+1+\alpha)\,\Gamma'(b+1) - \Gamma(b+1)\,\Gamma'(b+1+\alpha)}{\Gamma^2(b+1+\alpha)}\Big|_{b=0}$$

$$= -\frac{\alpha\,\Gamma(\alpha)}{\lambda}\frac{\Gamma(1+\alpha)\,\Gamma'(1) - \Gamma'(1+\alpha)}{\Gamma^2(1+\alpha)}.$$

5

$$\mathrm{Var}(X)$$

$$= E(X^2) - (E(X))^2$$

$$= \frac{\alpha}{\lambda^2} \frac{d^2}{db^2} B(b+1, \alpha) \Big|_{b=0}$$

$$- (E(X))^2$$

[8] Suppose X is the only observation.

$$L(\alpha, \lambda) = \alpha \lambda e^{-\lambda x} [1 - e^{-\lambda x}]^{\alpha - 1}$$

$$\log L(\alpha, \lambda) = \log(\alpha\lambda) - \lambda x + (\alpha - 1)\log[1 - e^{-\lambda x}]$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{1}{\alpha} + \log[1 - e^{-\lambda x}] = 0 \quad \text{--- (1)}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{1}{\lambda} - x + (\alpha - 1)\frac{x e^{-\lambda x}}{1 - e^{-\lambda x}} = 0 \quad \text{--- (2)}$$

$$(1) \Rightarrow 1 - e^{-\lambda x} = e^{-\frac{1}{\alpha}}$$

$$\Rightarrow \lambda = -\frac{1}{x}\log\left(1 - e^{-\frac{1}{\alpha}}\right) \quad \text{--- (3)}$$

Sub (3) into (2)

$$-\frac{x}{\log\left(1 - e^{-\frac{1}{\alpha}}\right)} - x + (\alpha - 1)\frac{x \cdot \left(1 - e^{-\frac{1}{\alpha}}\right)}{e^{-\frac{1}{\alpha}}} = 0 \quad \text{--- (4)}$$

$\hat{\alpha}$ is the root of (4)

Sub $\hat{\alpha}$ into (3) to get $\hat{\lambda}$.

LECTURE

27 OCTOBER

9:00-10:00AM

MATH3/4/68181

Next week
is Reading week

Mon to Fri

11am to 5pm

ATB 2,223

0161 275 5912

FINANCIAL

RISK

MEASURES

What is a risk measure?

A risk measure gives probabilities associated with a give loss.

Ex

X = Loss

$P(X > £1 \text{ million}) > 0.9$

$\Rightarrow$ will not invest

$P(X > £1 \text{ million}) < 10^{-10}$

$\Rightarrow$ may be ok to invest

Definition of a risk measure

A risk measure $\rho(\cdot)$ from a class of random variables to $(0, \infty)$ must satisfy

(i) $\rho(0) = 0$ [normalised property]

(ii) $\rho(X + c) = \rho(X) + c$ [translative property]

(iii) $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$

[monotone property]

Definition of a coherent risk measure

A risk measure ρ is coherent if it satisfies (i) – (iii) and

$$\text{(iv)}\quad \rho(cX) = c\,\rho(X)$$

[positive homogeneity]

$$\text{(v)}\quad \rho(X+Y) \leq \rho(X) + \rho(Y)$$

[sub additive]

The two most popular risk measures

1) Value at Risk (VaR)

introduced by J. P. Morgan in the 1980s.

Suppose X has CDF $F_X(\cdot)$

Then

$$\boxed{VaR_p(X)} = \inf \{u : F_X(u) \geq p\}$$

Interpretation : Δ is the loss exceeded with probability $1-p$.

2) Expected Shortfall (ES)

introduced in the 1990s by Artzner et al.

$$ES_p(X) = \frac{1}{p}\Big[E\big(X\, I\{X \leq VaR_p(X)\}\big) + p\, VaR_p(X) - VaR_p(X) \cdot P\big(X \leq VaR_p(X)\big) \Big]$$

$$I\{A\} = \begin{cases} 1 & A \text{ is true} \\ 0 & A \text{ is false} \end{cases}$$

" indicator function "

Interpretation : $ES_p(X)$ is the average loss given that the loss does not exceed $VaR_p(X)$.

Are VaR & ES coherent risk measures ?

<u>VaR</u>

(i) ✓

(ii) ✓

(iii) ✓

(iv) ✓

(v) X

⇒ VaR is <u>not</u> a coherent risk measure

<u>ES</u>

(i) ✓

(ii) ✓

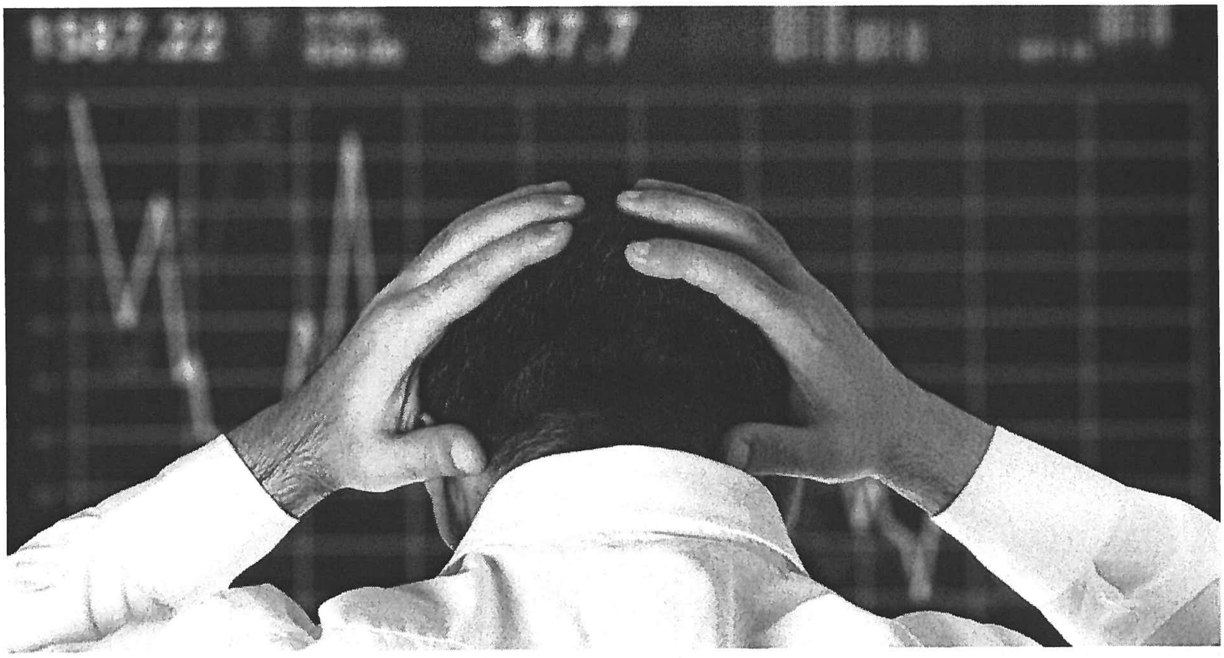
(iii) ✓

(iv) ✓

(v) ✓

⇒ ES is a coherent risk measure.

News - Did Value at Risk cause the crisis it was meant to avert?



News

Did Value at Risk cause the crisis it was meant to avert?

12 May 2016

What were the causes of the crisis of 2008? We show that managing risk using the procedure recommended by Basel II, which is called *Value at Risk*, may have played a central role. We make a very simple model for the banking system that captures the key elements of risk management under Value at Risk. Providing the banks' only take modest risks, the financial system remains stable. But if they take higher risks, or if the banking sector gets larger, the market begins to spontaneously oscillate, in a way that resembles the period leading up to and including the Global Financial Crisis. For about 10 - 15 years prices and leverage slowly rise while volatility slowly falls, then prices and leverage suddenly crash and volatility

VaR and its Role in the Credit Crisis

by Mark Kirkland, VP Treasury, Bombardier Transportation

The causes of the credit crisis of 2009 will be discussed by many for numerous years to come, although probably for fewer years than we now think. People have a unique ability to forget, perhaps black out, the worst episodes. I have sat down on a number of occasions and tried to think, what were the possible causes of the crisis? An inherent weakness in accounting of results, large numbers of over the counter derivatives with large fair values, weak governance by regulatory bodies or even that bankers were paid too much? In the end, I believe that none of the above was a key contributor to the crisis. In my mind there are two unrelated causes.

> It is now clear that very few shareholders of banks understood the risks thtat some banks were in fact taking.

The first is the mode of compensation in the financial industry. Not the amounts. Most bankers receive a kind of option pay out. If the firm makes a large profit (based on the mark to market of future uncertain cash flows), the employees receive large cash bonuses. If the firm makes a loss, in the worst case, staff may receive no bonus. Clearly, for a betting man, this gives carte blanche to load up the company with significant risk. Since most bonuses are not discussed with the owners of the company (the shareholders) but set by a compensation committee, often chaired by senior employees, there is a tendency to overpay since this justifies the compensation of the very people making the decisions. I will not dwell on this cause much longer – except to stress that the whole model encourages large risk taking.

The second is the point of this article. Risk was and still is, very badly understood, managed and reported. It is now clear that very few shareholders of banks understood the risks that some banks were in fact taking. In part, this is because disclosure of risk is unclear. A more fundamental issue, however, is that it appears that some of the banks did not fully comprehend the risk and actually outsourced much of their risk assessment to the rating agencies and then used flawed measures such as Value at Risk (VaR) not only to manage risk but also to report to management and shareholders alike.

IP&E

Is VaR to blame for the downturn?

IP Asia May 2009 **By Richard Newell**

Author and derivatives specialist Nassim Nicholas Taleb was recently quoted in a *New York Times* article entitled "Risk Mis-management". He made some valid points with regard to the usefulness of risk metrics at times of extreme market behaviour. But while VaR certainly has its laundry list of problems, Taleb takes VaR out of context by focusing on only one version of it; the Gaussian based parametric VaR, which he rightly points out is severely constrained by the dangerous assumption that asset returns follow a normal bell-shaped distribution.

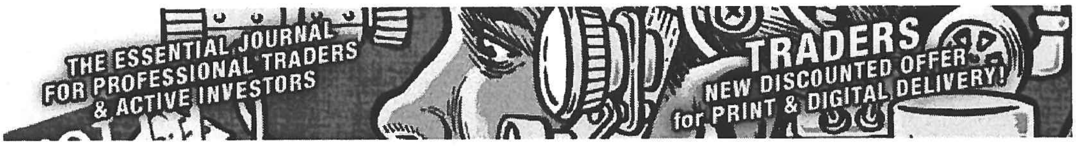
In fact, he even goes so far as to state that VaR was highly responsible for the current financial crises. This is rather disturbing, as his claims seem to have gained a wider currency, thus detracting from the infinitely more important issues behind the crisis. If we look back in history, we can see quite clearly that most "blow-ups" were not due to poor allocation decisions based on an over-reliance on risk measurement and optimisation models, but were about leverage, unchecked greed, operational disaster and outright fraud.

While VaR is a requirement for a bank, most traders and fund managers would laugh if you asked them if they took VaR seriously. The reality, alarmingly, is that risk managers have hardly any clout when it comes to strong-arming a trader or liquidity. Risk manager warnings are often ignored or overridden as senior management tends to focus purely on profitability, not risk. This is not a risk model problem, but a corporate governance problem. Instead of bashing risk managers, we should be giving them more independence, capabilities and authority to identify and limit excessive risk taking.

Long Term Capital Management was leveraged 100 times at one point and Bear Stearns' credit hedge funds over 40 times. A simple cap on gross exposure would have helped to avoid the problems they encountered with leverage. Of course, this would have interfered with a strategy that depended heavily on leverage to 'boost' minuscule returns. Back in the 1990s, Nick Leeson at Barings, the Orange County debacle, events in Mexico and Korea - all of these events had excessive leverage in common. The problems that lie within VaR are its inability to fully capture leverage and liquidity risk. Good risk managers are fully aware of this shortcoming and, as a result, VaR is only one in a whole repertoire of tools, both quantitative and qualitative, that risk managers use to get a sense of the risks they are taking on.

Taleb gives the impression that risk managers are only managing risk according to Gaussian principles, where probabilities are assumed to be normally distributed. There is more to the story than he lets on. Interestingly enough, Taleb seems to be a big fan of Monte Carlo simulations (a method that does not need to assume normality in asset return distributions) as seen in his use of Monte Carlo in the book 'Fooled by Randomness'. Taleb suggests Monte Carlo simulators allow us to learn from the simulated future which is superior to learning from the past, because the past has a survivorship bias, and we also tend to denigrate the past by claiming misfortune had by others will not happen to us. Most sophisticated risk managers use Monte Carlo very much in the same way he does.

To his credit, though, managers at LTCM were proponents of parametric VaR, which severely underestimated risk and was largely responsible for its catastrophic collapse. But even in the



VaR: The number that killed us

By Pablo Triana

December 1, 2010 • Reprints

FROM THE ARCHIVES

On Sept. 10, 2009 former trader and bestselling author Nassim Taleb did something that he very seldom does: he wore a tie. Taleb has oftentimes publicly expressed his distaste for the blood-constraining artifacts, as well as for those who tend to don them, so the Lebanese-American let the world know that was a very special day for him by betraying a sacred personal disposition.

So what prompted the composer of "The Black Swan" to button his shirt all the way up on that fall date? He had been invited to a very solemn venue by very distinguished hosts. And that was an invitation that Taleb had every intention of accepting. In fact, he had been waiting and expecting it for more than a decade. The raison d'etre of the event for which his company was now being required had been close to Taleb´s heart for most of his professional and intellectual life. It represented a central theme in his actions and ideas, close to an obsession. He had through the years incessantly warned as to the havoc that might be wreaked should others massively act in a manner counter to his convictions. Such concerns typically went unheeded (to the detriment, it turned out, of society), but now he was being offered a pulpit that seemed irresistible. This time, the world would have no option but to listen attentively.

As Taleb entered the Rayburn Building of the U.S. House of Representatives on Capitol Hill that September morning, he must have felt vindication. As he approached the sober room where several men and women awaited the start of the House Committee on Science and Technology´s hearing on the responsibility of mathematical model Value at Risk (VaR) for the terrible economic and financial crisis that had caused so much misery, Taleb probably reflected proudly on all those times when, indefatigably and in the face of harsh opposition, he alerted us of the lethal threat to the system posed by the widespread use of VaR in finance. Now that the damage wrought by VaR seemed so inescapably obvious that lawmakers had been motivated into investigating the device, Taleb no longer seemed like a lone wolf howling at the moon.

What is so wrong about VaR, and why was Taleb so concerned about its impact? More importantly, why should VaR be held responsible for the crisis? VaR is a number that purports to estimate future losses derived from a portfolio of financial assets, and presents two major problems: 1) it is doomed to being a very wrong estimate, because of its analytical foundations and the realities of real-life markets; 2) in spite of such (well-known) deficiencies, it has for the past two decades become an ubiquitously influential force in the financial world, capable of directing decision-making inside the most important banks. In other words, by letting trading activity be guided by VaR, we have essentially exposed our economic fate to a deeply flawed mechanism. Such flawedness, as was the case not only in this crisis but also before, can yield untold malaise.

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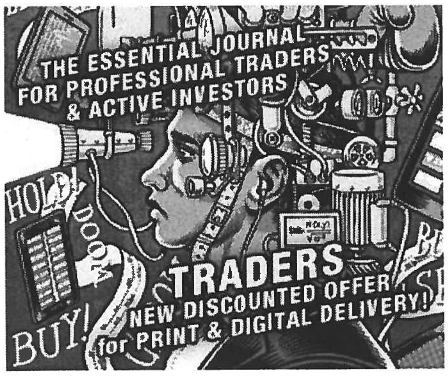

Commodities Equities Bonds Forex

Symbol	Last	Change	(%)
Emini	2576.50	+3.00	(+0.12%)
Euro	1.17715	−0.00485	(−0.41%)
Gold	1274.2	−7.6	(−0.59%)
Oil	52.19	+0.72	(+1.40%)
Gas	2.991	+0.081	(+2.78%)
Corn	344'6	+0'4	(+0.15%)

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Simpler expressions for VaR & ES

Suppose X is an absolutely continuous random variable.

Then

$$VaR_p(X) = F_X^{-1}(p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p F_X^{-1}(t)\,dt$$

Ex 1 Suppose X is $N(\mu, \sigma^2)$.

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

(Φ: CDF of $N(0,1)$)

Set $F_X(x) = p$

$$\Rightarrow \Phi\left(\frac{x-\mu}{\sigma}\right) = p$$

$$\Rightarrow \frac{x-\mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow x = \mu + \sigma\,\Phi^{-1}(p)$$

$$\Rightarrow \mathrm{VaR}_p(X) = \mu + \sigma\,\Phi^{-1}(p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \left[\mu + \sigma\,\Phi^{-1}(t)\right] dt$$

$$= \frac{1}{p}\left[\mu p + \sigma \int_0^p \Phi^{-1}(t)\,dt\right]$$

$$= \mu + \frac{\sigma}{p}\int_0^p \Phi^{-1}(t)\,dt$$

Ex 2

Suppose X is $Uni\,[-\theta, \theta]$.

$$F_X(x) = \frac{x+\theta}{2\theta}$$

Set $F_X(x) = p$

$$\Rightarrow \quad \frac{x+\theta}{2\theta} = p$$

$$\Rightarrow \quad x = 2\theta p - \theta$$

$$\Rightarrow \quad VaR_p(X) = 2\theta p - \theta$$

$$ES_p(X) = \frac{1}{p}\int_0^p (2\theta t - \theta)\,dt$$

$$= \frac{1}{p}\left[\theta t^2 - \theta t\right]_0^p$$

$$= \frac{1}{p}\left(\theta p^2 - \theta p\right)$$

$$= \theta p - \theta$$

Properties satisfied by VaR

(i) $VaR_p(0) = 0$

(ii) $VaR_p(X+c) = VaR_p(X) + c$

(iii) $X \leq Y \Rightarrow VaR_p(X) \leq VaR_p(Y)$

(iv) $VaR_p(cX) = c\,VaR_p(X)$

Homework: Try and prove these.

EXAMPLE CLASS

6 NOVEMBER

12:00-13:00PM

MATH3/4/68181

If X is an absolutely continuous RV then

$$VaR_p(X) = F^{-1}(p)$$

and
$$ES_p(X) = \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

Q1

$$F(x) = 1 - e^{-\lambda x}$$

$$1 - e^{-\lambda x} = p$$

$$\Rightarrow \quad e^{-\lambda x} = 1 - p$$

$$\Rightarrow \quad -\lambda x = \log(1-p)$$

$$\Rightarrow \quad x = -\frac{1}{\lambda}\log(1-p)$$

$$\Rightarrow \quad VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \left(-\frac{1}{\lambda}\log(1-t)\right) dt$$

$$= -\frac{1}{\lambda p}\int_0^p 1 \cdot \log(1-t)\, dt$$

Int by Parts

$$= -\frac{1}{\lambda p}\left\{\Big[t \cdot \log(1-t)\Big]_0^p - \int_0^p \frac{t \cdot (-1)}{1-t}\, dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\log(1-p) - 0 + \int_0^p \frac{t}{1-t}\, dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\log(1-p) + \int_0^p \frac{(t-1)+1}{1-t}\, dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\log(1-p) + \int_0^p \left(-1 + \frac{1}{1-t}\right) dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\log(1-p) + \Big[-t - \log(1-t)\Big]_0^p\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\log(1-p) - p - \log(1-p)\right\}.$$

Q2

$$F(x) = x^a$$

$$x^a = p$$

$$\Rightarrow \quad x = p^{\frac{1}{a}}$$

$$\Rightarrow \quad VaR_p(X) = p^{\frac{1}{a}}$$

$$ES_p(X) = \frac{1}{p}\int_0^p t^{\frac{1}{a}}\,dt$$

$$= \frac{1}{p}\left[\frac{t^{\frac{1}{a}+1}}{\frac{1}{a}+1}\right]_0^p$$

$$= \frac{1}{p}\ \frac{p^{\frac{1}{a}+1} - 0}{\frac{1}{a}+1}$$

$$= \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1} .$$

Q3

$$F(x) = \frac{x - a}{b - a}$$

$$\frac{x - a}{b - a} = p$$

$$\Rightarrow \quad x = a + p(b - a)$$

$$\Rightarrow \quad VaR_p(X) = a + p(b-a)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \left[a + t(b-a)\right] dt$$

$$= \frac{1}{p}\left[at + \frac{t^2}{2}(b-a)\right]_0^p$$

$$= \frac{1}{p}\left[ap + \frac{p^2}{2}(b-a)\right]$$

$$= a + \frac{p}{2}(b-a).$$

Q4

$$F(x) = 1 - \left(\frac{k}{x}\right)^a$$

$$1 - \left(\frac{k}{x}\right)^a = p$$

$$\Rightarrow \quad \frac{k}{x} = (1-p)^{\frac{1}{a}}$$

$$\Rightarrow \quad x = k(1-p)^{-\frac{1}{a}}$$

$$\Rightarrow \quad VaR_p(X) = k(1-p)^{-\frac{1}{a}}$$

$$ES_p(X) = \frac{k}{p}\int_0^p (1-t)^{-\frac{1}{a}}\,dt$$

$$= \frac{k}{p}\left[\frac{(1-t)^{1-\frac{1}{a}}}{(-1)\left(1-\frac{1}{a}\right)}\right]_0^p$$

$$= \frac{k}{p}\,\frac{(1-p)^{1-\frac{1}{a}} - 1}{\left(\frac{1}{a} - 1\right)}$$

Q6

$$F(x) = \left[1 + \left(\frac{x}{a}\right)^{-b}\right]^{-1}$$

$$F(x) = p$$

$$\Rightarrow \quad 1 + \left(\frac{x}{a}\right)^{-b} = \frac{1}{p}$$

$$\Rightarrow \quad \left(\frac{x}{a}\right)^{-b} = \frac{1-p}{p}$$

$$\Rightarrow \quad \frac{x}{a} = \left(\frac{1-p}{p}\right)^{-\frac{1}{b}}$$

$$\Rightarrow \quad x = a\left(\frac{1-p}{p}\right)^{-\frac{1}{b}}$$

$$\Rightarrow \quad VaR_p(X) = a\left(\frac{1-p}{p}\right)^{-\frac{1}{b}}$$

$$ES_p(X) = \frac{a}{p}\int_0^p t^{\frac{1}{b}}(1-t)^{-\frac{1}{b}}\,dt$$

Incomplete Beta Function

$$B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

$$= \frac{a}{p} B_p\left(\frac{1}{b}+1, 1-\frac{1}{b}\right).$$

Q7

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}$$

$$F(x) = p$$

$$\Rightarrow \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = 1 - p$$

$$\Rightarrow 1 + \frac{x}{\lambda} = (1-p)^{-\frac{1}{\alpha}}$$

$$\Rightarrow x = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right]$$

$$\Rightarrow VaR_p(X) = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right]$$

$$ES_p(X) = \frac{\lambda}{p}\int_0^p \left[(1-t)^{-\frac{1}{\alpha}} - 1\right] dt$$

$$= \frac{\lambda}{p}\left[\frac{(1-t)^{1-\frac{1}{\alpha}}}{\left(1-\frac{1}{\alpha}\right)(-1)} - t\right]_0^p$$

$$= \frac{\lambda}{p}\left[\frac{(1-p)^{1-\frac{1}{\alpha}}}{\left(\frac{1}{\alpha}-1\right)} - p - \frac{1}{\left(\frac{1}{\alpha}-1\right)}\right]$$

Q8

$$F(x) = e^{-\left(\frac{\sigma}{x}\right)^{\alpha}}$$

$$F(x) = p$$

$$\Rightarrow \quad -\left(\frac{\sigma}{x}\right)^{\alpha} = \log p$$

$$\Rightarrow \quad \frac{\sigma}{x} = (-\log p)^{\frac{1}{\alpha}}$$

$$\Rightarrow \quad x = \sigma\,(-\log p)^{-\frac{1}{\alpha}}$$

$$\Rightarrow \quad VaR_p(X) = \sigma\,(-\log p)^{-\frac{1}{\alpha}}$$

$$ES_p(X) = \frac{\sigma}{p}\int_0^p (-\log t)^{-\frac{1}{\alpha}}\,dt$$

Set $y = -\log t \Rightarrow t = e^{-y}$

$$\frac{dt}{dy} = -e^{-y}$$

$$= \frac{\sigma}{p}\int_{\infty}^{-\log p} y^{-\frac{1}{\alpha}}\,(-e^{-y})\,dy$$

$$= \frac{\sigma}{p}\int_{-\log p}^{\infty} y^{-\frac{1}{\alpha}}\,e^{-y}\,dy$$

Comp Incomplete Gamma Function

$$\Gamma(a, x) \quad \int_x^{\infty} t^{a-1} e^{-t}\,dt$$

$$= \frac{\sigma}{p}\,\Gamma\left(1-\frac{1}{\alpha}, -\log p\right).$$

LECTURE

7 NOVEMBER

9:00-10:00AM

MATH3/4/68181

IN-CLASS TEST

14 NOV
TUESDAY

9:00 - 10:00 AM
(LEVEL 3)

9:00 - 10:30 AM
(LEVEL 4, 6)

Proof of (ii)

Assume X is absolutely cont RV.

$$\mathrm{VaR}_p(X+c) = \mathrm{VaR}_p(X) + c \quad (*)$$

$$\iff F_{X+c}^{-1}(p) = F_X^{-1}(p) + c$$

$$\iff F_{X+c}\left(F_{X+c}^{-1}(p)\right) = F_{X+c}\left(F_X^{-1}(p)+c\right)$$

$$\iff p = F_{X+c}\left(F_X^{-1}(p) + c\right)$$

$$\iff p = P\left(X + \cancel{c} \leq F_X^{-1}(p) + \cancel{c}\right)$$

$$\iff p = P\left(X \leq F_X^{-1}(p)\right)$$

$$\iff p = F_X\left(F_X^{-1}(p)\right)$$

$$\iff p = p$$

Hence, $(*)$ must hold.

Proof of (iv)

Assume that X is absolutely cont RV.

$$\mathrm{VaR}_p(cX) = c\,\mathrm{VaR}_p(X) \quad (*)$$

$$\Leftrightarrow \quad F_{cX}^{-1}(p) = c\,F_X^{-1}(p)$$

$$\Leftrightarrow \quad F_{cX}\left(F_{cX}^{-1}(p)\right) = F_{cX}\left(c\,F_X^{-1}(p)\right)$$

$$\Leftrightarrow \quad p = F_{cX}\left(c\,F_X^{-1}(p)\right)$$

$$\Leftrightarrow \quad p = P\left(\not{c}X \leq \not{c}\,F_X^{-1}(p)\right)$$

$$\Leftrightarrow \quad p = P\left(X \leq F_X^{-1}(p)\right)$$

$$\Leftrightarrow \quad p = F_X\left(F_X^{-1}(p)\right)$$

$$\Leftrightarrow \quad p = p$$

Hence, $(*)$ must hold.

Estimation methods for VaR

a) Parametric estimation
Level 3, 4, 6

b) Non-parametric "
Level 3, 4, 6

c) Semi-parametric "
Level 4, 6

a) Parametric estimation of VaR

1) Assume X = Loss has $N(\mu, \sigma^2)$ distribution

$$VaR_p(X) = \mu + \sigma \Phi^{-1}(p)$$

Suppose $x_1, x_2, \ldots, x_n$ are observed losses. Assumed to be IID from $N(\mu, \sigma^2)$.

By the method of maximum likelihood, the estimators of μ and σ are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}$$

and

$$\widehat{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 .$$

The MLE of $VaR_p(X)$ is

$$\widehat{VaR_p}(X) = \hat{\mu} + \hat{\sigma}\,\Phi^{-1}(p)$$

$$= \bar{x} + \sqrt{\frac{1}{n}\sum(x_i - \bar{x})^2}\,\Phi^{-1}(p)$$

Φ is the CDF of $N(0,1)$

Suppose $\hat{\theta}$ is an estimator of θ.

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

$$\text{MSE}(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$$

(mean squared error)

$\hat{\theta}$ is <u>unbiased</u> if $\text{Bias}(\hat{\theta}) = 0$

$\hat{\theta}$ is <u>consistent</u> if $\text{MSE}(\hat{\theta}) \to 0$ as $n \to \infty$

<u>Homework</u>: Show that $\widehat{VaR_p}(X)$ is biased.

2) Suppose X = Loss has the Uni $[a, b]$ distribution

$$VaR_p(X) = a + p(b-a)$$

Suppose $x_1, \ldots, x_n$ are observed losses IID from Uni $[a, b]$.

By MLE, the estimators of a and b are

$$\hat{a} = \min(x_1, \ldots, x_n)$$

and

$$\hat{b} = \max(x_1, \ldots, x_n).$$

So, the MLE of $VaR_p(X)$ is

$$\widehat{VaR_p}(X) = \hat{a} + p(\hat{b} - \hat{a})$$

$$= \min(x_1, \ldots, x_n) + p\left[\max(x_1, \ldots, x_n) - \min(x_1, \ldots, x_n)\right]$$

<u>Homework</u>: Show that $\widehat{VaR_p}(X)$ is a biased estimator.

3) Assume $X =$ Loss has the Weibull distribution with CDF

$$F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}}, \quad x > 0.$$

$$F(x) = p$$

$$\Rightarrow \quad 1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}} = p$$

$$\Rightarrow \quad e^{-\left(\frac{x}{\theta}\right)^{\beta}} = 1 - p$$

$$\Rightarrow \quad \left(\frac{x}{\theta}\right)^{\beta} = -\log(1-p)$$

$$\Rightarrow \quad x = \theta\left[-\log(1-p)\right]^{\frac{1}{\beta}}$$

$$\Rightarrow \quad VaR_p(X) = \theta\left[-\log(1-p)\right]^{\frac{1}{\beta}}.$$

Suppose $x_1, \ldots, x_n$ are observed losses assumed to be IID from the Weibull distribution.

The estimators $\hat{\theta}$ and $\hat{\beta}$ are given by

$$\hat{\theta} = \frac{\bar{x}}{\Gamma\left(1+\frac{1}{\hat{\beta}}\right)},$$

$$\frac{\bar{x}^2}{s^2} = \frac{\left[\Gamma\left(1+\frac{1}{\hat{\beta}}\right)\right]^2}{\Gamma\left(1+\frac{2}{\hat{\beta}}\right) - \left[\Gamma\left(1+\frac{1}{\hat{\beta}}\right)\right]^2}.$$

Hence, the estimator of $VaR_p(X)$ is

$$\widehat{VaR}_p(X) = \hat{\theta}\left[-\log(1-p)\right]^{\frac{1}{\hat{\beta}}}.$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2.$$

EXAMPLE CLASS

7 NOVEMBER

10:00-11:00AM

MATH3/4/68181

Q1

$$F(x) = 1 - e^{-\lambda x}$$

$$1 - e^{-\lambda x} = p$$

$$\Rightarrow \quad e^{-\lambda x} = 1 - p$$

$$\Rightarrow \quad -\lambda x = \log(1-p)$$

$$\Rightarrow \quad x = -\frac{1}{\lambda}\log(1-p)$$

$$\Rightarrow \quad VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

$$= -\frac{1}{\lambda p}\int_0^p 1\cdot\log(1-t)\,dt$$

int by parts

$$= -\frac{1}{\lambda p}\left\{\left[t\cdot\log(1-t)\right]_0^p - \int_0^p \frac{(-t)}{1-t}\,dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\cdot\log(1-p) - 0 + \int_0^p \frac{(t-1)+1}{1-t}\,dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\cdot\log(1-p) + \int_0^p\left(-1 + \frac{1}{1-t}\right)dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\cdot\log(1-p) + \left[-t - \log(1-t)\right]_0^p\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\cdot\log(1-p) - p - \log(1-p)\right\}.$$

Q2

$$F(x) = x^a$$

$$x^a = p$$

$$\Rightarrow x = p^{\frac{1}{a}}$$

$$\Rightarrow VaR_p(X) = p^{\frac{1}{a}}$$

$$ES_p(X) = \frac{1}{p} \int_0^p t^{\frac{1}{a}} \, dt$$

$$= \frac{1}{p} \left[\frac{t^{\frac{1}{a}+1}}{\frac{1}{a}+1} \right]_0^p$$

$$= \frac{1}{p} \frac{p^{\frac{1}{a}+1} - 0}{\frac{1}{a}+1}$$

$$= \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1} .$$

Q3

$$F(x) = \frac{x-a}{b-a}$$

$$F(x) = p$$

$$\Rightarrow \frac{x-a}{b-a} = p$$

$$\Rightarrow x = a + p(b-a)$$

$$\Rightarrow VaR_p(X) = a + p(b-a)$$

$$ES_p(X) = \frac{1}{p}\int_0^p [a + t(b-a)]\, dt$$

$$= \frac{1}{p}\left[at + \frac{t^2}{2}(b-a)\right]_0^p$$

$$= \frac{1}{p}\left[ap + \frac{p^2}{2}(b-a)\right]$$

$$= a + \frac{p}{2}(b-a).$$

Q4.

$$F(x) = 1 - \left(\frac{k}{x}\right)^{a}$$

$$1 - \left(\frac{k}{x}\right)^{a} = p$$

$$\Rightarrow \left(\frac{k}{x}\right)^{a} = 1 - p$$

$$\Rightarrow \frac{k}{x} = (1-p)^{\frac{1}{a}}$$

$$\Rightarrow x = k\,(1-p)^{-\frac{1}{a}}$$

$$ES_p(X) = \frac{k}{p} \int_0^p (1-t)^{-\frac{1}{a}}\,dt$$

$$= \frac{k}{p} \left[\frac{(1-t)^{1-\frac{1}{a}}}{\left(1-\frac{1}{a}\right)(-1)} \right]_0^p$$

$$= \frac{k}{p} \, \frac{(1-p)^{1-\frac{1}{a}} - 1}{\left(\frac{1}{a} - 1\right)}.$$

Q6

$$F(x) = \left[1 + \left(\frac{x}{a}\right)^{-b}\right]^{-1}$$

$$F(x) = p$$

$$\Rightarrow \left[1 + \left(\frac{x}{a}\right)^{-b}\right]^{-1} = p$$

$$\Rightarrow \left(\frac{x}{a}\right)^{-b} = \frac{1-p}{p}$$

$$\Rightarrow x = a\left(\frac{1-p}{p}\right)^{-\frac{1}{b}}$$

$$\Rightarrow VaR_p(X) = a\left(\frac{1-p}{p}\right)^{-\frac{1}{b}}$$

$$ES_p(X) = \frac{a}{p}\int_0^p \left(\frac{1-t}{t}\right)^{-\frac{1}{b}} dt$$

$$= \frac{a}{p}\int_0^p t^{\frac{1}{b}} (1-t)^{-\frac{1}{b}} dt$$

$$B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dt$$

Incomplete Beta Function

$$= \frac{a}{p} B_p\left(1+\frac{1}{b}, 1-\frac{1}{b}\right).$$

Q7

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}$$

$$1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = p$$

$$\Rightarrow \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = 1 - p$$

$$\Rightarrow 1 + \frac{x}{\lambda} = (1-p)^{-\frac{1}{\alpha}}$$

$$\Rightarrow x = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right]$$

$$\Rightarrow VaR_p(X) = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right]$$

$$ES_p(X) = \frac{\lambda}{p}\int_0^p \left[(1-t)^{-\frac{1}{\alpha}} - 1\right] dt$$

$$= \frac{\lambda}{p}\left[\frac{(1-t)^{1-\frac{1}{\alpha}}}{(-1)\left(1-\frac{1}{\alpha}\right)} - t\right]_0^p$$

$$= \frac{\lambda}{p}\left[\frac{(1-p)^{1-\frac{1}{\alpha}}}{\frac{1}{\alpha} - 1} - p + \frac{1}{1 - \frac{1}{\alpha}}\right]$$

LECTURE

10 NOVEMBER

9:00-10:00AM

MATH3/4/68181

Estimation methods for VaR

a) Parametric estimation
Level 3, 4, 6

b) Non-parametric "
Level 3, 4, 6

c) Semi-parametric "
Level 4, 6

4) Power function method

Suppose X = Loss has cdf $F(x) = x^a$, $0 < x < 1$.

$$x^a = p$$

$$\Rightarrow x = p^{\frac{1}{a}}$$

$$\Rightarrow \text{VaR}_p(X) = p^{\frac{1}{a}}$$

Suppose $x_1, \ldots, x_n$ are observed losses assumed to be IID with cdf $F(x) = x^a$. The MLE for a is

$$\hat{a} = -\left(\frac{1}{n} \sum_{i=1}^{n} \log x_i\right)^{-1}.$$

So, the MLE of $\text{VaR}_p(X)$ is

$$\widehat{\text{VaR}}_p(X) = p^{\left(-\frac{n}{\sum_{i=1}^{n} \log x_i}\right)^{-1}}$$

5) <u>Variance Covariance Method</u>

Suppose the loss of a portfolio is

$$X = \sum_{i=1}^{p} w_i X_i$$

(w_i: weights; X_i: loss on ith investment)

Suppose X_i are independent $N(\mu_i, \sigma_i^2)$. Then

$$X \sim N\left(\sum_{i=1}^{p} w_i \mu_i, \sum_{i=1}^{p} w_i^2 \sigma_i^2\right).$$

The VaR of X must satify

$$F_X(x) = q$$

$$\Rightarrow \Phi\left(\frac{x - \sum_{i=1} w_i \mu_i}{\sqrt{\sum_{i=1}^{p} w_i^2 \sigma_i^2}}\right) = q$$

$$\Rightarrow x = \sum_{i=1}^{p} w_i \mu_i + \sqrt{\sum_{i=1}^{p} w_i^2 \sigma_i^2}\, \Phi^{-1}(q)$$

$$\Rightarrow \text{VaR}_q(X) = \sum_{i=1}^{p} w_i \mu_i + \sqrt{\sum_{i=1}^{p} w_i^2 \sigma_i^2}\, \Phi^{-1}(q).$$

Suppose

$x_{1,1}, x_{1,2}, \ldots, x_{1,n_1}$ are IID obsns on X_1

$x_{2,1}, x_{2,2}, \ldots, x_{2,n_2}$ " " " " X_2

$\vdots$

$x_{p,1}, x_{p,2}, \ldots, x_{p,n_p}$ " " " " X_p.

The MLEs of μ_i and σ_i^2 are

$$\widehat{\mu_i} = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{i,j}$$

$$\widehat{\sigma_i^2} = \frac{1}{n_i} \sum_{j=1}^{n_i} \left(x_{i,j} - \widehat{\mu_i} \right)^2.$$

Hence, the MLE of $VaR_q(X)$ is

$$\widehat{VaR_q}(X) = \sum_{i=1}^{p} w_i \widehat{\mu_i} + \sqrt{\sum_{i=1}^{p} w_i^2 \widehat{\sigma_i^2}}\, \Phi^{-1}(q)$$

b) Non-parametric Estimation of VaR

1) Historical Method

Suppose $x_1, x_2, \ldots, x_n$ are the losses. Arrange the data as

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$$

$x_{(1)}$: smallest loss

$x_{(n)}$: largest loss

Then

$$\widehat{VaR}_p(X) = x_{(i)}$$

if $p \in \left(\frac{i-1}{n}, \frac{i}{n} \right]$.

Example

Losses: $-5, 5, -8, 1, 0$

$$\begin{array}{ccccc} -8 & -5 & 0 & 1 & 5 \\ \| & \| & \| & \| & \| \\ x_{(1)} & x_{(2)} & x_{(3)} & x_{(4)} & x_{(5)} \end{array}$$

$$\widehat{VaR}_{0.3} = x_{(2)} = -5$$

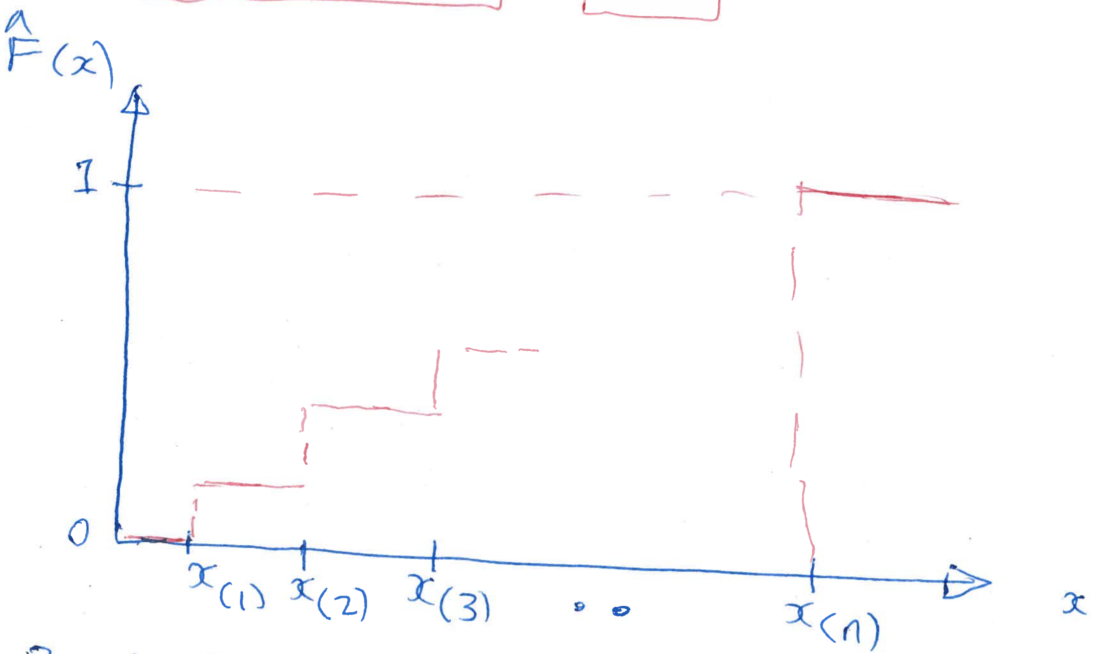
$$\widehat{VaR}_{0.9} = x_{(5)} = 5$$

2) Bootstrap Method

Suppose $x_1, x_2, \ldots, x_n$ are the observed losses. Let $\hat{F}$ denote the empirical cdf of the data.

$$\hat{F}(x) = \frac{1}{n} \sum_{j=1}^{n} I\{x_{(j)} \leq x\}$$

Empirical CDF



Bootstrap method:

i) simulate B samples each of size n from $\hat{F}$

ii) estimate VaRp by the historical method for each of the B samples, giving $\widehat{VaR}_p^{(1)}, \widehat{VaR}_p^{(2)}, \ldots, \widehat{VaR}_p^{(B)}$.

iii) Compute $\widehat{VaR}_p$ as the mean or median of $\widehat{VaR}_p^{(1)}, \ldots, \widehat{VaR}_p^{(B)}$.

3) Jackknife Method

i) estimate VaR_p by the historical method for $x_2, x_3, \ldots, x_n$, giving $\widehat{VaR}_p^{(1)}$

ii) estimate VaR_p by the historical method for $x_1, x_3, \ldots, x_n$, giving $\widehat{VaR}_p^{(2)}$.

$\vdots$

iii) estimate VaR_p by the historical method for $x_1, x_2, \ldots, x_{(n-1)}$, giving $\widehat{VaR}_p^{(n)}$.

iv) Compute $\widehat{VaR}_p$ as the mean or median of $\widehat{VaR}_p^{(1)}, \ldots, \widehat{VaR}_p^{(n)}$.

4) Kernel Method

Suppose $x_1, x_2, \ldots, x_n$ are the observed losses. The kernel estimate of the cdf of the data is

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} G\left(\frac{x - x_i}{h}\right)$$

where

$$G(x) = \int_{-\infty}^{x} K(u)\, du .$$

band width

kernel

$K(\cdot)$ is a symmetric PDF, usually $K(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$. The estimate of VaR_p is given by the root of

$$\hat{F}(x) = p$$

or

$$\widehat{VaR}_p(X) = \frac{\sum_{i=1}^{n} \hat{F}\left(\frac{i-\frac{1}{2}}{n} - p\right) x_{(i)}}{\sum_{i=1}^{n} \hat{F}\left(\frac{i-\frac{1}{2}}{n} - p\right)}$$

EXAMPLE CLASS

13 NOVEMBER

12:00-13:00PM

MATH3/4/68181

2014 Q2 (iv)

$$\lim_{k \to w(F)} \frac{P(X = k)}{1 - F(k-1)} \qquad w(F) = 2$$

$$= \frac{P(X=2)}{1 - F(1)}$$

$$= \frac{P(X=2)}{1 - P(X \leq 1)}$$

$$= \frac{P(X=2)}{P(X > 1)}$$

$$= \frac{P(X=2)}{P(X \geq 2)} = \frac{1}{1} = 1 \neq 0$$

ETT will not hold.

2014 Test Q2 (ii)

$$w(F) = +\infty$$

$$\frac{\left[1 - e^{-2x}\right]^2}{\frac{1}{2} + \frac{1}{2}\left[1 - e^{-2x}\right]^2} = 1$$

$$\Rightarrow \left[1 - e^{-2x}\right]^2 = \frac{1}{2} + \frac{1}{2}\left[1 - e^{-2x}\right]^2$$

$$\Rightarrow \frac{1}{2}\left[1 - e^{-2x}\right]^2 = \frac{1}{2}$$

$$\Rightarrow \left[1 - e^{-2x}\right]^2 = 1$$

$$\Rightarrow 1 - e^{-2x} = 1$$

$$\Rightarrow e^{-2x} = 0$$

$$\Rightarrow 2x = +\infty$$

$$\Rightarrow x = +\infty$$

(I):

$$\lim_{t\to\infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t\to\infty} \frac{\dfrac{1 - \left[1 - e^{-2(t+x\gamma(t))}\right]^2}{\frac{1}{2} + \frac{1}{2}\left[1 - e^{-2(t+x\gamma(t))}\right]^2}}{\dfrac{1 - \left[1 - e^{-2t}\right]^2}{\frac{1}{2} + \frac{1}{2}\left[1 - e^{-2t}\right]^2}}$$

(Marginal notes: $e^{-2(t+x\gamma(t))} \to 0$, $\frac{1}{2} + \frac{1}{2}\left[1 - e^{-2(t+x\gamma(t))}\right]^2 \to 1$; $e^{-2t} \to 0$, $\frac{1}{2} + \frac{1}{2}\left[1 - e^{-2t}\right]^2 \to 1$)

$$= \lim_{t\to\infty} \frac{1 - \left[1 - e^{-2(t+x\gamma(t))}\right]^2}{1 - \left[1 - e^{-2t}\right]^2}$$

$(1-x)^\alpha \approx 1 - \alpha x$ as $x \to 0$

$$= \lim_{t\to\infty} \frac{1 - \left[1 - 2\cdot e^{-2(t+x\gamma(t))}\right]}{1 - \left[1 - 2\cdot e^{-2t}\right]}$$

$$= \lim_{t\to\infty} \frac{e^{-2(t+x\gamma(t))}}{e^{-2t}}$$

$$= \lim_{t\to\infty} e^{-2x\gamma(t)}$$

$$= e^{-x} \quad \text{if} \quad \gamma(t) = \frac{1}{2}.$$

2016 Test Q1

Suppose G belongs to Gumbel domain.

$$\lim_{t \to w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to w(G)} \frac{\{1 - \{1 - [1 - G(t + x\gamma(t))]^2\}^3\}^4}{\{1 - \{1 - [1 - G(t)]^2\}^3\}^4}$$

($G(t + x\gamma(t)) \to 1$, $[1 - G(t+x\gamma(t))]^2 \to 0$; $G(t) \to 1$, $[1-G(t)]^2 \to 0$; $w(F)$ replaced by $w(G)$)

$(1-Z)^\alpha \approx 1 - \alpha Z$ as $Z \to 0$

$$= \lim_{t \to w(G)} \frac{\{1 - \{1 - 3[1 - G(t + x\gamma(t))]^2\}\}^4}{\{1 - \{1 - 3[1 - G(t)]^2\}\}^4}$$

$$= \lim_{t \to w(G)} \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^8$$

2015 Test Q2 (iii)

$$p(k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots$$

$$F(k) = \sum_{i=1}^{k} p(1-p)^{i-1}$$

$$= p \sum_{i=1}^{k} (1-p)^{i-1}$$

$$= 1 - (1-p)^k$$ by sum of a geometric series

$$\frac{p(k)}{1-F(k-1)} = \frac{p(1-p)^{k-1}}{1-[1-(1-p)^{k-1}]}$$

$$= p$$

$$\neq 0$$

ETT does not hold.

2014/15 Test Q1

i) Assume G belongs to Gumbel domain. Then there exist $\gamma(t) > 0$ such that

$$\lim_{t \to w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x} \quad \cdots (*)$$

Note

$$\lim_{t \to w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$\overset{LH}{=} \lim_{t \to w(F)} \frac{-f(t + x\gamma(t))\,(1 + x\gamma'(t))}{-f(t)}$$

$$= \lim_{t \to w(F)} \frac{\frac{1}{2}\left[-\log(1 - G(t + x\gamma(t)))\right]^2 g(t + x\gamma(t)) \cdot (1 + x\gamma'(t))}{\frac{1}{2}\left[-\log(1 - G(t))\right]^2 g(t)}$$

$$= \lim_{t \to w(F)} \left[\frac{\log(1 - G(t + x\gamma(t)))}{\log(1 - G(t))}\right]^2 \cdot \frac{g(t + x\gamma(t))\,(1 + x\gamma'(t))}{g(t)}$$

$$\overset{LH}{=} \lim_{t \to w(F)} \left[\frac{\frac{-g(t+x\gamma(t))(1+x\gamma'(t))}{1-G(t+x\gamma(t))}}{\frac{-g(t)}{1-G(t)}} \right]^2 \cdot \frac{g(t+x\gamma(t))(1+x\gamma'(t))}{g(t)}$$

$$= \lim_{t \to w(F)} \left[\frac{1-G(t)}{1-G(t+x\gamma(t))} \right]^2 \cdot \left[\frac{g(t+x\gamma(t))(1+x\gamma'(t))}{g(t)} \right]^3$$

$$\underset{\text{in reverse}}{\overset{LH}{=}} \lim_{t \to w(F)} \left[\frac{1-G(t)}{1-G(t+x\gamma(t))} \right]^2 \left[\frac{1-G(t+x\gamma(t))}{1-G(t)} \right]^3$$

$$\underset{w(G)}{\overset{w(F)=}{=}} \lim_{t \to w(G)} \frac{1-G(t+x\gamma(t))}{1-G(t)}$$

$$= e^{-x} \qquad \text{by } (*)$$

Hence, F also belong to the Gumbel domain.

2015 Test, Q2 (iv)

$$F(x) = 1$$

$$\Rightarrow \Phi^2(x) = 1$$

$$\Rightarrow \Phi(x) = 1$$

$$\Rightarrow x = \Phi^{-1}(1) = +\infty.$$

(I): $\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$

$$= \lim_{t \to \infty} \frac{1 - \Phi^2(t + x\gamma(t))}{1 - \Phi^2(t)}$$

$$\overset{LH}{=} \lim_{t \to \infty} \frac{-2\,\Phi(t + x\gamma(t))\,\phi(t + x\gamma(t))(1 + x\gamma'(t))}{-2\,\Phi(t)\,\phi(t)}$$

(red annotations: $\Phi(t + x\gamma(t)) \to 1$, $\Phi(t) \to 1$)

$$= \lim_{t \to \infty} \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(t + x\gamma(t))^2}{2}}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}} (1 + x\gamma'(t))$$

$$= \lim_{t \to \infty} e^{\frac{t^2}{2} - \frac{(t + x\gamma(t))^2}{2}} (1 + x\gamma'(t))$$

$$= \lim_{t \to \infty} e^{-t x\gamma(t) - \frac{x^2\gamma(t)^2}{2}} (1 + x\gamma'(t))$$

(red annotation: $t x\gamma(t) = x$)

Let $\gamma(t) = \frac{1}{t}$. $\gamma'(t) = -\frac{1}{t^2}$

$$= \lim_{t \to \infty} e^{-x - \frac{x^2}{2t^2}} \cdot \left(1 - \frac{x}{t^2}\right)$$

(red annotations: $\frac{x^2}{2t^2} \to 0$, $\frac{x}{t^2} \to 0$)

$$= e^{-x}.$$

2016 Test

Q2 (ii)

$$f(x) = 6x(1-x), \quad 0 < x < 1$$

$$w(F) = 1$$

(III)

$$\lim_{t \to 0} \frac{1 - F(1-tx)}{1 - F(1-t)}$$

$$\stackrel{LH}{=} \lim_{t \to 0} \frac{-x f(1-tx)}{-f(1-t)}$$

$$= \lim_{t \to 0} \frac{x \cdot 6 (1-tx) tx}{6 \cdot (1-t) \cdot t}$$

$$= x^2$$

LECTURE

17 NOVEMBER

9:00-10:00AM

MATH3/4/68181

Expected Shortfall

X = Loss

$$ES_p(X) = \frac{1}{p}\Big[E\big(X I\{X \leqslant VaR_p(X)\}\big) + p\, VaR_p(X) - VaR_p(X) \Pr\big(X \leqslant VaR_p(X)\big)\Big]$$

where $I\{\cdot\}$ denotes the indicator function

$$I\{A\} = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{" } A \text{ is false} \end{cases}$$

If X is absolutely continuous

$$ES_p(X) = \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

where F denotes the CDF of X.

Properties of ES

(i) if $X \leq Y$ then $ES_p(X) \leq ES_p(Y)$

(ii) $ES_p(X + c) = ES_p(X) + c$

where c is a constant

(iii) $ES_p(cX) = c\, ES_p(X)$

where c is a constant

Proof of (i)

Suppose X, Y ~~is~~ are absolutely continuous random variables. If $X \leq Y$ then

$$\mathrm{VaR}_q(X) \leq \mathrm{VaR}_q(Y) \quad \forall q \in (0,1)$$

$$\Rightarrow \quad F_X^{-1}(q) \leq F_Y^{-1}(q) \quad \forall q \in (0,1)$$

$$\Rightarrow \quad \int_0^p F_X^{-1}(q)\, dq \leq \int_0^p F_Y^{-1}(q)\, dq \quad \forall p \in (0,1)$$

$$\Rightarrow \quad \frac{1}{p}\int_0^p F_X^{-1}(q)\, dq \leq \frac{1}{p}\int_0^p F_Y^{-1}(q)\, dq \quad \forall p \in (0,1)$$

$$\Rightarrow \quad ES_p(X) \leq ES_p(Y) \quad \forall p \in (0,1)$$

Proof of (ii)

Suppose X is an absolutely continuous RV. Then

$$VaR_q(X+c) = VaR_q(X) + c \quad \forall q \in (0,1)$$

$$\Rightarrow \quad F_{X+c}^{-1}(q) = F_X^{-1}(q) + c \quad \forall q \in (0,1)$$

$$\Rightarrow \quad \int_0^p F_{X+c}^{-1}(q)\,dq = \int_0^p \left[F_X^{-1}(q) + c\right] dq \quad \forall p \in (0,1)$$

$$\Rightarrow \quad \frac{1}{p}\int_0^p F_{X+c}^{-1}(q)\,dq = \frac{1}{p}\int_0^p \left[F_X^{-1}(q) + c\right] dq \quad \forall p \in (0,1)$$

$$\Rightarrow \quad ES_p(X+c) = ES_p(X) + c$$

Proof of (iii)

Suppose X is an absolutely continuous RV. Then

$$\mathrm{VaR}_q(cX) = c\,\mathrm{VaR}_q(X) \quad \forall q \in (0,1)$$

$$\Rightarrow \quad F_{cX}^{-1}(q) = c\,F_X^{-1}(q) \quad \forall q \in (0,1)$$

$$\Rightarrow \quad \int_0^p F_{cX}^{-1}(q)\,dq = c\int_0^p F_X^{-1}(q)\,dq \quad \forall p \in (0,1)$$

$$\Rightarrow \quad \frac{1}{p}\int_0^p F_{cX}^{-1}(q)\,dq = \frac{c}{p}\int_0^p F_X^{-1}(q)\,dq \quad \forall p \in (0,1)$$

$$\Rightarrow \quad ES_p(cX) = c\,ES_p(X).$$

Estimation Methods for ~~VaR~~ ES

a) Parametric estimation methods
Levels 3, 4, 6

b) Non-parametric " "
Levels 3, 4, 6

c) Semi-parametric " "
Levels 4, 6

a) Parametric estimation of ES

i) Normal distribution method

$$X = \text{Loss} \sim N(\mu, \sigma^2)$$

$$VaR_p(X) = \mu + \sigma \Phi^{-1}(p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p [\mu + \sigma \Phi^{-1}(t)]\,dt$$

$$= \mu + \frac{\sigma}{p}\int_0^p \Phi^{-1}(t)\,dt$$

Suppose $x_1, \ldots, x_n$ are IID observed losses on X. Then the MLEs of μ and σ are

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

and

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}.$$

So, the MLE of $ES_p(X)$ is

$$\widehat{ES}_p(X) = \bar{x} + \frac{1}{p}\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}\int_0^p \Phi^{-1}(t)\,dt$$

ii) Uniform distribution method

$$X = \text{Loss} \sim \text{Uni}[a, b]$$

$$VaR_p(X) = a + (b-a)p$$

$$ES_p(X) = \frac{1}{p}\int_0^p [a + (b-a)t] \, dt$$

$$= a + \frac{p}{2}(b-a).$$

Suppose $x_1, \ldots, x_n$ are IID observed losses on X. Then the MLEs of a and b are

$$\widehat{a} = \min(x_1, \ldots, x_n)$$

$$\widehat{b} = \max(x_1, \ldots, x_n).$$

So, the MLE of $ES_p(X)$ is

$$\widehat{ES_p(X)} = \min(x_1, \ldots, x_n) + \frac{p}{2}\left[\max(x_1, \ldots, x_n) - \min(x_1, \ldots, x_n)\right]$$

(iii) Power function distribution method

X = Loss has CDF $F(x) = x^a$, $0 < x < 1$

$$VaR_p(X) = p^{\frac{1}{a}}$$

$$ES_p(X) = \frac{1}{p}\int_0^p t^{\frac{1}{a}}\,dt$$

$$= \frac{1}{p}\left[\frac{t^{\frac{1}{a}+1}}{\frac{1}{a}+1}\right]_0^p$$

$$= \frac{a p^{\frac{1}{a}}}{a+1}.$$

Suppose $x_1, \ldots, x_n$ are IID observed losses on X. Then the MLE of a is

$$\widehat{a} = -\frac{n}{\sum_{i=1}^{n} \log x_i}.$$

So, the MLE of $ES_p(X)$ is

$$\widehat{ES_p(X)} = \frac{p^{-\frac{\sum_{i=1}^{n} \log x_i}{n}}}{1 - \frac{\sum_{i=1}^{n} \log x_i}{n}}.$$

(iv) Weibull distribution method

$$X = \text{Loss} \sim F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}}$$

Set $F(x) = p$

$$\Rightarrow 1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}} = p$$

$$\Rightarrow \left(\frac{x}{\theta}\right)^{\beta} = -\log(1-p)$$

$$\Rightarrow VaR_p(X) = \theta\left[-\log(1-p)\right]^{\frac{1}{\beta}}$$

$$ES_p(X) = \frac{\theta}{p}\int_0^p \left[-\log(1-t)\right]^{\frac{1}{\beta}} dt$$

Set $x = -\log(1-t)$

$$\Rightarrow e^{-x} = 1 - t$$

$$\Rightarrow t = 1 - e^{-x}$$

$$\Rightarrow \frac{dt}{dx} = e^{-x}$$

$$= \frac{\theta}{p}\int_0^{-\log(1-p)} x^{\frac{1}{\beta}} e^{-x}\, dx$$

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t}\, dt$$

Incomplete gamma function

$$= \frac{\theta}{p}\,\gamma\left(\frac{1}{\beta} + 1, -\log(1-p)\right).$$

Suppose $x_1, \ldots, x_n$ are IID observed losses on X. The estimators of θ and β are given by

$$\widehat{\theta} = \frac{\overline{x}}{\Gamma\left(1 + \frac{1}{\widehat{\beta}}\right)}$$

and

$$\frac{\overline{x}^2}{s^2} = \frac{\left(\Gamma\left(1 + \frac{1}{\widehat{\beta}}\right)\right)^2}{\Gamma\left(1 + \frac{2}{\widehat{\beta}}\right) - \left(\Gamma\left(1 + \frac{1}{\widehat{\beta}}\right)\right)^2}$$

where

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

and

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2.$$

The estimator of $ES_p(X)$ is

$$\widehat{ES_p(X)} = \frac{\widehat{\theta}}{p} \gamma\left(\frac{1}{\widehat{\beta}} + 1, -\log(1-p)\right).$$

EXAMPLE CLASS

20 NOVEMBER

12:00-13:00PM

MATH3/4/68181

Q1

$X \sim Exp(\lambda)$

$$VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

$$ES_p(X) = -\frac{1}{\lambda p}\left[p\log(1-p) - p - \log(1-p)\right]$$

$$L(\lambda) = \prod_{i=1}^{n}\left[\lambda e^{-\lambda x_i}\right]$$

$$= \lambda^n e^{-\lambda\sum_{i=1}^{n} x_i}$$

$$\log L(\lambda) = n\log\lambda - \lambda\sum_{i=1}^{n} x_i$$

$$\frac{d\log L}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}.$$

Thus, the MLEs of VaR and ES are

$$\widehat{VaR}_p(X) = -\bar{x}\cdot\log(1-p)$$

$$\widehat{ES}_p(X) = -\frac{\bar{x}}{p}\left[p\cdot\log(1-p) - p - \log(1-p)\right]$$

Q2

$$\mathrm{VaR}_p(X) = p^{\frac{1}{a}}$$

$$\mathrm{ES}_p(X) = \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1}$$

$$L(a) = \prod_{i=1}^{n}\left[a x_i^{a-1}\right]$$

$$= a^n \left(\prod_{i=1}^{n} x_i\right)^{a-1}$$

$$\log L(a) = n \log a + (a-1)\sum_{i=1}^{n} \log x_i$$

$$\frac{d \log L}{da} = \frac{n}{a} + \sum_{i=1}^{n} \log x_i = 0$$

$$\hat{a} = -\frac{n}{\sum_{i=1}^{n} \log x_i}.$$

Thus, the MLEs of VaR and ES are

$$\widehat{\mathrm{VaR}}_p(X) = p^{-\frac{\sum_{i=1}^{n}\log x_i}{n}}$$

and

$$\widehat{\mathrm{ES}}_p(X) = \frac{p^{-\frac{\sum_{i=1}^{n}\log x_i}{n}}}{-\frac{\sum_{i=1}^{n}\log x_i}{n}+1}$$

Q3 $X \sim N(\mu, \sigma^2)$

$$\mathrm{VaR}_p(X) = \mu + \sigma \Phi^{-1}(p)$$

$$\mathrm{ES}_p(X) = \mu + \frac{\sigma}{p}\int_0^p \Phi^{-1}(t)\,dt$$

$$L(\mu, \sigma) = \prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}\right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$\log L(\mu, \sigma) = -\frac{n}{2}\log(2\pi) - n\log\sigma$$

$$-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial \log L}{\partial \mu} = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)\cdot 2\cdot(-1)$$

$$= \frac{1}{\sigma^2}\left(\left(\sum_{i=1}^{n} x_i\right) - n\mu\right) = 0 \quad \text{--- (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2 = 0$$

$$\text{--- (2)}$$

$$(1) \Rightarrow \widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$$(2) \Rightarrow \widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Thus, the MLEs of VaR and ES are

$$\widehat{VaR}_p(X) = \bar{x} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}\,\Phi^{-1}(p)$$

$$\widehat{ES}_p(X) = \bar{x} + \frac{1}{p}\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} \cdot \int_0^p \Phi^{-1}(t)\,dt$$

Q4

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right)$$

Set $F(x) = p$

$$\Rightarrow \Phi\left(\frac{\log x - \mu}{\sigma}\right) = p$$

$$\Rightarrow \frac{\log x - \mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow \log x = \mu + \sigma \Phi^{-1}(p)$$

$$\Rightarrow x = \boxed{e^{\mu + \sigma \Phi^{-1}(p)}}$$

$$ES_p(X) = \frac{1}{p} \int_0^p F^{-1}(t)\, dt$$

$$= \boxed{\frac{e^{\mu}}{p} \int_0^p e^{\sigma \Phi^{-1}(t)}\, dt}$$

$$L(\mu, \sigma) = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi}\,\sigma x_i} e^{-\frac{(\log x_i - \mu)^2}{2\sigma^2}} \right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n \left(\prod_{i=1}^{n} x_i \right)} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (\log x_i - \mu)^2}$$

$$\log L(\mu, \sigma) = -\frac{n}{2} \log(2\pi) - n \log \sigma$$

$$- \sum_{i=1}^{n} \log x_i - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (\log x_i - \mu)^2$$

$$\frac{\partial \log L}{\partial \mu} = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (\log x_i - \mu) \cdot 2 \cdot (-1)$$

$$= \frac{1}{\sigma^2} \left[\left(\sum_{i=1}^{n} \log x_i \right) - n\mu \right] = 0 \quad \text{— (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (\log x_i - \mu)^2 = 0 \quad \text{— (2)}$$

$$(1) \Rightarrow \hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \log x_i$$

$$(2) \Rightarrow \widehat{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} (\log x_i - \hat{\mu})^2$$

The MLEs of VaR and ES are

$$\widehat{VaR}_p(X) = e^{\frac{1}{n}\sum_{i=1}^{n}\log x_i + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\log x_i - \hat{\mu})^2}\,\Phi^{-1}(}$$

and

$$\widehat{ES}_p(X) = \frac{e^{\frac{1}{n}\sum_{i=1}^{n}\log x_i}}{p}\int_0^p e^{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\log x_i - \hat{\mu})^2}\,\Phi^{-1}(t)}\,dt$$

LECTURE

21 NOVEMBER

9:00-10:00AM

MATH3/4/68181

Estimation Methods for ~~VaR~~ ES

a) Parametric estimation methods
Levels 3, 4, 6

b) Non-parametric " "
Levels 3, 4, 6

c) Semi-parametric " "
Levels 4, 6

b) Non-parametric estimation methods for ES

i) Historical method

Suppose the data are $x_1, x_2, \ldots, x_n$

Order the data as

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$$

smallest ($x_{(1)}$) largest ($x_{(n)}$)

The historical estimator of $ES_p(X)$ is

$$\widehat{ES}_p(X) = \frac{\sum_{i=1}^{[np]} x_{(i)}}{[np]},$$

where $[x]$ denotes the largest integer not greater than x.

Ex

$[4.1] = 4$

$[5.6] = 5$

Ex

Observed losses: $-6, 7, 0, 1, 9$

$$\begin{matrix} -6 & 0 & 1 & 7 & 9 \\ \| & \| & \| & \| & \| \\ x_{(1)} & x_{(2)} & x_{(3)} & x_{(4)} & x_{(5)} \end{matrix}$$

$$\widehat{ES}_{0.2}(X) = \frac{\sum_{i=1}^{1} x_{(i)}}{1} = -6$$

$$\widehat{ES}_{0.8}(X) = \frac{\sum_{i=1}^{4} x_{(i)}}{4}$$

$$= \frac{-6+0+1+7}{4}$$

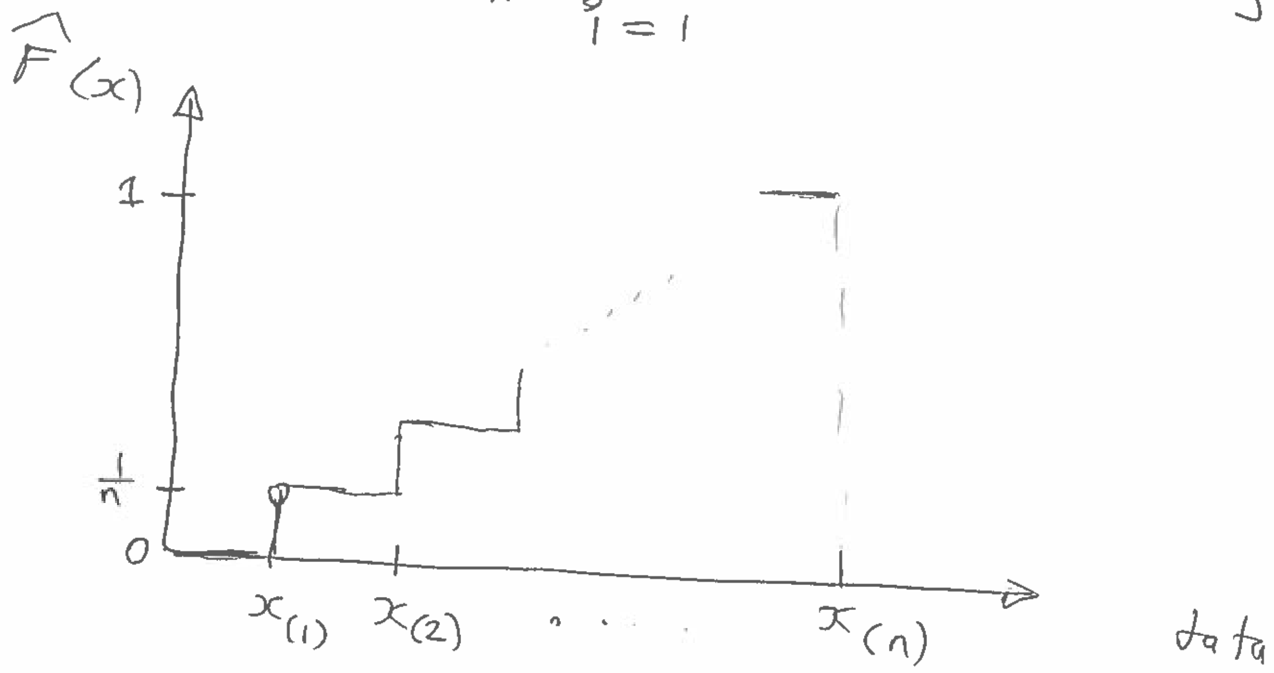
$$= 0.5$$

(ii) Bootstrap method

Suppose the data are $x_1, x_2, \ldots, x_n$.

The empirical cdf is

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} I\{x_i \leq x\}$$



1. Simulate B random samples each of size n from $\hat{F}$.
2. Estimate $ES_p(X)$ by the historical method for each of the B samples, resulting in $\widehat{ES}_p^{(1)}, ES_p^{(2)}, \ldots, \widehat{ES}_p^{(B)}$.
3. Compute $\widehat{ES}_p$ as the mean or median of $\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \ldots, \widehat{ES}_p^{(B)}$.

iii) Jackknife method

Soppose the data are $x_1, x_2, \ldots, x_n$.

1. Estimate $ES_p(X)$ by the historical estimator based on $x_2, x_3, \ldots, x_n$, resulting in $\widehat{ES}_p^{(1)}$.
2. Estimate $ES_p(X)$ by the historical estimator based on $x_1, x_3, \ldots, x_n$, resulting in $\widehat{ES}_p^{(2)}$.

$\vdots$

3. Estimate $ES_p(X)$ by the historical estimator based $x_1, x_2, \ldots, x_{n-1}$, resulting in $\widehat{ES}_p^{(n)}$.
4. Compute $\widehat{ES}_p$ as the mean or median of $\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \ldots, \widehat{ES}_p^{(n)}$.

iv) Kernel method

Suppose the data are $x_1, x_2, \ldots, x_n$.

Order the data as $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$.

The kernel estimator of $ES_p(X)$ is

$$\widehat{ES_p}(X) = \frac{1}{np} \sum_{i=1}^{n} x_i \, A\left(\frac{\hat{q}(p) - x_i}{h} \right),$$

where

$$A(x) = \int_{-\infty}^{x} K(u)\, du$$

and

$$\hat{q}(p) = \frac{1}{h} \sum_{i=1}^{n} \left[\int_{\frac{i-1}{n}}^{\frac{i}{n}} K\left(\frac{t-p}{h}\right) dt \right] x_{(i)}$$

$K(\cdot)$ is the kernel, eg

$$K(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$$

h = bandwidth

v) Richardson's method

(Richardson was a professor in Manchester. His photo in ground floor of ATB).

1. Compute the empirical cdf $\hat{F}$ of the observed data

2. simulate $x_1, x_2, .., x_N$ from $\hat{F}$

3. estimate ES_p by the historical method for $x_1, x_2, .., x_N$

4. repeat steps 2 and 3 1000 times and compute

$$m_N = \frac{1}{1000} \sum_{i=1}^{1000} \widehat{ES}_{N,i}$$

estimate obtained in the ith iteration

5. Set

$$S_q = m N_q$$

for $q = 1, 2, \ldots, k+1$

and for some $N_1, N_2, \ldots, N_{k+1}$.

6. Compute

$$\widehat{ES_p(X)} = \frac{\begin{vmatrix} S_1 & S_2 & \cdots & S_{k+1} \\ 1 & \frac{1}{2} & \cdots & \frac{1}{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1^k & \left(\frac{1}{2}\right)^k & \cdots & \left(\frac{1}{k+1}\right)^k \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \frac{1}{2} & \cdots & \frac{1}{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1^k & \left(\frac{1}{2}\right)^k & \cdots & \left(\frac{1}{k+1}\right)^k \end{vmatrix}}.$$

EXAMPLE CLASS

21 NOVEMBER

10:00-11:00AM

MATH3/4/68181

Q1

$$VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

$$ES_p(X) = -\frac{1}{\lambda p}\left[p\cdot\log(1-p) - p - \log(1-p)\right]$$

$$L(\lambda) = \prod_{i=1}^{n}\left[\lambda e^{-\lambda x_i}\right]$$

$$= \lambda^n e^{-\lambda\sum_{i=1}^{n} x_i}$$

$$\log L(\lambda) = n\log\lambda - \lambda\sum_{i=1}^{n} x_i$$

$$\frac{d\log L}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$$

$$\frac{d^2\log L}{d\lambda^2} = -\frac{n}{\lambda^2} < 0$$

So $\hat{\lambda} = \frac{1}{\bar{x}}$ is an MLE

Hence, the MLEs of VaR and ES are

$$\widehat{VaR}_p(X) = -\bar{x}\log(1-p)$$

$$\widehat{ES}_p(X) = -\frac{\bar{x}}{p}\left[p\log(1-p) - p - \log(1-p)\right]$$

Q 2

$$\mathrm{VaR}_p(X) = p^{\frac{1}{a}}$$

$$\mathrm{ES}_p(X) = \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1}$$

$$L(a) = \prod_{i=1}^{n}\left[a\, x_i^{a-1}\right] = a^n\left(\prod_{i=1}^{n} x_i\right)^{a-}$$

$$\log L(a) = n\log a + (a-1)\sum_{i=1}^{n}\log x_i$$

$$\frac{d\log L}{da} = \frac{n}{a} + \sum_{i=1}^{n}\log x_i = 0$$

$$\Rightarrow \hat{a} = -\frac{n}{\sum_{i=1}^{n}\log x_i}$$

$$\frac{d^2\log L}{da^2} = -\frac{n}{a^2} < 0$$

So, $\hat{a}$ is an MLE. Hence, the MLEs of VaR and ES are

$$\widehat{\mathrm{VaR}}_p(X) = p^{-\frac{\sum_{i=1}^{n}\log x_i}{n}}$$

and

$$\widehat{\mathrm{ES}}_p(X) = \frac{p^{-\frac{\sum_{i=1}^{n}\log x_i}{n}}}{1-\frac{\sum_{i=1}^{n}\log x_i}{n}}$$

Q3

$$VaR_p(X) = \mu + \sigma \Phi^{-1}(p)$$

$$ES_p(X) = \mu + \frac{\sigma}{p} \int_0^p \Phi^{-1}(t)\,dt$$

$$L(\mu, \sigma) = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} \right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2}$$

$$\log L(\mu, \sigma) = -\frac{n}{2} \log(2\pi) - n \log \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2$$

$$\frac{\partial \log L}{\partial \mu} = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu) \cdot 2 \cdot (-1)$$

$$= \frac{1}{\sigma^2} \left[\left(\sum_{i=1}^{n} x_i \right) - n\mu \right] = 0 \quad \text{--- (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (x_i - \mu)^2 = 0 \quad \text{--- (2)}$$

$$(1) \Rightarrow \widehat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}$$

$$(2) \Rightarrow \widehat{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

The MLEs of VaR and ES are

$$\widehat{VaR}_p(X) = \bar{x} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}\, \Phi^{-1}(p)$$

and

$$\widehat{ES}_p(X) = \bar{x} + \frac{1}{p}\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} \int_0^p \Phi^{-1}(t)\,dt$$

Q4

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right) = p$$

$$\Rightarrow \frac{\log x - \mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow \log x = \mu + \sigma \Phi^{-1}(p)$$

$$\Rightarrow x = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$\Rightarrow \mathrm{VaR}_p(X) = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$ES_p(X) = \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

$$= \frac{e^{\mu}}{p}\int_0^p e^{\sigma \Phi^{-1}(t)}\,dt$$

$$L(\mu, \sigma) = \prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi}\,\sigma x_i} e^{-\frac{(\log x_i - \mu)^2}{2\sigma^2}}\right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n \prod_{i=1}^{n} x_i} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\log x_i - \mu)^2}$$

$$\log L(\mu, \sigma) = -\frac{n}{2}\log(2\pi) - n\log\sigma - \sum_{i=1}^{n}\log x_i - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\log x_i - \mu)^2$$

$$\frac{\partial \log L}{\partial \mu} = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\log x_i - \mu)\cdot 2\cdot(-1)$$

$$= \frac{1}{\sigma^2}\left(\left(\sum_{i=1}^{n}\log x_i\right) - n\mu\right) = 0 \quad \text{—— (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(\log x_i - \mu)^2 = 0$$

$$\text{—— (2)}$$

$$(1) \Rightarrow \quad \widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n}\log x_i$$

$$(2) \Rightarrow \quad \widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(\log x_i - \widehat{\mu})^2$$

The MLEs of VaR and ES are

$$\widehat{VaR}_p(X) = e^{\frac{1}{n}\sum_{i=1}^{n}\log x_i + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\log x_i - \widehat{\mu})^2}\,\Phi^{-1}(p}$$

and

$$\widehat{ES}_p(X) = \frac{e^{\frac{1}{n}\sum_{i=1}^{n}\log x_i}}{p}\int_0^p e^{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\log x_i - \widehat{\mu})^2}\,\Phi^{-1}(t)}\, dt$$

MATH3/4/68181: Extreme values and financial risk
Semester 1
Problem sheet 10

1. If $x_1, x_2, \ldots, x_n$ is a random sample from Exp (λ) find the maximum likelihood estimates of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$.

2. If $x_1, x_2, \ldots, x_n$ is a random sample from the power function distribution with pdf $f(x) = ax^{a-1}$, $0 < x < 1$, $a > 0$ find the maximum likelihood estimates of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$.

3. If $x_1, x_2, \ldots, x_n$ is a random sample from the normal distribution $N(\mu, \sigma^2)$ find the maximum likelihood estimates of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$.

4. If $x_1, x_2, \ldots, x_n$ is a random sample from the log-normal distribution $LN(\mu, \sigma^2)$ find the maximum likelihood estimates of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$.

5. If $x_1, x_2, \ldots, x_n$ is a random sample from a distribution with pdf $f(x) = \theta_2 x^{\theta_2 - 1} \theta_1^{-\theta_2}$, $0 < x < \theta_1$, $\theta_1 > 0$, $\theta_2 > 0$ find the maximum likelihood estimates of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$.

6. If $x_1, x_2, \ldots, x_n$ is a random sample from the uniform $[\mu - \delta, \mu + \delta]$ distribution find the maximum likelihood estimates of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$.

Indicator function approach

LECTURE

24 NOVEMBER

9:00-10:00AM

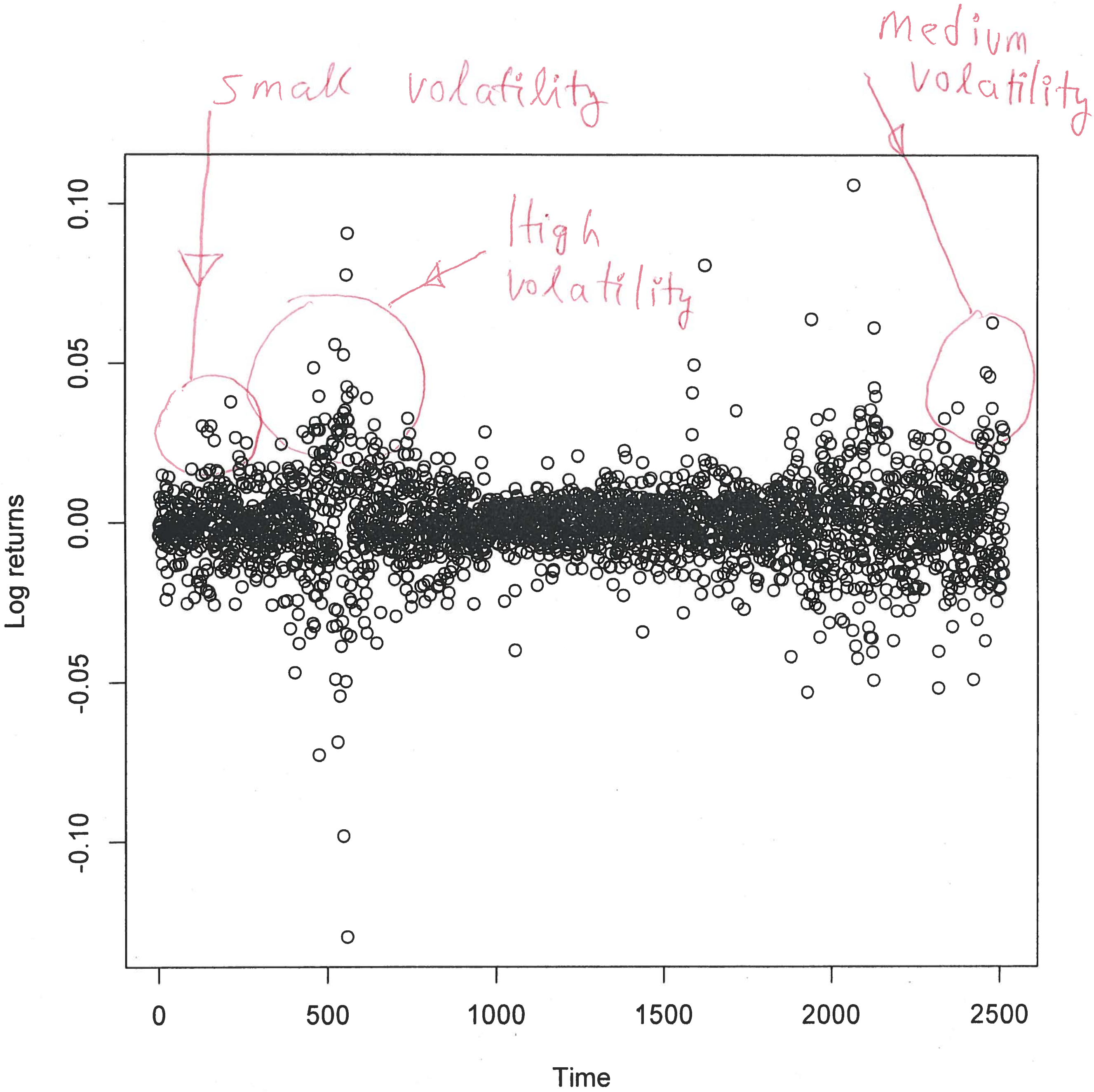
MATH3/4/68181

Models for Stock Returns

i) Model I

ii) Model II

iii) Model III



the amount of volatility at a given time is a variable itself.

i) Model I

Let X = stock return at a given time

V = volatility at a given time

X is an observable RV

V is not an " "

But X depends on V.

What is the actual distribution of X?

The PDF of X is

$$f_X(x) = \int_0^{\infty} f_{X|V}(x|v)\, g(v)\, dv$$

[total prob rule]

$f_{X|V}(x|v)$: PDF of X given V

$g(v)$: PDF of V

The CDF of X is

$$F_X(x) = \int_0^\infty F_{X|V}(x|v)\, g(v)\, dv$$

[total prob rule]

CDF of X given V

PDF of V

The nth moment of X is

$$E[X^n] = E\big[E[X^n | V]\big]$$

$$= \int_0^\infty E[X^n | V]\, g(v)\, dv$$

For example,

$$E[X] = E\big[E[X|V]\big],$$

$$\mathrm{Var}[X] = E\big[E[X^2|V]\big] - \Big(E\big[E[X|V]\big]\Big)^2$$

Ex

$$X \mid V \sim N(0, \boxed{\sigma^2})$$

$\sigma^2 = V$

V is a RV

Let $g(\cdot)$ denote the PDF of σ.

The actual PDF of X is

$$f_X(x) = \int_0^\infty \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}} \underbrace{g(\sigma)}\, d\sigma$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sigma} e^{-\frac{x^2}{2\sigma^2}} g(\sigma)\, d\sigma$$

Suppose

$$g(\sigma) = \frac{2}{\sigma^3} e^{-\frac{1}{\sigma^2}}, \quad \sigma > 0$$

Then

$$f_X(x) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sigma^4} e^{-\frac{x^2}{2\sigma^2} - \frac{1}{\sigma^2}} d\sigma$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{\sigma^4} e^{-\left(\frac{x^2}{2} + 1\right)\frac{1}{\sigma^2}} d\sigma$$

Set $y = \left(\frac{x^2}{2}+1\right)\frac{1}{\sigma^2}$

$$\sigma^2 = \left(\frac{x^2}{2}+1\right)\frac{1}{y}$$

$$\sigma = \sqrt{\frac{x^2}{2}+1}\ \frac{1}{\sqrt{y}}$$

$$\frac{d\sigma}{dy} = -\frac{1}{2}\sqrt{\frac{x^2}{2}+1}\ \frac{1}{y^{3/2}}$$

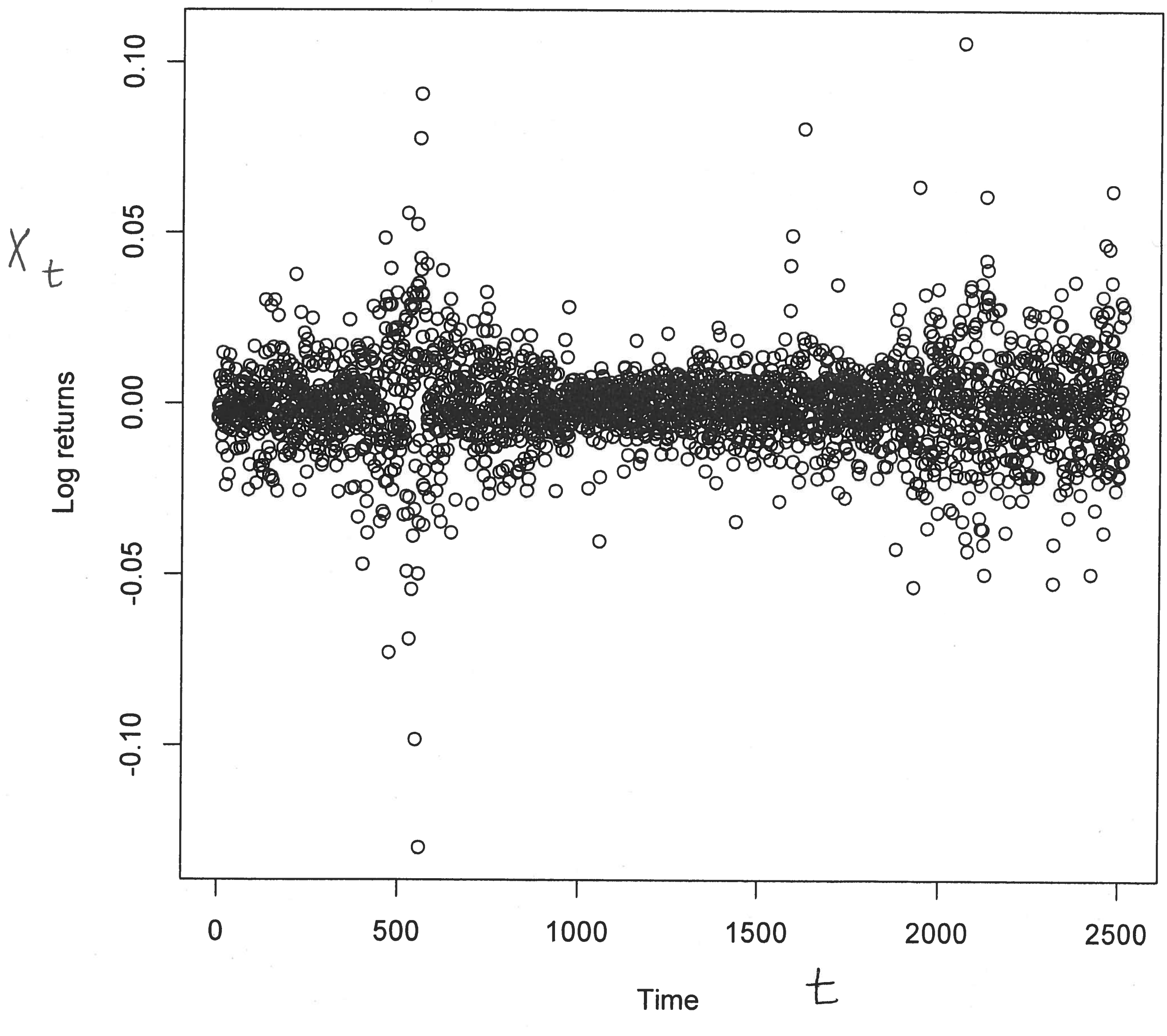
$$f_X(x) = \sqrt{\frac{2}{\pi}}\int_{\infty}^{0} \frac{y^2}{\left(\frac{x^2}{2}+1\right)^2} e^{-y}\left(-\frac{1}{2}\right)\sqrt{\frac{x^2}{2}+1}\cdot\frac{1}{y^{3/2}}\,dy$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{x^2}{2}+1\right)^{-\frac{3}{2}} \int_0^{\infty} y^{\frac{1}{2}} e^{-y}\,dy$$

$$\int_0^{\infty} y^{\frac{1}{2}} e^{-y}\,dy = \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{x^2}{2}+1\right)^{-\frac{3}{2}} \frac{\sqrt{\pi}}{2}$$

$$= \frac{1}{2\sqrt{2}}\left(\frac{x^2}{2}+1\right)^{-\frac{3}{2}}$$



X_t
Log returns
0.10
0.05
0.00
-0.05
-0.10
0
500
1000
1500
2000
2500
Time
t

ii) Model II

Let X_t = stock return at t

X_0 = stock return at time 0

Assume X_0 is known (a fixed number).

$$
\begin{aligned}
X_t = \; & (X_t - X_{t-1}) && = Z_t \\
& + (X_{t-1} - X_{t-2}) && = Z_{t-1} \\
& + (X_{t-2} - X_{t-3}) && = Z_{t-2} \\
& + \vdots \\
& + (X_1 - X_0) && = Z_1 \\
& + X_0
\end{aligned}
$$

$$= \left(\sum_{i=1}^{t} Z_i \right) + X_0$$

$$E[X_t] = \sum_{i=1}^{t} E(Z_i) + X_0$$

$$E[X_t^2] = E\left\{\left[\left(\sum_{i=1}^{t} Z_i\right) + X_0\right]^2\right\}$$

$$= E\left[\left(\sum_{i=1}^{t} Z_i\right)^2\right]$$

$$+ 2X_0 E\left[\sum_{i=1}^{t} Z_i\right]$$

$$+ X_0^2$$

$$= \sum_{i=1}^{t} E(Z_i^2) + \sum_{i \neq j} E(Z_i Z_j)$$

$$+ 2X_0 \sum_{i=1}^{t} E(Z_i)$$

$$+ X_0^2$$

$$Var[X_t] = E[X_t^2] - (E[X_t])^2$$

$$= \sum_{i=1}^{t} E(Z_i^2) + \sum_{i \neq j} E(Z_i Z_j)$$

$$- \left[\sum_{i=1}^{t} E(Z_i)\right]^2.$$

Suppose Z_i are IID.

Then

$$E[X_t] = t\,E(Z) + X_0,$$

$$\begin{aligned} E[X_t^2] = {} & t\,E(Z^2) \\ & + t(t-1)\,E(Z^2) \\ & + 2X_0\,t\,E(Z) \\ & + X_0^2, \end{aligned}$$

$$\begin{aligned} \mathrm{Var}[X_t] = {} & t\,E[Z^2] \\ & + t(t-1)\,E(Z^2) \\ & - t^2\left(E(Z)\right)^2 \end{aligned}$$

Ex

Suppose Z_i are independent $N(\mu_i, \sigma_i^2)$. What is the distribution of X_t?

$$X_t = \left(\sum_{i=1}^{t} Z_i\right) + X_0$$

$$= \left(\sum_{i=1}^{t} N(\mu_i, \sigma_i^2)\right) + X_0$$

$$= N\left(\sum_{i=1}^{t} \mu_i, \sum_{i=1}^{t} \sigma_i^2\right) + X_0$$

$$= N\left(X_0 + \sum_{i=1}^{t} \mu_i, \sum_{i=1}^{t} \sigma_i^2\right).$$

$$E[X_t] = X_0 + \sum_{i=1}^{t} \mu_i$$

$$Var[X_t] = \sum_{i=1}^{t} \sigma_i^2$$

EXAMPLE CLASS

27 NOVEMBER

12:00-13:00PM

MATH3/4/68181

Q1

X = stock returns

$X \mid \lambda \sim Exp(\lambda)$ — λ: volatility

$\lambda \sim Exp(a)$

$$f_X(x) = \int_0^\infty \underbrace{f_{X|\lambda}(x|\lambda)}_{\text{cond PDF of } X \text{ given } \lambda} \underbrace{g(\lambda)}_{\text{PDF of } \lambda} d\lambda$$

$$= \int_0^\infty \lambda e^{-\lambda x} \, a e^{-a\lambda} \, d\lambda$$

$$= a \int_0^\infty \lambda e^{-(x+a)\lambda} \, d\lambda$$

Set $y = (x+a)\lambda$

$\lambda = \frac{y}{x+a}$

$d\lambda = \frac{dy}{x+a}$

$$= a \int_0^\infty \frac{y}{x+a} e^{-y} \frac{dy}{x+a}$$

$$= \frac{a}{(x+a)^2} \underbrace{\int_0^\infty y e^{-y} \, dy}_{\Gamma(2)}$$

$$f_X(x) = \frac{a}{(x+a)^2}.$$

Suppose $x_1, \ldots, x_n$ are IID observations on X.

$$L(a) = \prod_{i=1}^{n} \left[\frac{a}{(x_i + a)^2} \right]$$

$$= a^n \left[\prod_{i=1}^{n} (x_i + a) \right]^{-2}$$

$$\log L(a) = n \log a - 2 \sum_{i=1}^{n} \log (x_i + a)$$

$$\frac{d \log L(a)}{da} = \frac{n}{a} - 2 \sum_{i=1}^{n} \frac{1}{x_i + a}$$

So $\hat{a}$ (the MLE of a) is the root of

$$\frac{n}{a} = 2 \sum_{i=1}^{n} \frac{1}{x_i + a}$$

Q2

$$X \mid \lambda \sim \text{Exp}(\lambda)$$

$$\lambda \sim \text{Uni}[a, b]$$

$$f_X(x) = \int_a^b f_{X|\lambda}(x|\lambda)\, g(\lambda)\, d\lambda$$

$$= \int_a^b \lambda e^{-\lambda x} \frac{1}{b-a}\, d\lambda$$

Set $y = \lambda x$

$$\lambda = \frac{y}{x}$$

$$d\lambda = \frac{dy}{x}$$

$$= \frac{1}{b-a} \int_{ax}^{bx} \frac{y}{x} e^{-y} \frac{dy}{x}$$

$$= \frac{1}{(b-a)x^2} \int_{ax}^{bx} y e^{-y}\, dy$$

Int by parts

$$= \frac{1}{(b-a)x^2} \left\{ \left[y(-e^{-y}) \right]_{ax}^{bx} + \int_{ax}^{bx} e^{-y}\, dy \right\}$$

$$= \frac{1}{(b-a)x^2} \left[-bx e^{-bx} + ax e^{-ax} + e^{-ax} - e^{-bx} \right]$$

Q3

$$X \mid \lambda \sim \text{Exp}(\lambda)$$

$$\lambda \sim \text{PDF} = a\lambda^{a-1}, \quad 0 < \lambda < 1$$

$$f_X(x) = \int_0^1 f_{X|\lambda}(x|\lambda)\, g(\lambda)\, d\lambda$$

$$= \int_0^1 \lambda e^{-\lambda x}\, a\lambda^{a-1}\, d\lambda$$

$$= a\int_0^1 \lambda^a e^{-\lambda x}\, d\lambda$$

Set $y = \lambda x$

$$\lambda = \frac{y}{x}$$

$$d\lambda = \frac{dy}{x}$$

$$= a\int_0^x \left(\frac{y}{x}\right)^a e^{-y}\, \frac{dy}{x}$$

$$= \frac{a}{x^{a+1}} \int_0^x y^a e^{-y}\, dy$$

$$= \frac{a}{x^{a+1}}\, \gamma(a+1, x)$$

$$\gamma(\beta, x) = \int_0^x t^{\beta-1} e^{-t}\, dt$$

Incomplete Gamma Function

Q4

$$X \mid \lambda \sim \mathrm{Exp}(\lambda)$$

$$\lambda \sim \text{PDF} \quad \frac{a k^a}{\lambda^{a+1}}, \lambda > k$$

$$f_X(x) = \int_k^{\infty} f_{X|\lambda}(x \mid \lambda)\, g(\lambda)\, d\lambda$$

$$= \int_k^{\infty} \lambda e^{-\lambda x} \frac{a k^a}{\lambda^{a+1}} d\lambda$$

$$= a k^a \int_k^{\infty} \lambda^{-a} e^{-\lambda x} d\lambda$$

Set $y = \lambda x$

$\lambda = \frac{y}{x}$

$d\lambda = \frac{dy}{x}$

$$= a k^a \int_{kx}^{\infty} \left(\frac{y}{x}\right)^{-a} e^{-y} \frac{dy}{x}$$

$$= a k^a x^{a-1} \int_{kx}^{\infty} y^{-a} e^{-y} dy$$

$$= a k^a x^{a-1} \Gamma(1-a, kx)$$

$$\Gamma(\beta, x) = \int_x^{\infty} t^{\beta-1} e^{-t} dt$$

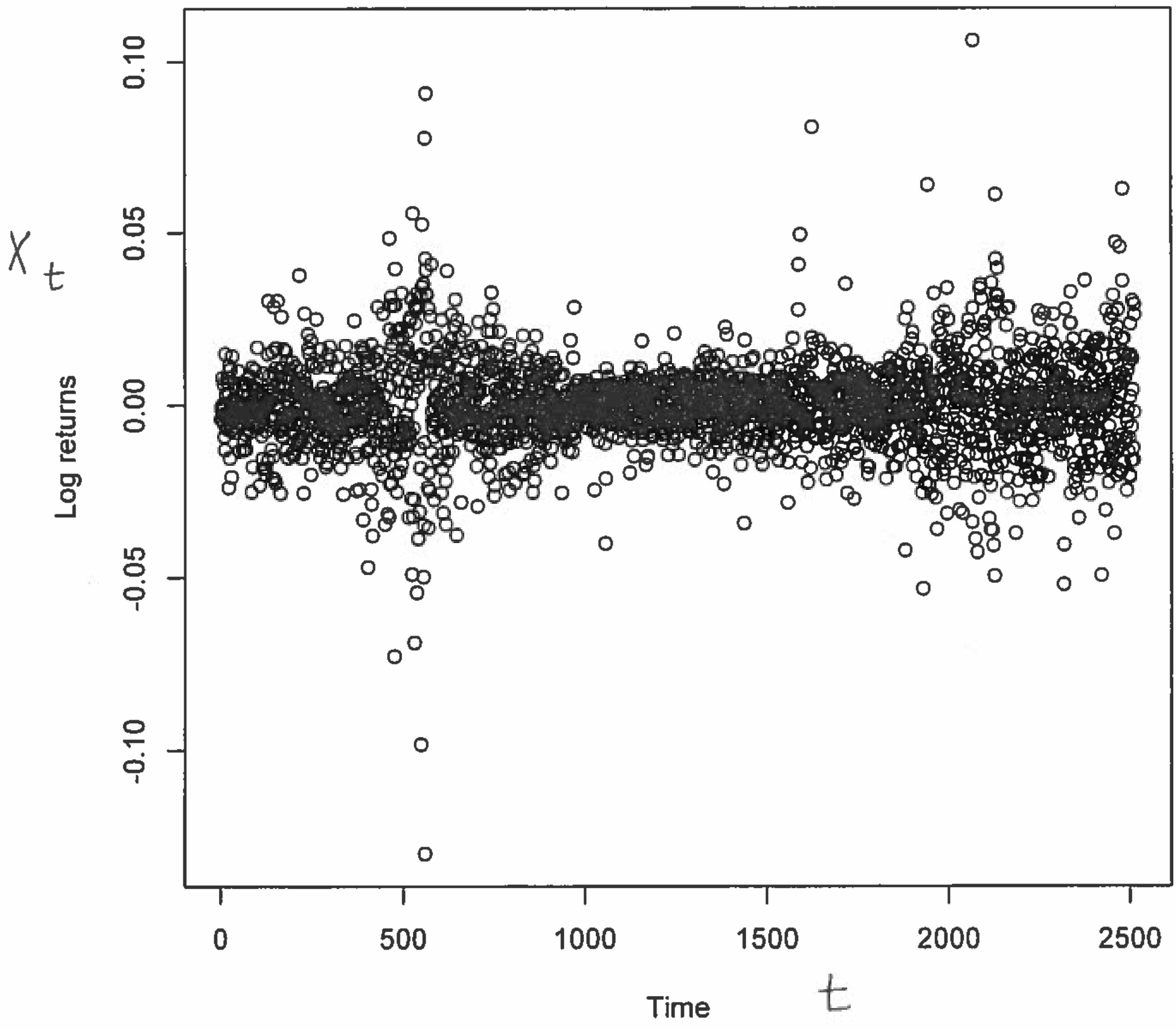
Complementary Incomplete Gamma Function

LECTURE

28 NOVEMBER

9:00-10:00AM

MATH3/4/68181



X_t
0.10
0.05
0.00
-0.05
-0.10
Log returns
0
500
1000
1500
2000
2500
Time
t

ii) Model II

Let X_t = stock return at t

X_0 = stock return at time 0

Assume X_0 is known (a fixed number).

$$
\begin{aligned}
X_t = \; & (X_t - X_{t-1}) = Z_t \\
& + (X_{t-1} - X_{t-2}) = Z_{t-1} \\
& + (X_{t-2} - X_{t-3}) = Z_{t-2} \\
& + \vdots \\
& + (X_1 - X_0) = Z_1 \\
& + X_0
\end{aligned}
$$

$$
= \left(\sum_{i=1}^{t} Z_i \right) + X_0
$$

$$E[X_t] = \sum_{i=1}^{t} E(Z_i) + X_0$$

$$E[X_t^2] = E\left\{\left[\left(\sum_{i=1}^{t} Z_i\right) + X_0\right]^2\right\}$$

$$= E\left[\left(\sum_{i=1}^{t} Z_i\right)^2\right]$$

$$+ 2X_0 \, E\left[\sum_{i=1}^{t} Z_i\right]$$

$$+ X_0^2$$

$$= \sum_{i=1}^{t} E(Z_i^2) + \sum_{i \neq j} E(Z_i Z_j)$$

$$+ 2X_0 \sum_{i=1}^{t} E(Z_i)$$

$$+ X_0^2$$

$$Var[X_t] = E[X_t^2] - \left(E[X_t]\right)^2$$

$$= \sum_{i=1}^{t} E(Z_i^2) + \sum_{i \neq j} E(Z_i Z_j)$$

$$- \left[\sum_{i=1}^{t} E(Z_i)\right]^2.$$

$$\left(\sum_{j=1}^{n} a_j\right)^2$$

$$= \sum_{j=1}^{n} a_j^2 + \sum_{i \neq j} a_i a_j$$

Suppose Z_i are IID.

Then

$$E[X_t] = t\,E(Z) + X_0,$$

$$E[X_t^2] = t\,E(Z^2) + t(t-1)\,E(Z^2) + 2X_0 t\,E(Z) + X_0^2,$$

$$\mathrm{Var}[X_t] = t\,E[Z^2] + t(t-1)\,E(Z^2) - t^2 (E(Z))^2$$

Ex

Suppose Z_i are independent $N(\mu_i, \sigma_i^2)$. What is the distribution of X_t ?

$$X_t = \left(\sum_{i=1}^{t} Z_i\right) + X_0$$

$$= \left(\sum_{i=1}^{t} N(\mu_i, \sigma_i^2)\right) + X_0$$

$$= N\left(\sum_{i=1}^{t} \mu_i, \sum_{i=1}^{t} \sigma_i^2\right) + X_0$$

$$= N\left(X_0 + \sum_{i=1}^{t} \mu_i, \sum_{i=1}^{t} \sigma_i^2\right).$$

$$E[X_t] = X_0 + \sum_{i=1}^{t} \mu_i$$

$$Var[X_t] = \sum_{i=1}^{t} \sigma_i^2$$

Ex (contd)

$$P(X_t < x)$$

$$= \Phi\left(\frac{x - X_0 - \sum_{i=1}^{t} \mu_i}{\sqrt{\sum_{i=1}^{t} \sigma_i^2}}\right)$$

$\Phi(\cdot)$ is the CDF of $N(0,1)$

Answer questions like:

$$P(X_{5000} > x) = 0.95$$

$$\Rightarrow P(X_{5000} \leq x) = 0.05$$

$$\Rightarrow \Phi\left(\frac{x - X_0 - \sum_{i=1}^{5000} \mu_i}{\sqrt{\sum_{i=1}^{5000} \sigma_i^2}}\right) = 0.05$$

$$\Rightarrow x = X_0 + \sum_{i=1}^{5000} \mu_i + \Phi^{-1}(0.05)\sqrt{\sum_{i=1}^{5000} \sigma_i^2}$$

Ex 2

Suppose Z_i are independent gamma random variables with parameters (a_i, b). What is the distribution of X_t?

$$X_t = \left(\sum_{i=1}^{t} Z_i\right) + X_0$$

The MGF Approach

$$M_{X_t}(s) = E\left(e^{sX_t}\right)$$

$$= E\left(e^{s\left(\left(\sum_{i=1}^{t} Z_i\right) + X_0\right)}\right)$$

$$= e^{sX_0} E\left(e^{s\sum_{i=1}^{t} Z_i}\right)$$

$$= e^{sX_0} E\left(e^{sZ_1}\right) \cdots E\left(e^{sZ_t}\right)$$

Math 20802

$$= e^{sX_0} \left(\frac{b}{b-s}\right)^{a_1} \cdots \left(\frac{b}{b-s}\right)^{a_t}$$

$$= e^{sX_0} \left(\frac{b}{b-s}\right)^{\sum_{i=1}^{t} a_i}$$

= MGF of a gamma RV with paras $(\sum a_i, b)$

=

$$\Rightarrow \quad X_t = Z + X_0 \leftarrow \text{constant}$$

where Z is a gamma RV with parameters $\left(\sum_{i=1}^{t} a_i ,\ b\right)$

$$E(X_t) = E(Z) + X_0$$

$$= \frac{\sum_{i=1}^{t} a_i}{b} + X_0$$

$$Var(X_t) = Var(Z + X_0)$$

$$= Var(Z)$$

$$= \frac{\sum_{i=1}^{t} a_i}{b^2}$$

$$P(X_t < x) = q$$

$$\Rightarrow P(Z + X_0 < x) = q$$

$$\Rightarrow P(Z < x - X_0) = q$$

$$\Rightarrow \frac{\gamma\left(\sum_{i=1}^{t} a_i, b(x - X_0)\right)}{\Gamma\left(\sum_{i=1}^{t} a_i\right)} = q$$

where $\gamma(a, x)$ denotes the incomplete gamma function.

Model III

Suppose X_0 is again a fixed number.

$$X_t = \left(\frac{X_t}{X_{t-1}}\right) = Z_t$$

$$\times \left(\frac{X_{t-1}}{X_{t-2}}\right) = Z_{t-1}$$

$$\times \left(\frac{X_{t-2}}{X_{t-3}}\right) = Z_{t-2}$$

$$\vdots$$

$$\times \left(\frac{X_1}{X_0}\right) = Z_1$$

$$\times \quad X_0$$

$$= \left(\prod_{i=1}^{t} Z_i\right) \times X_0$$

Suppose Z_i are indep RVs

$$E(X_t) = X_0 \cdot E\left(\prod_{i=1}^{t} Z_i\right)$$

$$= X_0 \prod_{i=1}^{t} E(Z_i)$$

$$E(X_t^2) = X_0^2 \prod_{i=1}^{t} E(Z_i^2)$$

$$\vdots$$

$$E(X_t^n) = X_0^n \prod_{i=1}^{t} E(Z_i^n)$$

$$Var(X_t) = X_0^2 \left[\prod_{i=1}^{t} E(Z_i^2) - \prod_{i=1}^{t} \left(E(Z_i)\right)^2 \right].$$

Suppose Z_i are IID

$$E(X_t) = X_0 \left(E(Z)\right)^t$$

$$E(X_t^2) = X_0^2 \left(E(Z^2)\right)^t$$

$$\vdots$$

$$E(X_t^n) = X_0^n \left(E(Z^n)\right)^t$$

$$Var(X_t) = X_0^2 \left[\left(E(Z^2)\right)^t - \left(E(Z)\right)^{2t}\right]$$

Ex

Suppose Z_i are $LN(\mu_i, \sigma_i^2)$.
What is the distribution of X_t?

$$X_t = X_0 \prod_{i=1}^{t} LN(\mu_i, \sigma_i^2)$$

$$\Rightarrow \log X_t = \log X_0 + \sum_{i=1}^{t} \log LN(\mu_i, \sigma_i^2)$$

$$\Rightarrow \log X_t = \log X_0 + \sum_{i=1}^{t} N(\mu_i, \sigma_i^2)$$

$$= \log X_0 + N\left(\sum_{i=1}^{t} \mu_i, \sum_{i=1}^{t} \sigma_i^2\right)$$

$$= N\left(\log X_0 + \sum_{i=1}^{t} \mu_i, \sum_{i=1}^{t} \sigma_i^2\right)$$

$$\Rightarrow X_t \equiv LN\left(\log X_0 + \sum_{i=1}^{t} \mu_i, \sum_{i=1}^{t} \sigma_i^2\right)$$

EXAMPLE CLASS

27 NOVEMBER

10:00-11:00AM

MATH3/4/68181

Q1

X = stock returns

$X \mid \lambda \sim \text{Exp}(\lambda)$

$\lambda \sim \text{Exp}(a)$

$$f_X(x) = \int_0^{\infty} f_{X|\lambda}(x \mid \lambda)\, g(\lambda)\, d\lambda$$

$$= \int_0^{\infty} \lambda e^{-\lambda x}\, a e^{-a\lambda}\, d\lambda$$

$$= a \int_0^{\infty} \lambda e^{-\lambda(x+a)}\, d\lambda$$

Set $y = \lambda(x+a)$

$\lambda = y/(x+a)$

$\frac{d\lambda}{dy} = 1/(x+a)$

$$= \frac{a}{(x+a)^2} \int_0^{\infty} y e^{-y}\, dy = \Gamma(2) = 1!$$

$$= \frac{a}{(x+a)^2}.$$

Q2 $X \mid \lambda \sim \text{Exp}(\lambda)$

$\lambda \sim \text{Uni}[a, b]$

$$f_X(x) = \int_a^b f_{X|\lambda}(x|\lambda)\, g(\lambda)\, d\lambda$$

$$= \int_a^b \lambda e^{-\lambda x} \cdot \frac{1}{b-a}\, d\lambda$$

$$= \frac{1}{b-a} \int_a^b \lambda e^{-\lambda x}\, d\lambda$$

Int by Parts

$$= \frac{1}{b-a} \left\{ \left[\lambda \frac{e^{-\lambda x}}{(-x)} \right]_a^b + \frac{1}{x} \int_a^b e^{-\lambda x}\, d\lambda \right\}$$

$$= \frac{1}{b-a} \left\{ \frac{b e^{-bx} - a e^{-ax}}{(-x)} + \frac{1}{x} \left[\frac{e^{-\lambda x}}{(-x)} \right]_a^b \right\}$$

$$= \frac{1}{b-a} \left\{ \frac{b e^{-bx} - a e^{-ax}}{(-x)} + \frac{1}{x} \cdot \frac{e^{-bx} - e^{-ax}}{(-x)} \right\}.$$

Q4

$X \mid \lambda \sim Exp(\lambda)$

λ has PDF $\dfrac{a k^a}{\lambda^{a+1}}, \lambda > k$

$$f_X(x) = \int_k^\infty \lambda e^{-\lambda x} \cdot \frac{a k^a}{\lambda^{a+1}} \, d\lambda$$

$$= a k^a \int_k^\infty \frac{1}{\lambda^a} e^{-\lambda x} \, d\lambda$$

Set $y = \lambda x \Rightarrow \lambda = \frac{y}{x} \Rightarrow d\lambda = \frac{dy}{x}$

$$= \frac{a k^a}{x^{a+1}} \int_{kx}^\infty \frac{1}{y^a} e^{-y} \, dy$$

$$\Gamma(\beta, u) = \int_u^\infty t^{\beta-1} e^{-t} \, dt$$

Complementary Incomp Gamma Fun

$$= \frac{a k^a}{x^{a+1}} \Gamma(1-a, kx).$$

Q3

$X | \lambda \sim Exp(\lambda)$

λ has PDF $a\lambda^{a-1}, \; 0<\lambda<1$

$$f_X(x) = \int_0^1 \lambda e^{-\lambda x} \cdot a\lambda^{a-1} \, d\lambda$$

$$= a\int_0^1 \lambda^a e^{-\lambda x} \, d\lambda$$

set $y = \lambda x$

$\lambda = \frac{y}{x}$

$d\lambda = \frac{dy}{x}$

$$= \frac{a}{x^{a+1}} \int_0^x y^a e^{-y} \, dy$$

$$\gamma(\beta, u) = \int_0^u t^{\beta-1} e^{-t} \, dt$$

Incomplete Gamma Fun

$$= \frac{a}{x^{a+1}} \gamma(a+1, x).$$

LECTURE

1 DECEMBER

9:00-10:00AM

MATH3/4/68181

ANNOUNCEMENTS

1) the syllabus for Math 38181 will be completed today.

the syllabus for Math 4/68181 will be completed on Thurs 7 Dec.

2) Revision classes

- Mon 4 Dec 12-1 Chem G53
- Tues 5 Dec 9-10 Rutherford
- " " " 10-11 Hum Bridge
- Fri 8 Dec 9-10 Roscoe B
- Mon 11 Dec 12-1 Chem G53
- Tues 12 Dec 9-10 Rutherford
- " " " 10-11 Hum Bridge
- Thurs 14 Dec 12-1 ATB G205
  [Levels 4 & 6 only]
- Fri 15 Dec 9-10 Roscoe B

3) Check your email ~~later~~ today for what to be expected in the final exam in Jan 2018.

4) Bonus question — no bonus question this semester.

5) UEQ (Unit Evalvation Questionnaire) will go on-line on Mon 4 Dec.

Income Moteling

X = True income

Z = Reported income

X is not observable

Z is observable

i) Under reporting

eg $X = £\ 50000.22$

$Z = £\ 50000$

Under reported if $Z < X$

ii) Over reporting

eg $X = £\ 39999.51$

$Z = £\ 40000$

Over reporting if $Z > X$.

What is the distribution of X ?

i) <u>Under reporting</u>

$$Z = XY, \quad Y \in (0, 1)$$

Assume Y has the CDF

$$F_Y(y) = y^c, \quad 0 < y < 1$$

Assume Z has the Pareto distribution with CDF

$$F_Z(z) = 1 - \left(\frac{K}{z}\right)^a, \quad z \geq K.$$

[Pareto is the first distribution used to model income data]

Assume also Z and Y are indep RVs.

Then X is Pareto distributed if and only if Z is also Pareto distributed.

Proof Suppose X is Pareto distributed with CDF

$$F_X(x) = 1 - \left(\frac{L}{x}\right)^b, \quad x \geq L.$$

Then

$$F_Z(z) = P(Z \leq z)$$

$$= P(XY \leq z)$$

$$= P\left(X \leq \frac{z}{Y}\right)$$

$$= \int_0^1 F_X\left(\frac{z}{y}\right) f_Y(y)\, dy$$

[total prob rule]

$$= \int_0^1 \left[1 - \left(\frac{Ly}{z}\right)^b\right] c\, y^{c-1}\, dy$$

$$= c\int_0^1 y^{c-1}\, dy - \frac{cL^b}{z^b}\int_0^1 y^{b+c-1}\, dy$$

$$= 1 - \frac{cL^b}{z^b}\left[\frac{y^{b+c}}{b+c}\right]_0^1$$

$$= 1 - \frac{cL^b}{(b+c)\, z^b} = 1 - \frac{\left[\frac{c^{\frac{1}{b}} L}{(b+c)^{\frac{1}{b}}}\right]^b}{z^b}$$

$$= 1 - \frac{k^b}{z^b}, \quad \text{Pareto CDF}$$

Suppose Z is Pareto distributed with CDF

$$F_Z(z) = 1 - \left(\frac{K}{z}\right)^a, \quad z \geq K.$$

Then $F_X(x) = P(X \leq x)$

$$= P\left(\frac{Z}{Y} \leq x\right)$$

$$= P(Z \leq xY)$$

$$= \int_0^1 F_Z(xy) \, f_Y(y) \, dy$$

[total prob rule]

$$= \int_0^1 \left[1 - \left(\frac{K}{xy}\right)^a\right] c\, y^{c-1} \, dy$$

$$= c\int_0^1 y^{c-1} \, dy - \frac{cK^a}{x^a} \int_0^1 y^{c-a-1} \, dy$$

$$= 1 - \frac{cK^a}{x^a} \left[\frac{y^{c-a}}{c-a}\right]_0^1$$

$$= 1 - \frac{cK^a}{x^a} \cdot \frac{1}{c-a}$$

$$= 1 - \frac{\left[\frac{c^{\frac{1}{a}} K}{(c-a)^{1/a}}\right]^a}{x^a} = 1 - \left(\frac{L}{x}\right)^a, \text{ Pareto CDF}$$

ii) Over reporting

$$Z = \frac{X}{Y}, \quad Y \in (0, 1)$$

Assume Y has the CDF

$$F_Y(y) = y^c, \quad 0 < y < 1.$$

Assume also Y and Z are indep RVs

Then, X is Pareto distributed if and only if Z is also Pareto distributed

Proof Assume X is Pareto distributed with CDF

$$F_X(x) = 1 - \left(\frac{k}{x}\right)^a, \quad x \geq k.$$

Then

$$F_Z(z) = P(Z \leq z)$$

$$= P\left(\frac{X}{Y} \leq z\right) = P(X \leq Yz)$$

$$= \int_0^1 F_X(yz)\, f_Y(y)\, dy$$

[total prob rule]

$$= \int_0^1 \left[1 - \left(\frac{k}{yz}\right)^a\right] c\, y^{c-1}\, dy$$

$$= c\int_0^1 y^{c-1}\, dy - \frac{c k^a}{z^a}\int_0^1 y^{a+c-1}\, dy$$

$$= 1 - \frac{c k^a}{z^a}\left[\frac{y^{a+c}}{a+c}\right]_0^1$$

$$= 1 - \frac{c k^a}{z^a} \cdot \frac{1}{(a+c)}$$

$$= 1 - \frac{\left[\frac{c^{\frac{1}{a}} k}{(a+c)^{1/a}}\right]^a}{z^a} = 1 - \left(\frac{L}{z}\right)^a, \text{ Pareto CDF}$$

Assume Z is Pareto distributed with CDF

$$F_Z(z) = 1 - \left(\frac{L}{z}\right)^b, \quad z \geq L$$

Prove that X is also. Homework

REVISION CLASS

4 DECEMBER

12:00-13:00PM

MATH3/4/68181

REVISION CLASS

Q4, 2016/17 Exam

(a)

$$\mathrm{VaR}_p(X) = F^{-1}(p)$$

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

(b) (i)

$$F(x) = \int_{-a}^{x} \frac{3y^2}{2a^3}\,dy$$

$$= \frac{3}{2a^3}\left[\frac{y^3}{3}\right]_{-a}^{x}$$

$$= \frac{3}{2a^3}\left[\frac{x^3}{3} - \frac{(-a)^3}{3}\right]$$

$$= \frac{x^3 + a^3}{2a^3}.$$

(ii)

$$F(x) = p$$

$$\Rightarrow \frac{x^3 + a^3}{2a^3} = p$$

$$\Rightarrow x^3 = (2p-1)\,a^3$$

$$\Rightarrow x = (2p-1)^{\frac{1}{3}}\,a$$

$$\Rightarrow \mathrm{VaR}_p(X) = (2p-1)^{\frac{1}{3}}\,a$$

(iii)

$$ES_p(X) = \frac{1}{p}\int_0^p (2t-1)^{\frac{1}{3}} a \, dt$$

$$= \frac{a}{p}\left[\frac{(2t-1)^{\frac{4}{3}}}{2\cdot(4/3)}\right]_0^p$$

$$= \frac{3a}{8p}\left[(2p-1)^{\frac{4}{3}} - 1\right]$$

(c) (i)

$$L(a) = \prod_{i=1}^{n}\left[\frac{3x_i^2}{2a^3} I\{-a < x_i < a\}\right]$$

$$= \frac{3^n}{2^n a^{3n}}\left[\prod_{i=1}^{n} I\{-a < x_i < a\}\right]$$

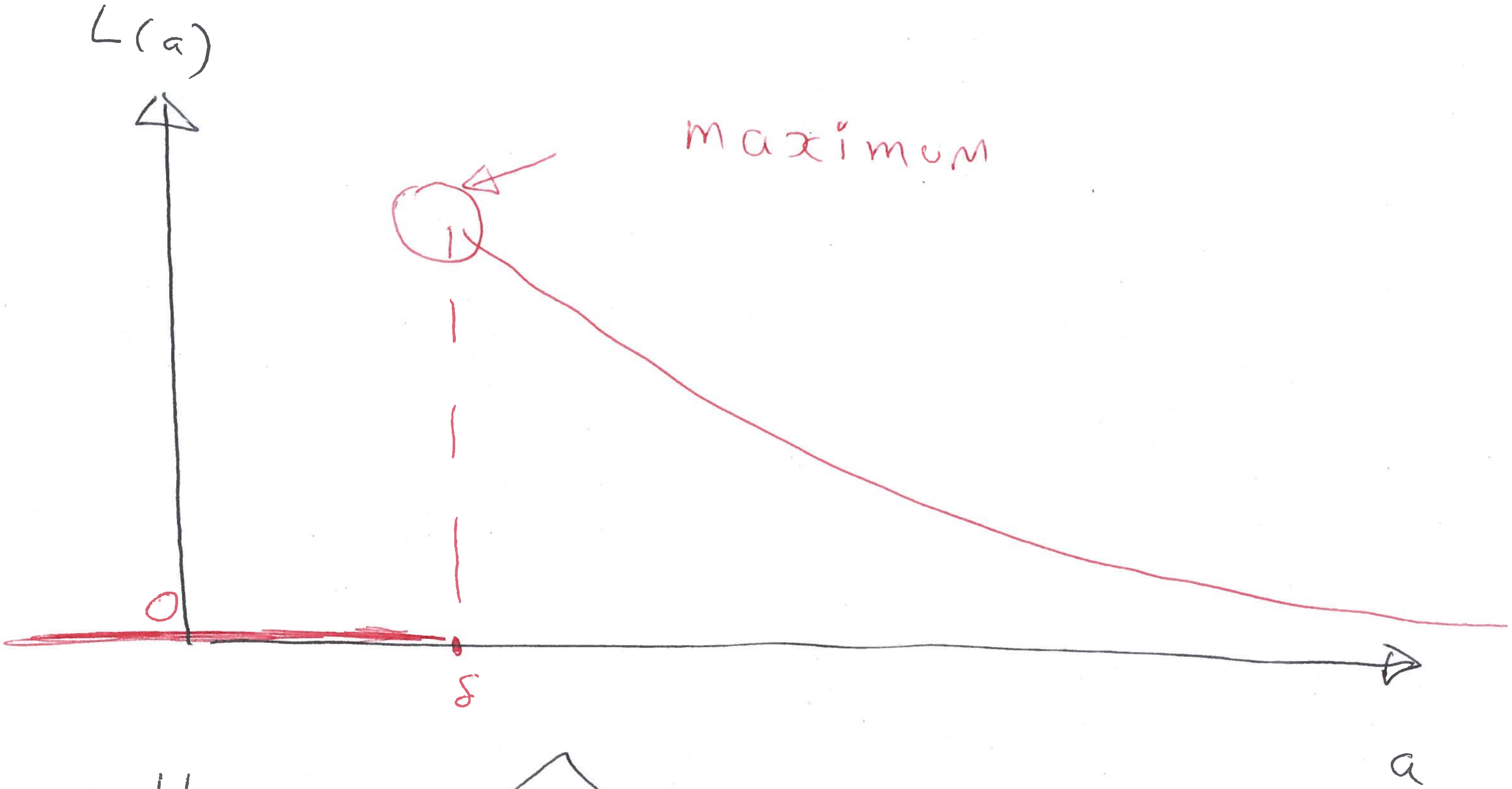
Math20802

$$= \frac{3^n}{2^n a^{3n}} I\{\max(x_1,\dots,x_n) < a \text{ and } \min(x_1,\dots,x_n) > -a\}$$

$$= \frac{3^n}{2^n a^{3n}} I\{a > \max(x_1,\dots,x_n) \text{ and } a > -\min(x_1,\dots,x_n)\}$$

δ

$$= \frac{3^n}{2^n a^{3n}} I\{a > \max[\max(x_1,\dots,x_n), -\min(x_1,\dots,x_n)]\}$$



Hence, $\widehat{a} = \delta$.

(iii) $$\widehat{VaR_p(X)} = (2p-1)^{\frac{1}{3}}\,\delta$$

$$\widehat{ES}_p(X) = \frac{3\delta}{8p}\left[(2p-1)^{\frac{4}{3}} - 1\right]$$

c(iv)

$$Z = \max[\max(X_1, \dots, X_n), -\min(X_1, \dots, X_n)]$$

$$F_Z(z) = P(Z \leq z)$$

$$= P[\max[\max(X_1, \dots, X_n), -\min(X_1, \dots, X_n)] \leq z]$$

$$= P[\max(X_1, \dots, X_n) \leq z, -\min(X_1, \dots, X_n) \leq z]$$

$$= P[X_1 \leq z, \dots, X_n \leq z, \min(X_1, \dots, X_n) \geq -z]$$

$$= P[X_1 \leq z, \dots, X_n \leq z, X_1 \geq -z, \dots, X_n \geq -z]$$

$$= P[-z \leq X_1 \leq z, \dots, -z \leq X_n \leq z]$$

indep

$$= P[-z \leq X_1 \leq z] \cdots P[-z \leq X_n \leq z]$$

$$= \{P[-z \leq X \leq z]\}^n$$

$$= \left\{ F(z) - F(-z) \right\}^n$$

b(i)

$$= \left\{ \frac{z^3 + a^3}{2a^3} - \frac{(-z)^3 + a^3}{2a^3} \right\}^n$$

$$= \left[\frac{z^3}{a^3} \right]^n .$$

2015/16 Exam, Q3 (d)

$$f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$$

$$\omega(F) = +\infty$$

(I):

$$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$\overset{LH}{=} \lim_{t \to \infty} \frac{-f(t + x\gamma(t)) \cdot (1 + x\gamma'(t))}{-f(t)}$$

$$= \lim_{t \to \infty} \frac{\frac{1}{2} e^{-|t + x\gamma(t)|} (1 + x\gamma'(t))}{\frac{1}{2} e^{-|t|}}$$

$$= \lim_{t \to \infty} \frac{e^{-(t + x\gamma(t))} (1 + x\gamma'(t))}{e^{-t}}$$

$$= \lim_{t \to \infty} e^{-x\gamma(t)} (1 + x\gamma'(t))$$

Choose $\gamma(t) \equiv 1$
$\gamma'(t) = 0$

$= e^{-x} \Rightarrow F$ belongs to the Gumbel max domain.

REVISION CLASS

5 DECEMBER

9:00-10:00AM

MATH3/4/68181

REVISION CLASS

(TUES 5 DEC 9-10 AM)

Exam 2012/2013, Q6

(a)

$$\text{VaR}_p(X) = F^{-1}(p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

(b)

$$F(x) = \frac{x-a}{b-a}$$

$$F(x) = p$$

$$\Rightarrow \frac{x-a}{b-a} = p$$

$$\Rightarrow x = a + p(b-a)$$

$$\Rightarrow \text{VaR}_p(X) = a + p(b-a)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \left[a + t(b-a)\right] dt$$

$$= \frac{1}{p}\left[a.t + \frac{t^2}{2}(b-a)\right]_0^p$$

$$= a + \frac{p}{2}(b-a).$$

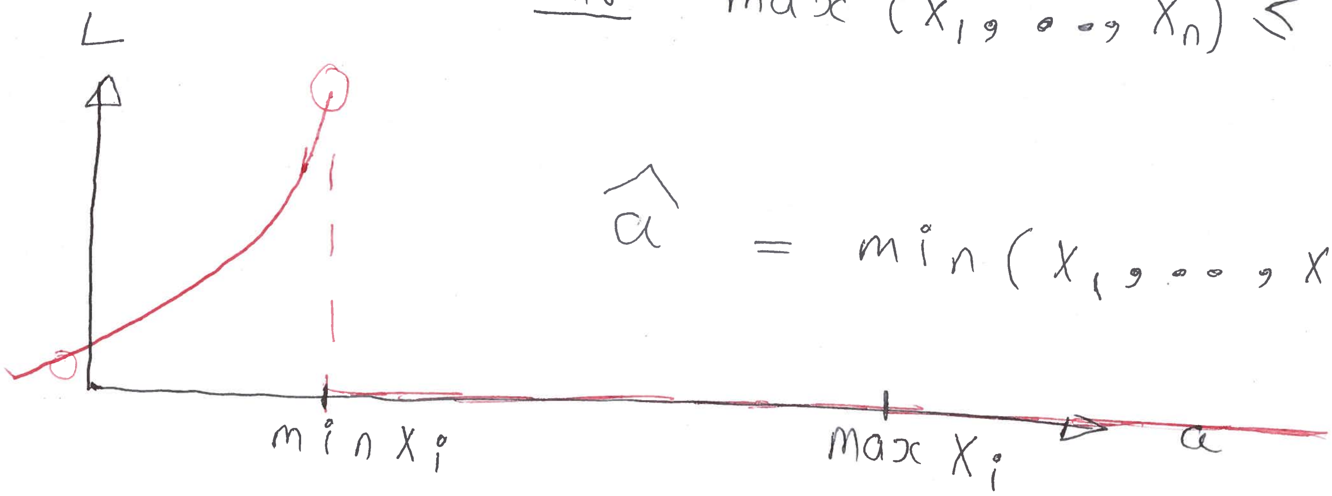
(c) (i) – (iii)

$$L(a,b) = \prod_{i=1}^{n} \left[\frac{1}{b-a} I\{a \leq x_i \leq b\} \right]$$

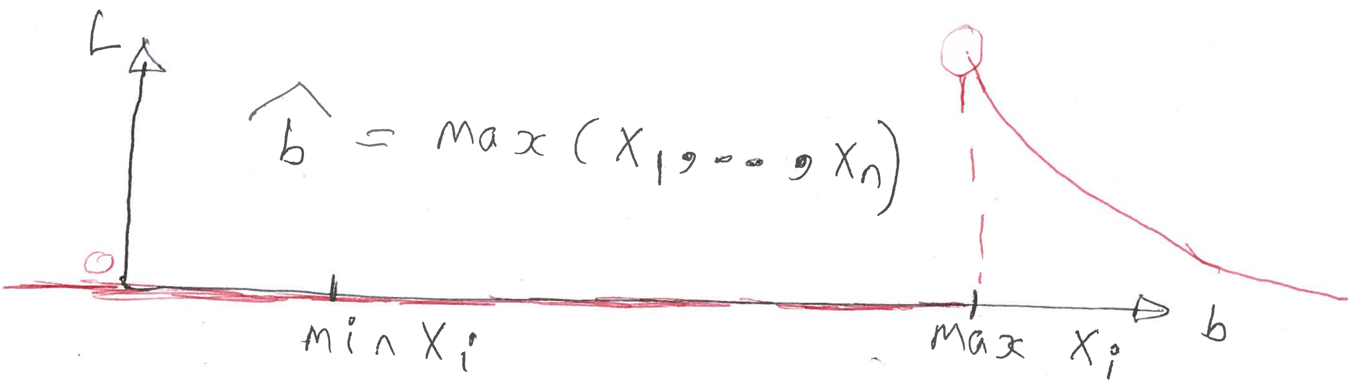
$$= \frac{1}{(b-a)^n} \prod_{i=1}^{n} I\{a \leq x_i \leq b\}$$

Math20802

$$= \boxed{\frac{1}{(b-a)^n}} I\{\min(x_1, \ldots, x_n) \geq a \text{ and } \max(x_1, \ldots, x_n) \leq b\}$$



$$\hat{a} = \min(X_1, \ldots, X_n)$$



$$\hat{b} = \max(X_1, \ldots, X_n)$$

(iv)

$$\widehat{VaR}_p(X) = \min(X_1, \ldots, X_n) + p\left[\max(X_1, \ldots, X_n) - \min(X_1, \ldots, X_n)\right]$$

$$\widehat{ES}_p(X) = \min(X_1, \ldots, X_n) + \frac{p}{2}\left[\max(X_1, \ldots, X_n) - \min(X_1, \ldots, X_n)\right]$$

(v) $E\left[\widehat{VaR}_p(X)\right] \neq VaR_p(X)$

(vi) $E\left[\widehat{ES}_p(X)\right] \neq ES_p(X)$

2013/14 Exam Q6

(i) $F(x) = 1 - e^{1-(1+\lambda x)^{\alpha}}, \quad x > 0$

$w(F) = +\infty$

(I)

$$\lim_{t\to\infty} \frac{1-F(t+x\gamma(t))}{1-F(t)}$$

$$= \lim_{t\to\infty} \frac{1-\{1-e^{1-(1+\lambda t+\lambda x\gamma(t))^{\alpha}}\}}{1-\{1-e^{1-(1+\lambda t)^{\alpha}}\}}$$

$$= \lim_{t\to\infty} e^{(1+\lambda t)^{\alpha}-(1+\lambda t+\lambda x\gamma(t))^{\alpha}}$$

$$= \lim_{t\to\infty} e^{(1+\lambda t)^{\alpha}-(1+\lambda t)^{\alpha}\left[1+\frac{\lambda x\gamma(t)}{1+\lambda t}\right]^{\alpha}}$$

$$= \lim_{t\to\infty} e^{(1+\lambda t)^{\alpha}\left\{1-\left[1+\frac{\lambda x\gamma(t)}{1+\lambda t}\right]^{\alpha}\right\}}$$

$(1+x)^{a} \approx 1+ax$; $\frac{\lambda x\gamma(t)}{1+\lambda t} \to 0$

$$= \lim_{t\to\infty} e^{(1+\lambda t)^{\alpha}\left\{1-\left[1+\frac{\alpha\lambda x\gamma(t)}{1+\lambda t}\right]\right\}}$$

$$= \lim_{t\to\infty} e^{-(1+\lambda t)^{\alpha-1}\cdot\alpha\lambda x\gamma(t)}$$

$\gamma(t) = \frac{1}{\alpha\lambda}(1+\lambda t)^{1-\alpha}$

$= e^{-x} \Rightarrow F$ belongs to Gumbel domain

2013/14 Exam Q6

(v) $F(x) = 1 - q^{(x+1)^a}$, $0 < q < 1$
$a > 1$
$x = 0, 1, 2, \ldots$

$w(F) = +\infty$

$$\lim_{k \to \infty} \frac{P(X = k)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{F(k) - F(k-1)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{1 - q^{(k+1)^a} - [1 - q^{k^a}]}{1 - [1 - q^{k^a}]}$$

$$= \lim_{k \to \infty} \frac{q^{k^a} - q^{(k+1)^a}}{q^{k^a}}$$

$$= \lim_{k \to \infty} 1 - q^{(k+1)^a - k^a}$$

$$= \lim_{k \to \infty} 1 - q^{k^a(1+\frac{1}{k})^a - k^a}$$

$$= \lim_{k \to \infty} 1 - q^{k^a[(1+\frac{1}{k})^a - 1]}$$

$$= \lim_{k \to \infty} 1 - q^{k^a \left[1 + \frac{a}{k} + \frac{a(a-1)}{2k^2} + \dots - 1 \right]}$$

Binomial Exp

$$= \lim_{k \to \infty} 1 - q^{a k^{a-1}}$$

$+\infty$

0

$$= 1 \neq 0$$

Hence ETT does not hold.

REVISION CLASS

5 DECEMBER

10:00-11:00AM

MATH3/4/68181

REVISION CLASS

(TUES 5 DEC 10-11 AM)

EXAM 2016/17 Q5

(a)

$$\bar{F}(x_1, \ldots, x_k)$$

$$= \left(1 + \frac{1}{\theta}\sum_{i=1}^{k} x_i\right)^{-\alpha}$$

$$f(x_1, \ldots, x_k) = \frac{(-1)^k \partial^k}{\partial x_1 \ldots \partial x_k} \bar{F}(x_1, \ldots, x_k)$$

$$\frac{\partial}{\partial x_1}\bar{F} = (-\alpha) \cdot \left(1 + \frac{1}{\theta}\sum_{i=1}^{k} x_i\right)^{-\alpha-1} \left(\frac{1}{\theta}\right)$$

$$\frac{\partial^2 \bar{F}}{\partial x_1 \partial x_2} = (-\alpha)(-\alpha-1)\left(1 + \frac{1}{\theta}\sum_{i=1}^{k} x_i\right)^{-\alpha-2} \left(\frac{1}{\theta^2}\right)$$

$$\frac{\partial^3 \bar{F}}{\partial x_1 \partial x_2 \partial x_3} = (-\alpha)(-\alpha-1)(-\alpha-2)\left(1 + \frac{1}{\theta}\sum_{i=1}^{k} x_i\right)^{-\alpha-3} \left(\frac{1}{\theta^3}\right)$$

$$\vdots$$

$$\frac{\partial^k}{\partial x_1 \ldots \partial x_k} = (-\alpha)(-\alpha-1)\ldots(-\alpha-k+1)\left(1 + \frac{1}{\theta}\sum_{i=1}^{k} x_i\right)^{-\alpha-k} \left(\frac{1}{\theta^k}\right)$$

$$f(x_1, \ldots, x_k) = \frac{\alpha(\alpha+1)\ldots(\alpha+k-1)}{\theta^k}\left(1 + \frac{1}{\theta}\sum_{i=1}^{k} x_i\right)^{-\alpha-k}$$

(b) $$f_S(s) = \underbrace{\int\int\cdots\int}_{x_1+\cdots+x_k=s} f(x_1,\ldots,x_k)\, dx_k \cdots dx_2\, dx_1$$

$$= \underbrace{\int\int\cdots\int}_{x_1+\cdots+x_k=s} \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k}\left(1+\frac{s}{\theta}\right)^{-\alpha-k} dx_k \cdots dx_2\, dx_1$$

$$= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k}\left(1+\frac{s}{\theta}\right)^{-\alpha-k} \cdot \underbrace{\int\int\cdots\int}_{x_1+\cdots+x_k=s} 1\cdot dx_k \cdots dx_2\, dx_1$$

$$\underbrace{\int\int\cdots\int}_{x_1+\cdots+x_k=s} 1\cdot dx_k \cdots dx_2\, dx_1 = \frac{s^k}{k!}$$

$$= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k}\left(1+\frac{s}{\theta}\right)^{-\alpha-k}\frac{s^k}{k!}$$

(c) $$F_S(s) = \int_0^s f_{S'}(t)\,dt$$

$$= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k k!}\int_0^s \left(1+\frac{t}{\theta}\right)^{-\alpha-k} t^k\,dt$$

Set $u = 1-\left(1+\frac{t}{\theta}\right)^{-1}$

$\Rightarrow 1+\frac{t}{\theta} = \frac{1}{1-u}$

$\Rightarrow \frac{t}{\theta} = \frac{u}{1-u}$

$\Rightarrow t = \theta\frac{u}{1-u}$

$\Rightarrow \frac{dt}{du} = +\frac{\theta}{(1-u)^2}$

$$= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k k!}\int_0^{\frac{s}{s+\theta}} (1-u)^{\alpha+k}\frac{\theta^k u^k}{(1-u)^k}\frac{\theta}{(1-u)^2}\,du$$

$$= \frac{\theta\,\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}\int_0^{\frac{s}{s+\theta}} (1-u)^{\alpha-2} u^k\,du$$

$$= \frac{\theta\, \alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} B_{\frac{s}{s+\theta}}(k+1, \alpha-1)$$

REVISION CLASS

8 DECEMBER

9:00-10:00AM

MATH3/4/68181

REVISION

CLASS

(FRI 8 DEC 9-10 AM)

Exam 2014/2015 Q6 (6)

$$F(x) = 1 - (1-x^b)^a, \quad 0 < x < 1$$

$$F(x) = 1$$

$$\Rightarrow 1 - (1-x^b)^a = 1$$

$$\Rightarrow (1-x^b)^a = 0$$

$$\Rightarrow 1 - x^b = 0$$

$$\Rightarrow x^b = 1 \Rightarrow x = 1$$

$$\Rightarrow \omega(F) = 1$$

(III): $\lim_{t \to 0} \dfrac{1 - F(1-tx)}{1 - F(1-t)}$

$$= \lim_{t \to 0} \frac{1 - [1 - (1-(1-tx)^b)^a]}{1 - [1 - (1-(1-t)^b)^a]}$$

$$= \lim_{t \to 0} \frac{(1-(1-tx)^b)^a}{(1-(1-t)^b)^a} \quad (tx \to 0,\ t \to 0)$$

$$= \lim_{t \to 0} \frac{(1-(1-b\cdot tx))^a}{(1-(1-b\cdot t))^a}$$

$$= \lim_{t \to 0} \frac{(btx)^a}{(bt)^a} = x^a \Rightarrow$$ F belongs to Weibull max domain

Exam 2014/15 Q6 (e)

$$F(x) = e^{-e^{-x}}, \quad -\infty < x < \infty$$

$$F(x) = 1$$

$$\Rightarrow \quad e^{-e^{-x}} = 1$$

$$\Rightarrow \quad -e^{-x} = 0$$

$$\Rightarrow \quad e^{-x} = 0$$

$$\Rightarrow \quad -x = \log 0 = -\infty$$

$$\Rightarrow \quad x = +\infty \Rightarrow \omega(F) = +\infty$$

(I) :

$$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{1 - e^{-e^{-t - x\gamma(t)}}}{1 - e^{-e^{-t}}}$$

(circled: $e^{-t - x\gamma(t)} \to 0$, $e^{-t} \to 0$)

$$= \lim_{t \to \infty} \frac{1 - [1 - e^{-t - x\gamma(t)}]}{1 - [1 - e^{-t}]}$$

$$= \lim_{t \to \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t \to \infty} e^{-x\gamma(t)} = e^{-x}$$

if $\gamma(t) \equiv 1$

F belongs to the Gumbel max domain

Exam 2014/15 Q8 (e)

$$F(y) = 1 - \left(\frac{k}{y}\right)^{a\alpha} = p$$

$$\Rightarrow \left(\frac{k}{y}\right)^{a\alpha} = 1 - p$$

$$\Rightarrow y = k(1-p)^{-\frac{1}{a\alpha}}$$

$$ES_p(Y) = \frac{k}{p}\int_0^p (1-t)^{-\frac{1}{a\alpha}}\,dt$$

$$= \frac{k}{p} \cdot \frac{1}{(-1)\left(1 - \frac{1}{a\alpha}\right)}\left[(1-t)^{1-\frac{1}{a\alpha}}\right]_0^p$$

$$= \frac{K}{p}\,\frac{1}{\left(\frac{1}{a\alpha} - 1\right)}\left[(1-p)^{1-\frac{1}{a\alpha}} - 1\right]$$

Exam 2014/15 Q7

(b) (i)

$x > 0$

$$F(x) = \int_{-\infty}^{x} f(y)\, dy$$

$$= \int_{-\infty}^{0} f(y)\, dy + \int_{0}^{x} f(y)\, dy$$

$$= \int_{-\infty}^{0} \frac{1}{2\lambda} e^{-\frac{|y|}{\lambda}}\, dy + \int_{0}^{x} \frac{1}{2\lambda} e^{-\frac{|y|}{\lambda}}\, dy$$

$$= \frac{1}{2\lambda} \int_{-\infty}^{0} e^{\frac{y}{\lambda}}\, dy + \frac{1}{2\lambda} \int_{0}^{x} e^{-\frac{y}{\lambda}}\, dy$$

$$= \frac{1}{2\lambda} \left[\frac{e^{\frac{y}{\lambda}}}{\frac{1}{\lambda}} \right]_{-\infty}^{0} + \frac{1}{2\lambda} \left[\frac{e^{-\frac{y}{\lambda}}}{\left(-\frac{1}{\lambda}\right)} \right]_{0}^{x}$$

$$= \frac{1}{2}(1 - 0) + \frac{1}{2}\left(-e^{-\frac{x}{\lambda}} + 1\right)$$

$$= 1 - \frac{1}{2} e^{-\frac{x}{\lambda}}$$

$x \leq 0$

$$F(x) = \int_{-\infty}^{x} f(y)\, dy$$

$$= \int_{-\infty}^{x} \frac{1}{2\lambda} e^{-\frac{|y|}{\lambda}}\, dy$$

$$= \frac{1}{2\lambda} \int_{-\infty}^{x} e^{+\frac{y}{\lambda}}\, dy$$

$$= \frac{1}{2\lambda} \left[\frac{e^{\frac{y}{\lambda}}}{\frac{1}{\lambda}} \right]_{-\infty}^{x}$$

$$= \frac{1}{2} \left[e^{\frac{x}{\lambda}} - 0 \right]$$

$$= \frac{1}{2} e^{\frac{x}{\lambda}}$$

$$F(x) = \begin{cases} 1 - \frac{1}{2} e^{-\frac{x}{\lambda}} & x > 0 \quad F(x) > \frac{1}{2} \\ \frac{1}{2} e^{\frac{x}{\lambda}} & x \leq 0 \quad F(x) \leq \frac{1}{2} \end{cases}$$

$$F(0) = \frac{1}{2} e^{\frac{0}{\lambda}} = \frac{1}{2}$$

b (ii)

$$F(x) = p$$

$p > \frac{1}{2}$

$$1 - \frac{1}{2} e^{-\frac{x}{\lambda}} = p$$

$$\Rightarrow \frac{1}{2} e^{-\frac{x}{\lambda}} = 1 - p$$

$$\Rightarrow e^{-\frac{x}{\lambda}} = 2(1-p)$$

$$\Rightarrow x = -\lambda \log [2(1-p)]$$

$p \leq \frac{1}{2}$

$$\frac{1}{2} e^{\frac{x}{\lambda}} = p$$

$$\Rightarrow e^{\frac{x}{\lambda}} = 2p$$

$$\Rightarrow x = \lambda \log [2p]$$

$$VaR_p(X) = \begin{cases} -\lambda \log [2(1-p)] & p > \frac{1}{2} \\ \lambda \log [2p] & p \leq \frac{1}{2} \end{cases}$$

b (iii)

$$ES_p(X) = \frac{1}{p} \int_0^p VaR_t(X)\, dt$$

$p > \frac{1}{2}$

$$ES_p(X) = \frac{1}{p} \left(\int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^{p} \right) VaR_t(X)\, dt$$

$$= \frac{1}{p} \left[\int_0^{\frac{1}{2}} \lambda \log(2t)\, dt + \int_{\frac{1}{2}}^{p} (-\lambda) \log[2(1-t)]\, dt \right]$$

$$= \frac{1}{p} \left[\lambda \int_0^{\frac{1}{2}} [\log 2 + \log t]\, dt - \lambda \int_{\frac{1}{2}}^{p} [\log 2 + \log(1-t)]\, dt \right]$$

$$= \frac{1}{2} \left[\frac{\lambda}{2} \log 2 + \lambda \int_0^{\frac{1}{2}} \log t\, dt - \lambda \left(p - \frac{1}{2}\right) \log 2 - \lambda \int_{\frac{1}{2}}^{p} \log(1-t)\, dt \right]$$

$$= \frac{1}{2}\Big[\ \frac{\lambda}{2}\log 2 + \big[\log t \cdot t\big]_0^{\frac{1}{2}}$$

int by parts

$$- \frac{\lambda}{2}$$

$$- \lambda\left(\rho - \frac{1}{2}\right)\cdot \log 2$$

$$- \lambda\ \big[\log(1-t)\cdot t\big]_{\frac{1}{2}}^{\rho}$$

$$- \lambda \int_{\frac{1}{2}}^{\rho} \frac{t}{1-t}\, dt\ \Big]$$

$$= \ldots$$

REVISION CLASS

11 DECEMBER

12:00-13:00PM

MATH3/4/68181

REVISION CLASS

(MON 11 DEC 12-1 PM)

Exam 2016/17 Q5

(a)

$$f(x_1, \dots, x_k)$$

$$= (-1)^k \frac{\partial^k}{\partial x_1 \cdots \partial x_k} \bar{F}(x_1, \dots, x_k) \quad (*)$$

$$\frac{\partial}{\partial x_1} \bar{F} = (-\alpha) \cdot \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i\right)^{-\alpha-1} \cdot \frac{1}{\theta}$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} \bar{F} = (-\alpha)(-\alpha-1) \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i\right)^{-\alpha-2} \left(\frac{1}{\theta}\right)^2$$

$$\frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} \bar{F} = (-\alpha)(-\alpha-1)(-\alpha-2) \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i\right)^{-\alpha-3} \cdot \left(\frac{1}{\theta}\right)^3$$

$$\vdots$$

$$\frac{\partial^k}{\partial x_1 \cdots \partial x_k} \bar{F} = (-\alpha)(-\alpha-1) \cdots (-\alpha-k+1) \cdot \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i\right)^{-\alpha-k} \cdot \left(\frac{1}{\theta}\right)^k$$

By $(*)$, $f(x_1, \dots, x_k)$

$$= \alpha(\alpha+1) \cdots (\alpha+k-1) \left(1 + \frac{1}{\theta} \sum_{i=1}^{k} x_i\right)^{-\alpha-k} \left(\frac{1}{\theta}\right)^k$$

(b)

$$f_S(s) = \underbrace{\int\int\cdots\int}_{x_1+x_2+\cdots+x_k=s} f(x_1, x_2, \ldots, x_k)\, dx_k \cdots dx_2\, dx_1$$

$$= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k} \cdot \underbrace{\int\int\cdots\int}_{x_1+\cdots+x_k=s} \left(1+\frac{1}{\theta}\sum_{i=1}^{k} x_i\right)^{-\alpha-k} dx_k \cdots dx_2\, dx_1$$

($\sum_{i=1}^{k} x_i = s$)

$$= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k}\left(1+\frac{s}{\theta}\right)^{-\alpha-k} \cdot \underbrace{\int\int\cdots\int}_{x_1+\cdots+x_k=s} 1\cdot dx_k \cdots dx_2\, dx_1$$

($\int\int\cdots\int 1\cdot dx_k \cdots dx_2\, dx_1 = \frac{s^k}{k!}$)

$$= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k}\left(1+\frac{s}{\theta}\right)^{-\alpha-k} \frac{s^k}{k!}.$$

(c)

$$F_S(s) = \int_0^s f_{S'}(t)\, dt$$

$$= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k\, k!} \int_0^s \left(1+\frac{t}{\theta}\right)^{-\alpha-k} t^k\, dt$$

Set $y = \dfrac{t}{\theta+t}$

$\Rightarrow \quad y\theta + yt = t$

$\Rightarrow \quad t = \dfrac{y\theta}{1-y}$

$\Rightarrow \quad \dfrac{dt}{dy} = \dfrac{\theta}{(1-y)^2}$

$$= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k\, k!} \int_0^{\frac{s}{(\theta+s)}} \left(1+\frac{y}{1-y}\right)^{-\alpha-k} \frac{y^k\theta^k}{(1-y)^k} \cdot \frac{\theta}{(1-y)^2}\, dy$$

$$= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\theta^k\, k!}\, \theta^{k+1} \int_0^{\frac{s}{\theta+s}} y^k (1-y)^{\alpha-2}\, dy$$

$$= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)\,\theta}{k!}\, B_{\frac{s}{\theta+s}}(k+1,\ \alpha-1)$$

Exam 2014/15 Q5 (c)

i) Assume G belongs to Gumbel max domain, that is there exists $\gamma(t) > 0$ such that

$$\lim_{t \to w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x} \quad (*)$$

$$\lim_{t \to w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to w(G)} \frac{1 - \{1 - [1 - (G(t + x\gamma(t)))^{\theta}]^2\}^{\alpha}}{1 - \{1 - [1 - (G(t))^{\theta}]^2\}^{\alpha}}$$

(corrections: $w(F)$ struck out and replaced by $w(G)$; annotations: $(G(t+x\gamma(t)))^{\theta} \to 1$, $[\ldots]^2 \to 0$; $(G(t))^{\theta} \to 1$, $[\ldots]^2 \to 0$)

$$= \lim_{t \to w(G)} \frac{1 - \{1 - \alpha \cdot [1 - ((G(t + x\gamma(t)))^{\theta}]^2\}}{1 - \{1 - \alpha [1 - (G(t))^{\theta}]^2\}}$$

(the $1-$, $1-$ and α are struck out in numerator and denominator)

$(1 - z)^{\alpha} \approx 1 - \alpha \cdot z$ as $z \to 0$

$$= \lim_{t \to w(G)} \left[\frac{1 - (G(t + x\gamma(t)))^{\theta}}{1 - (G(t))^{\theta}}\right]^2$$

$$= \lim_{t \to w(G)} \left[\frac{1 - \{1 - [1 - G(t + x\gamma(t))]\}^{\theta}}{1 - \{1 - [1 - G(t)]\}^{\theta}} \right]^2$$

(each $[1 - G(\cdot)] \to 0$)

$$= \lim_{t \to w(G)} \left[\frac{1 - \{1 - \theta [1 - G(t + x\gamma(t))]\}}{1 - \{1 - \theta [1 - G(t)]\}} \right]^2$$

$(1 - z)^{\theta} \approx 1 - \theta \cdot z$ as $z \to 0$

$$= \lim_{t \to w(G)} \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)} \right]^2$$

$$\overset{\circledast}{=} e^{-2x}$$

So F belongs to also the Gumbel max domain.

REVISION CLASS

12 DECEMBER

9:00-10:00AM

MATH3/4/68181

$$\overline{F}_{X,Y,Z}(x,y,z) = 1 - F_X(x) - F_Y(y) - F_Z(z) + F_{X,Y}(x,y) + F_{X,Z}(x,z) + F_{Y,Z}(y,z) - F_{X,Y,Z}(x,y,z) \quad (*)$$

$$F_{X,Y,Z}(x,y,z) = 1 - \overline{F}_X(x) - \overline{F}_Y(y) - \overline{F}_Z(z) + \overline{F}_{X,Y}(x,y) + \overline{F}_{X,Z}(x,z) + \overline{F}_{Y,Z}(y,z) - \overline{F}_{X,Y,Z}(x,y,z) \quad (**)$$

REVISION

CLASS

(TUES 12 DEC
9 - 10 AM)

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

$$\bar{F}_{X,Y}(x,y) = P(X > x, Y > y)$$

$$\boxed{\bar{F}_{X,Y}(x,y) = 1 - F_X(x) - F_Y(y) + F_{X,Y}(x,y)}$$

$$F_{X,Y}(x,y) = \bar{F}_{X,Y}(x,y) - 1 + F_X(x) + F_Y(y)$$

$$= \bar{F}_{X,Y}(x,y) - 1 + 1 - \bar{F}_X(x) + 1 - \bar{F}_Y(y)$$

$$\boxed{F_{X,Y}(x,y) \quad 1 - \bar{F}_X(x) - \bar{F}_Y(y) + \bar{F}_{X,Y}(x,y)}$$

Exam 2012/13 Q4

(a)

$$F_M(m)$$

$$= P(M \leqslant m)$$

$$= P(\max(X, Y, Z) \leqslant m)$$

$$= P(X \leqslant m, Y \leqslant m, Z \leqslant m)$$

$$= F_{X,Y,Z}(m, m, m)$$

(**)

$$= 1 - \bar{F}_X(m) - \bar{F}_Y(m) - \bar{F}_Z(m) + \bar{F}_{X,Y}(m, m) + \bar{F}_{X,Z}(m, m) + \bar{F}_{Y,Z}(m, m) - \bar{F}_{X,Y,Z}(m, m, m)$$

$$= 1 - \left[1 + \frac{m}{a}\right]^{-d} - \left[1 + \frac{m}{b}\right]^{-d} - \left[1 + \frac{m}{c}\right]^{-d} + \left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-d} + \left[1 + \frac{m}{a} + \frac{m}{c}\right]^{-d} + \left[1 + \frac{m}{b} + \frac{m}{c}\right]^{-d} - \left[1 + \frac{m}{a} + \frac{m}{b} + \frac{m}{c}\right]^{-d}$$

(b)

$$f_M(m) = \frac{d}{dm} F_M(m)$$

$$= \frac{d}{a}\left[1+\frac{m}{a}\right]^{-d-1} + \frac{d}{b}\left[1+\frac{m}{b}\right]^{-d-1}$$

$$+ \frac{d}{c}\left[1+\frac{m}{c}\right]^{-d-1}$$

$$- d\left(\frac{1}{a}+\frac{1}{b}\right)\left[1+\frac{m}{a}+\frac{m}{b}\right]^{-d-1}$$

$$- d\left(\frac{1}{a}+\frac{1}{c}\right)\left[1+\frac{m}{a}+\frac{m}{c}\right]^{-d-1}$$

$$- d\left(\frac{1}{b}+\frac{1}{c}\right)\left[1+\frac{m}{b}+\frac{m}{c}\right]^{-d-1}$$

$$+ d\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left[1+\frac{m}{a}+\frac{m}{b}+\frac{m}{c}\right]^{-d-1}$$

Exam 2016/17 Q3(e)

$$F(x) = 1 - \left[1 - e^{-\frac{1}{x}}\right]^a, \quad x > 0$$

$$w(F) = +\infty$$

(II):

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{1 - \left\{1 - \left[1 - e^{-\frac{1}{tx}}\right]^a\right\}}{1 - \left\{1 - \left[1 - e^{-\frac{1}{t}}\right]^a\right\}}$$

$$= \lim_{t \to \infty} \frac{\left[1 - e^{-\frac{1}{tx}}\right]^a}{\left[1 - e^{-\frac{1}{t}}\right]^a}$$

$\frac{1}{tx} \to 0$, $\frac{1}{t} \to 0$

$$e^{-y} \approx 1 - y$$

$$= \lim_{t \to \infty} \frac{\left[1 - \left(1 - \frac{1}{tx}\right)\right]^a}{\left[1 - \left(1 - \frac{1}{t}\right)\right]^a}$$

$$= x^{-a}$$

$\Rightarrow$ F belongs to the Frechet max domain

Exam 2016/17 Q3 (b)

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$k = 0, 1, \dots, n$$

$$w(F) = n$$

$$\lim_{k \to w(F)} \frac{P(X=k)}{1-F(k-1)}$$

$$= \frac{P(X=n)}{1-F(n-1)}$$

$$= \frac{P(X=n)}{1-P(X \leq n-1)}$$

$$= \frac{P(X=n)}{P(X > n-1)}$$

$$= \frac{P(X=n)}{P(X=n)} = 1 \neq 0$$

EVT does not hold.

Exam 2016/17, Q3 (c)

$$f(x) = \frac{1}{(\log b - \log a)\, x}, \quad a < x < b$$

$$w(F) = b$$

(III):

$$\lim_{t \to 0} \frac{1 - F(b - tx)}{1 - F(b - t)}$$

$$\overset{LH}{=} \lim_{t \to 0} \frac{-f(b - tx) \cdot (+x)}{-f(b - t)}$$

$$= \lim_{t \to 0} \frac{\frac{1}{(\log b - \log a)} \cdot \frac{1}{(b - tx)} \cdot x}{\frac{1}{(\log b - \log a)} \cdot \frac{1}{b - t}}$$

$$= \lim_{t \to 0} \frac{b - t}{b - tx} \cdot x$$

$$= x$$

$\Rightarrow$ F belongs to the Weibull max domain.

REVISION

CLASS

(FRI 15 DEC 9-10 AM)

Exam date : 24 Jan Wed 9:45am

Away from Manchester between 3 Jan – 11 Jan

(feel free to email me during this period)

Happy to meet you during the period the ATB will be closed. Please email me date / time.

Exam 2015/16 Q1

(a)

$$M_{X_i}(t) = E[e^{tX_i}]$$

$$= \int_0^\infty e^{tx} \cdot \underbrace{\lambda e^{-\lambda x}} \, dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x} \, dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^\infty$$

$$= \lambda \left[0 - \frac{1}{-(\lambda - t)} \right]$$

$$= \frac{\lambda}{\lambda - t} .$$

(b) $M_{T|N=n}(s) = E[e^{sT} | N=n]$

$$= E[e^{s(X_1 + \cdots + X_N)} | N=n]$$

$$= E[e^{s(X_1 + \cdots + X_n)}]$$

$$= \underbrace{E[e^{sX_1}]} \cdots E[e^{sX_n}]$$

$$= \frac{\lambda}{\lambda - s} \cdots \frac{\lambda}{\lambda - s}$$

$$= \left(\frac{\lambda}{\lambda - s}\right)^n$$

(c) $T \mid N = n$ has the gamma distribution with scale parameter λ and shape parameter n.

$$f_{T \mid N=n}(t) = \frac{\lambda (\lambda x)^{n-1} e^{-\lambda x}}{\Gamma(n)}$$

(d) $$f_T(t) = \sum_{n=1}^{\infty} f_{T/N=n}(t) \; P(N=n)$$

$$= \sum_{n=1}^{\infty} \frac{\lambda(\lambda x)^{n-1} e^{-\lambda x}}{\Gamma(n)} \cdot \theta (1-\theta)^{n-1}$$

$$= \theta \lambda e^{-\lambda x} \sum_{n=1}^{\infty} \frac{[\lambda x (1-\theta)]^{n-1}}{(n-1)!}$$

Set $m = n-1$

$$= \theta \lambda e^{-\lambda x} \sum_{m=0}^{\infty} \frac{[\lambda x(1-\theta)]^m}{m!} \quad = e^{\lambda x(1-\theta)}$$

$$= \theta \lambda e^{-\theta \lambda x}$$

$\Rightarrow \quad T \sim \text{Exp}(\theta \lambda).$

Exam 2015/16 Q3 (a)

$$w(F) = 1$$

(III):

$$\lim_{t \downarrow 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)}$$

$$\overset{LH}{=} \lim_{t \downarrow 0} \frac{(-1) f(1 - tx) \cdot (-x)}{(-1) f(1 - t) (-1)}$$

$$= \lim_{t \downarrow 0} \frac{C (1 - tx)^{\alpha - 1} \cdot (tx)^{\beta - 1} \cdot x}{C (1 - t)^{\alpha - 1} t^{\beta - 1}}$$

(annotations: $(1 - tx) \to 1$, $(1 - t) \to 1$)

$$= x^{\beta}$$

$\Rightarrow$ F belongs to the Weibull max domain.

Exam 2015/16 Q 4 (c)

(i)

$$L(a, k) = \prod_{i=1}^{n} \left[a k^a x_i^{-a-1} I\{x_i > k\} \right]$$

$$= a^n k^{na} \left(\prod_{i=1}^{n} x_i \right)^{-a-1} \cdot \prod_{i=1}^{n} I\{x_i > k\}$$

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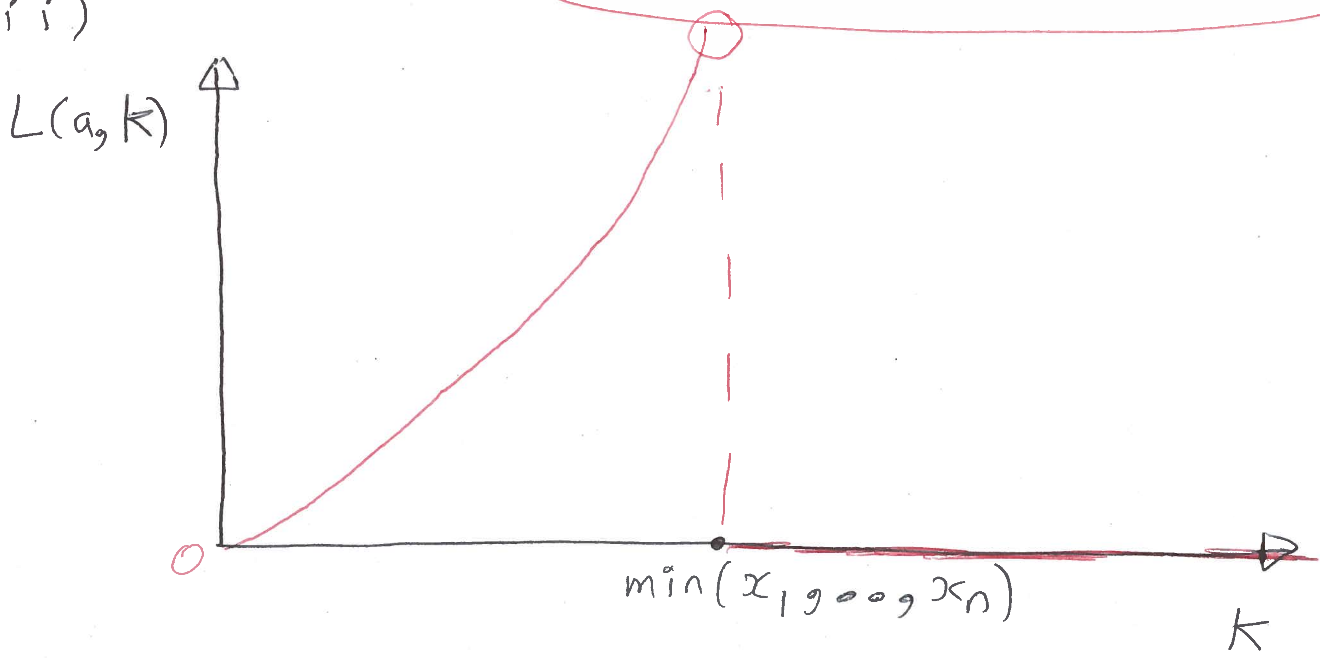
$$= a^n k^{na} \left(\prod_{i=1}^{n} x_i \right)^{-a-1} \cdot I\{\min(x_1, \dots, x_n) > k\}$$

(ii)



$$\Rightarrow \hat{k} = \min(x_1, \dots, x_n)$$

(iii) $\log L(a, k) = n \log a + na \log k$

$+ (a-1) \sum_{i=1}^{n} \log x_i$

$+ \log I\{\min(x_1, \ldots, x_n) > k\}$

$$\frac{d \log L}{da} = \frac{n}{a} + n \log k + \sum_{i=1}^{n} \log x_i = 0$$

$$\Rightarrow \hat{a} = \frac{n}{-n \log \hat{k} + \sum_{i=1}^{n} \log x_i}$$

(iv)

$$VaR_p(X) = k(1-p)^{-\frac{1}{a}}$$

$$VaR_0(X) = k(1-0)^{-\frac{1}{a}} = k$$

$$\widehat{VaR_0}(X) = \hat{k}$$

$$ES_p(X) = \frac{1}{p} \int_0^p VaR_t(X)\, dt$$

$$ES_0(X) = \lim_{p \to 0} \frac{1}{p} \int_0^p VaR_t(X)\, dt$$

($p \to 0$; $\int_0^p VaR_t(X)\, dt \to 0$)

$$\overset{LH}{=} \lim_{p \to 0} \frac{VaR_p(X)}{1}$$

$$= VaR_0(X) = k$$

$$\widehat{ES_0}(X) = \hat{k}$$