LECTURE

27 SEPTEMBER

9:00-10:00AM

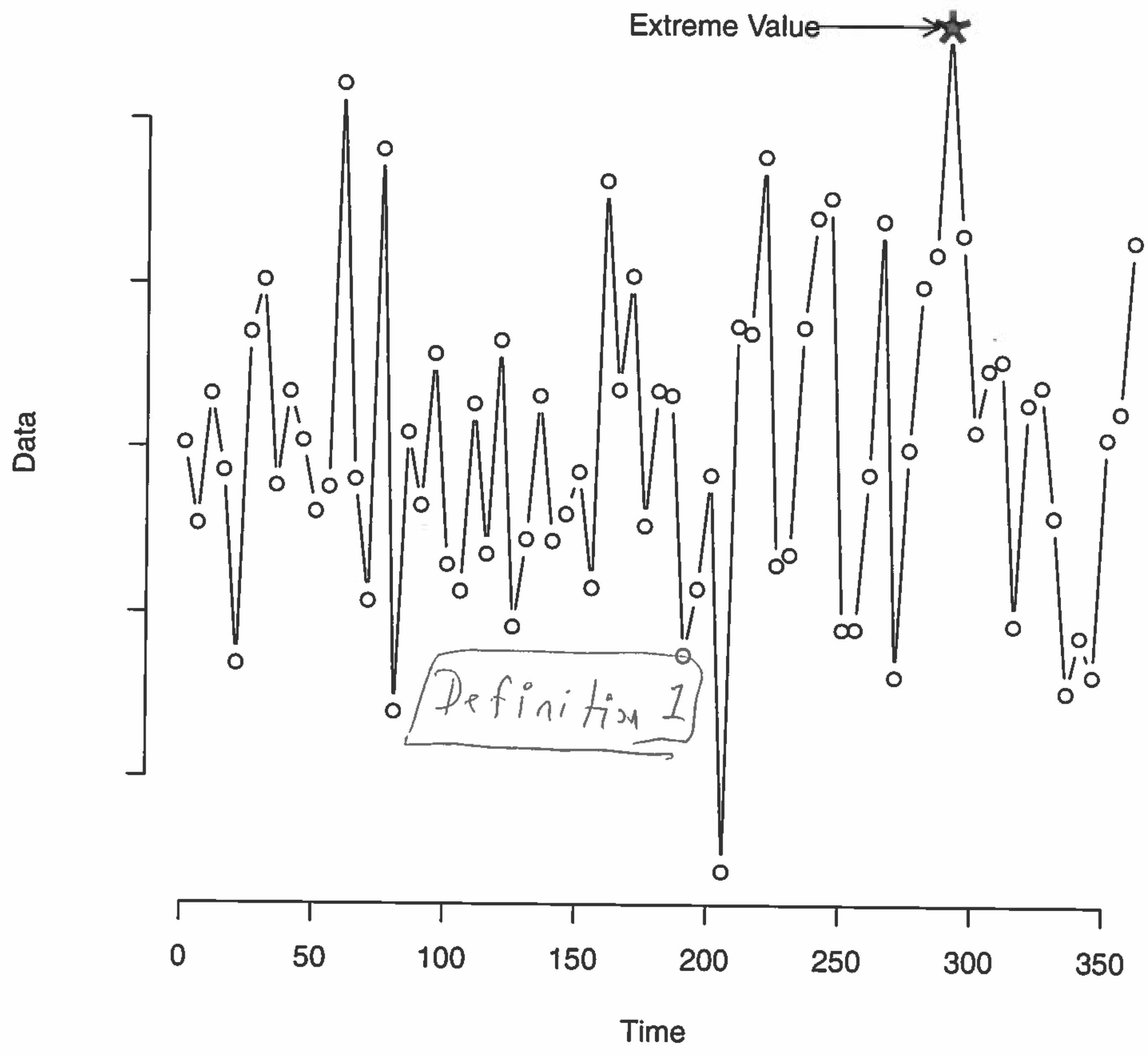
MATH3/4/68181

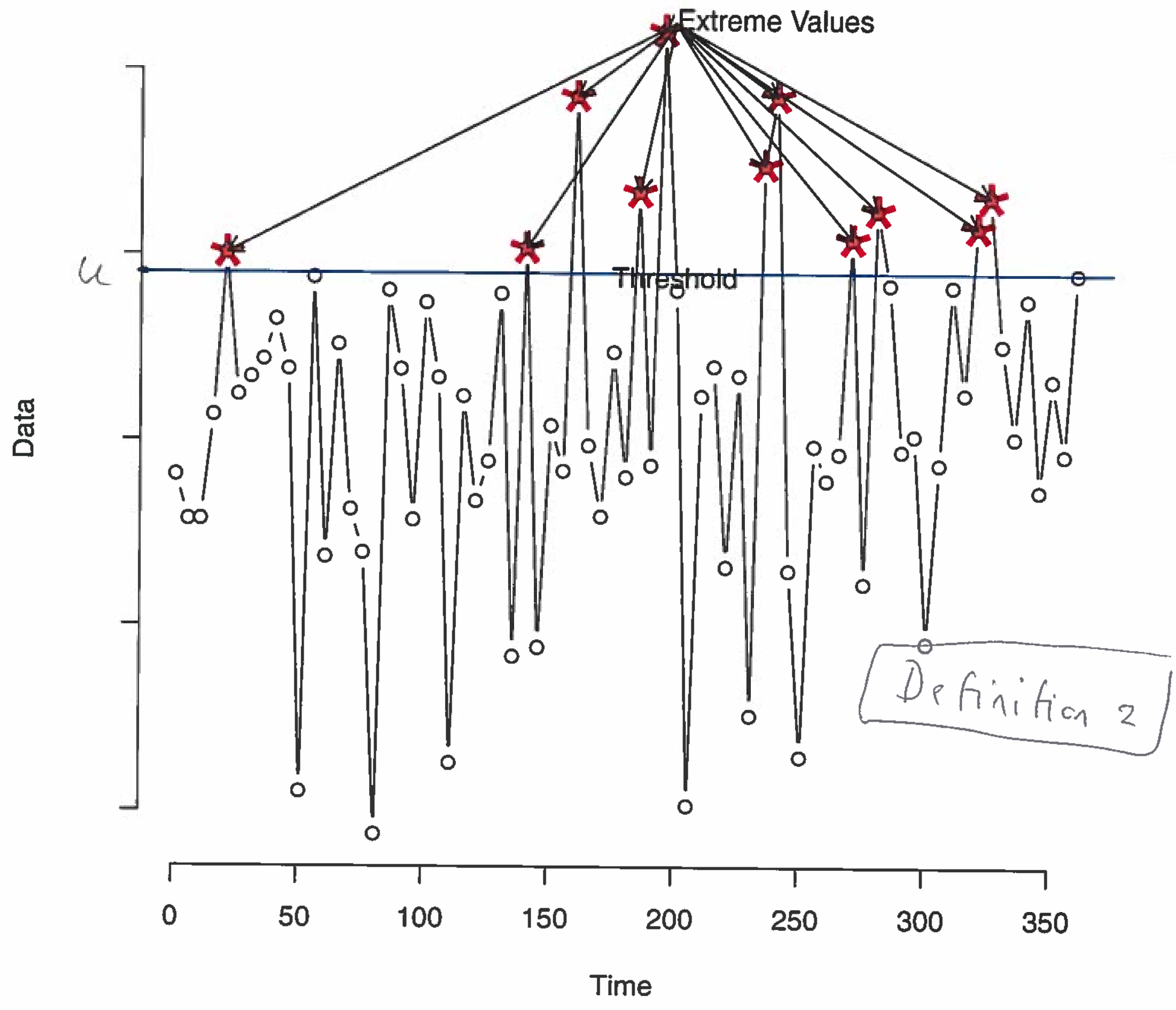
WELCOME TO

MATH 3/4/68/81

What is an extreme value?

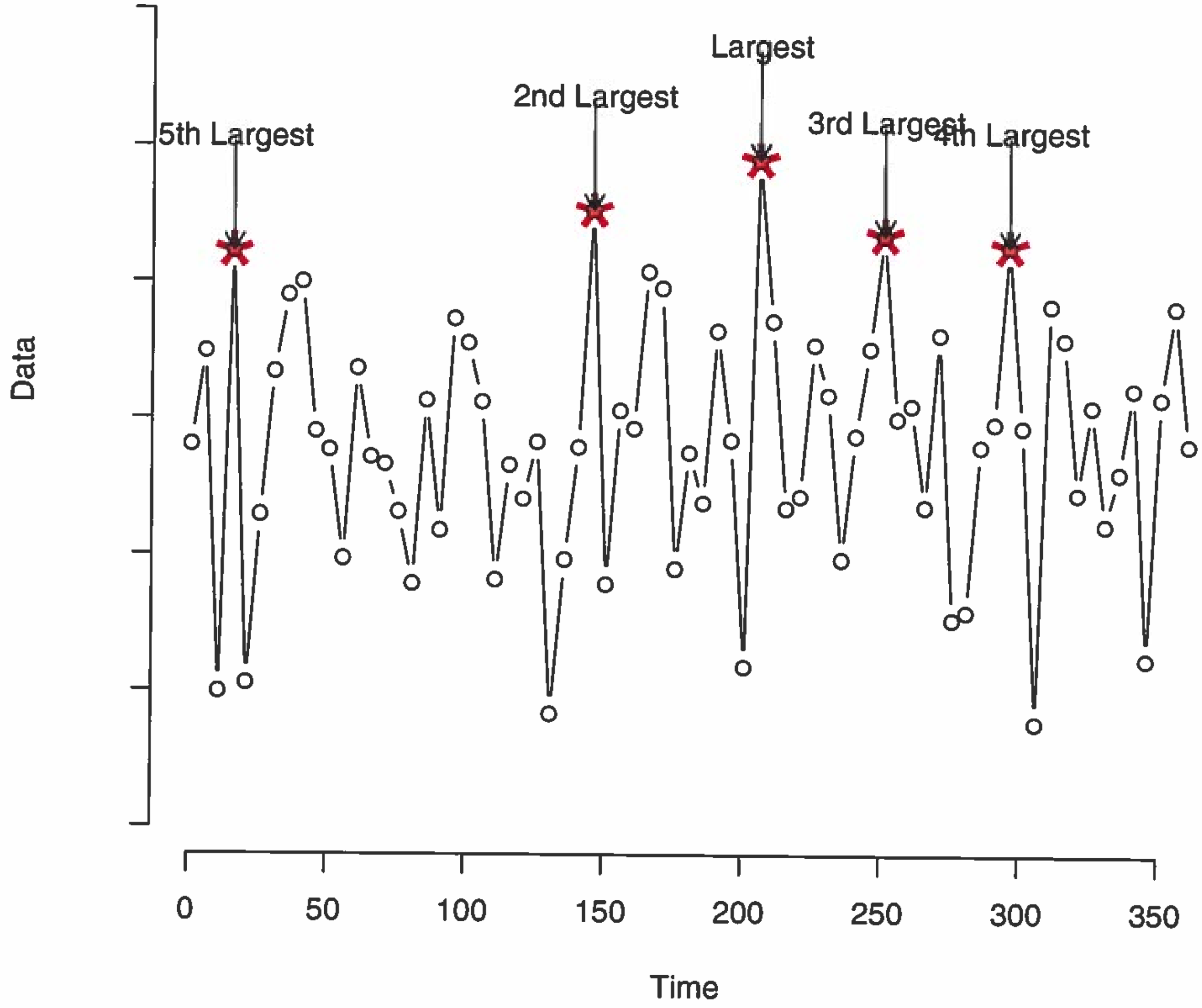
Three definitions


Extreme Value
Definition 1
Data
0
50
100
150
200
250
300
350
Time



Extreme Values
Threshold
u
Data
Time
0
50
100
150
200
250
300
350
Definition 2

r-largest method - Definition 3



Suppose $X_1, X_2, \ldots, X_n$ are IID with CDF F.

By definition 1, the extreme value is

$$M_n = \max(X_1, X_2, \ldots, X_n).$$

What is the distribution of M_n?

$$\begin{aligned}
&\Pr(M_n \leqslant x) \\
&= \Pr(\max(X_1, \ldots, X_n) \leqslant x) \\
&= \Pr(X_1 \leqslant x, \ldots, X_n \leqslant x) \\
&= \Pr(X_1 \leqslant x) \cdots \Pr(X_n \leqslant x) \\
&= F(x) \cdots F(x) \\
&= F^n(x).
\end{aligned}$$

$$\Rightarrow \quad P(M_n \leq x) = F^n(x).$$

$$\Rightarrow \quad \lim_{n\to\infty} P(M_n \leq x) = \lim_{n\to\infty} F^n(x)$$

$$= \begin{cases} 1 & \text{if } F(x) = 1 \\ 0 & \text{if } F(x) < 1 \end{cases}$$

"degenerate" limit

$$Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = Pr\left(M_n \leq a_n x + b_n\right)$$

$$= Pr\left(\max(X_1, \ldots, X_n) \leq a_n x + b_n\right)$$

$$= Pr\left(X_1 \leq a_n x + b_n, \ldots, X_n \leq a_n x + b_n\right)$$

indep

$$= Pr(X_1 \leq a_n x + b_n) \cdots Pr(X_n \leq a_n x + b_n)$$

$$= F(a_n x + b_n) \cdots F(a_n x + b_n)$$

$$= F^n(a_n x + b_n)$$

Suppose $X_1, X_2, \ldots, X_n$ are IID with CDF F. Let

$$\bar{X} = \frac{X_1 + \cdots + X_n}{n}$$

$$\bar{X} \xrightarrow[n \to \infty]{} \mu \quad \text{(population mean)}$$

SLLN

$$\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \xrightarrow[n \to \infty]{} N(0,1)$$

CLT

$$\Rightarrow \Pr\left(\frac{M_n - b_n}{a_n} \le x\right) = F^n(a_n x + b_n)$$

$$\Rightarrow \lim_{n \to \infty} \Pr\left(\frac{M_n - b_n}{a_n} \le x\right)$$

$$= \lim_{n \to \infty} F^n(a_n x + b_n) \quad \text{———} \quad (*)$$

(*) can be of the <u>same</u> <u>type</u> as one of the following

<u>Gumbel</u> (I) $\Lambda(x) = e^{-e^{-x}}, \quad -\infty < x < \infty$

<u>Frechet</u> (II) $\Phi_\alpha(x) = \begin{cases} 0 & x < 0 \\ e^{-x^{-\alpha}} & x \ge 0 \end{cases}$

<u>Weibull</u> (III) $\Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha} & x < 0 \\ 1 & x \ge 0 \end{cases}$

"Extremal Types Theorem"

"<u>same</u> <u>type</u>"

$G(x)$ CDF

$G^*(x) = G(ax+b)$, $a > 0, b \in \mathbb{R}$

then G & G^* are of the <u>same</u> <u>type</u>.

LECTURE

30 SEPTEMBER

9:00-10:00AM

MATH3/4/68181

Extremal Types Thm

Suppose $X_1, X_2, \dots, X_n$ are IID with CDF F. Let $M_n = \max(X_1, X_2, \dots, X_n)$.

If there exists [Definition I] $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\Pr\left(\frac{M_n - b_n}{a_n} < x\right) \longrightarrow G(x)$$

as $n \to \infty$ then $G(x)$ must be of the $\boxed{\text{same}}$ $\boxed{\text{type}}$ as

"Gumbel" $\quad G(x) = e^{-e^{-x}}, \quad -\infty < x < \infty$

"Fréchet" $\quad G(x) = \begin{cases} 0 & x < 0 \\ e^{-x^{-\alpha}} & x \geq 0 \end{cases}$

"Weibull" $\quad G(x) = \begin{cases} e^{-(-x)^{\alpha}} & x < 0 \\ 1 & x \geq 0 \end{cases}$

Same type: Two CDFs G_1 & G_2 are of the same type if $G_1(x) = G_2(ax+b)$ for all x, $a > 0$ & $b \in \mathbb{R}$.

eg i)

$$G_1(x) = e^{-e^{-x}}$$

$$G_2(x) = e^{-e^{-2x+3}}$$

are of the same type.

ii)

$$G_1(x) = e^{-x^{-2}}$$

$$G_2(x) = e^{-(5x+1)^{-2}}$$

are of the same type

iii)

$$G_1(x) = e^{-x^2}$$

$$G_2(x) = e^{-(-2x+3)^2}$$

are not of the same type.

Q: Given a CDF F (population CDF), which of the three limits will be attained if any?

<u>Answer</u>:

Let $w(F) = \sup\{x : F(x) < 1\}$

" upper end point of F "

"Gumbel" limit will be attained if

$$\lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = e^{-x}$$

for a ~~non-negative~~ positive function $\gamma(t)$.

"Frechet" limit will be attained if

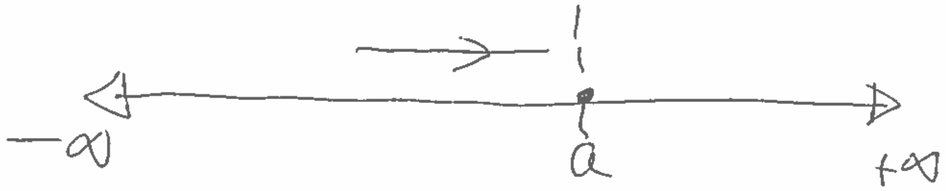
$$w(F) = \infty \quad \& \quad \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \alpha > 0$$

"Weibull" limit will be attained if

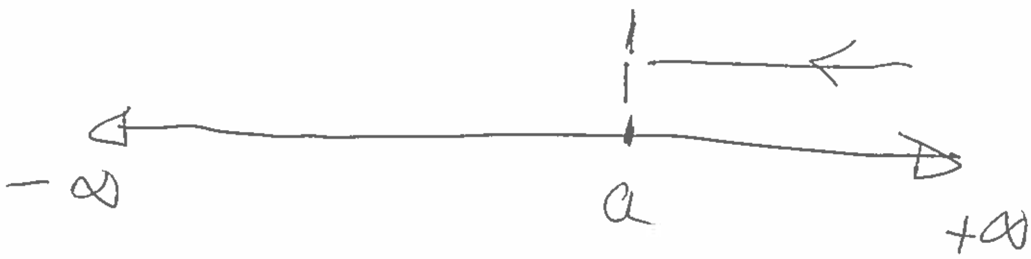
$$w(F) < \infty \quad \& \quad \lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^{\alpha}, \alpha > 0$$

Limits

$t \uparrow a$



$t \downarrow a$



Q: How to choose a_n & b_n?

Answer:

"Gumbel" limit

$$a_n = \gamma\left(F^{-1}\left(1-\frac{1}{n}\right)\right)$$
$$b_n = F^{-1}\left(1-\frac{1}{n}\right)$$

"Fréchet" limit

$$a_n = F^{-1}\left(1-\frac{1}{n}\right)$$
$$b_n \equiv 0$$

"Weibull" limit

$$a_n = w(F) - F^{-1}\left(1-\frac{1}{n}\right)$$
$$b_n = w(F).$$

Ex 1

$$F(x) = 1 - e^{-x} \quad , \quad x > 0$$

(Exponential)

$$w(F) = +\infty$$

$$\boxed{\begin{aligned} &F(x) = 1 \Rightarrow 1 - e^{-x} = 1 \Rightarrow -e^{-x} = 0 \\ &\Rightarrow e^{-x} = 0 \Rightarrow -x = \log 0 \Rightarrow -x = -\infty \Rightarrow x = +\infty \end{aligned}}$$

"Gumbel"

$$\frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \frac{\cancel{1} - [\cancel{1} - e^{-t - x\gamma(t)}]}{\cancel{1} - [\cancel{1} - e^{-t}]}$$

$$= \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= e^{-x\gamma(t)} \quad = e^{-x} \quad \text{if } \gamma(t) \equiv 1$$

$\Rightarrow$ Gumbel limit is attained

$$F^{-1}(y) = -\log(1-y)$$

$$a_n = \gamma\left(F^{-1}\left(1 - \tfrac{1}{n}\right)\right) = 1$$

$$b_n = F^{-1}\left(1 - \tfrac{1}{n}\right) = -\log\left(1 - \left(1 - \tfrac{1}{n}\right)\right)$$

$$= \log n$$

$$\Rightarrow \boxed{P\left(\frac{M_n - \log n}{1} < x\right) \to e^{-e^{-x}} \quad \text{as } n \to \infty}$$

Ex 2

$$F(x) = 1 - \frac{1}{x^2}, \quad x \geq 1$$

(Pareto)

$$\boxed{\begin{aligned} &F(x) = 1 \Rightarrow 1 - \frac{1}{x^2} = 1 \Rightarrow \frac{1}{x^2} = 0 \\ &\Rightarrow x = +\infty \Rightarrow w(F) = +\infty \end{aligned}}$$

$w(F) = \infty$

$$\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t \uparrow \infty} \frac{\cancel{1} - \left(\cancel{1} - \frac{1}{t^2 x^2}\right)}{\cancel{1} - \left(\cancel{1} - \frac{1}{t^2}\right)}$$

$$= \lim_{t \uparrow \infty} \frac{\frac{1}{\cancel{t^2} x^2}}{\frac{1}{t^2}} = x^{-2}, \quad \alpha = 2$$

$\Rightarrow$ Frechet limit is attained

$$F^{-1}(y) = (1-y)^{-\frac{1}{2}}$$

$$a_n = F^{-1}\left(1 - \frac{1}{n}\right) = \left(1 - \left(1 - \frac{1}{n}\right)\right)^{-\frac{1}{2}}$$

$$= n^{\frac{1}{2}}$$

$$b_n = 0$$

$\Rightarrow$ By ETT,

$$P\left(\frac{M_n - 0}{\sqrt{n}} < x\right) \to \begin{cases} e^{-x^{-2}} & x > 0 \\ 0 & x < 0 \end{cases}$$

EXAMPLE CLASS

3 OCTOBER

12:00-13:00PM

MATH3/4/68181

Ex 1

$$\Lambda(x) = e^{-e^{-x}}$$

$$\Lambda'(x) = e^{-x} \, e^{-e^{-x}}$$

$$\Phi_\alpha(x) = e^{-x^{-\alpha}}, \quad x \geq 0$$

$$\Phi_\alpha'(x) = e^{-x^{-\alpha}} (-1)(-\alpha) \, x^{-\alpha-1}$$

$$= \alpha x^{-\alpha-1} e^{-x^{-\alpha}}, \quad x \geq 0$$

$$\Psi_\alpha(x) = e^{-(-x)^\alpha}, \quad x \leq 0$$

$$\Psi_\alpha'(x) = e^{-(-x)^\alpha} (-1) \, \alpha (-x)^{\alpha-1} (-1)$$

$$= \alpha(-x)^{\alpha-1} e^{-(-x)^\alpha}$$

$$\left.\frac{d y^a}{d a}\right|_{a=0} = \left. y^a \log y \right|_{a=0}$$

$$\Rightarrow \left.\frac{d y^a}{d a}\right|_{a=0} = \log y \quad \text{—}(*)$$

Gamma function:

$$\Gamma(a) = \int_0^{+\infty} t^{a-1} e^{-t}\, dt$$

$$\left.\frac{d^2 y^a}{d a^2}\right|_{a=0} = \left. y^a \log^2 y \right|_{a=0}$$

$$\left.\frac{d^2 y^a}{d^2 a}\right|_{a=0} = \log^2 y \qquad (*)$$

Q2

$$\Lambda'(x) = e^{-x} e^{-e^{-x}}$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot e^{-x} \cdot e^{-e^{-x}} \, dx$$

$$\boxed{\begin{aligned} \text{Set} \quad y &= e^{-x} \\ x &= -\log y \\ \frac{dx}{dy} &= -\frac{1}{y} \end{aligned}}$$

$$= \int_{+\infty}^{0} (-\log y) \cdot \cancel{y} \cdot e^{-y} \left(-\frac{dy}{\cancel{y}}\right)$$

$$= \int_{+\infty}^{0} \log y \; e^{-y} \, dy$$

$$= \int_{+\infty}^{0} \frac{d\, y^a}{da}\Big|_{a=0} e^{-y} \, dy \qquad \text{by } (*)$$

$$= \frac{d}{da} \int_{+\infty}^{0} y^a e^{-y} \, dy \Big|_{a=0}$$

$$= -\frac{d}{da} \boxed{\int_{0}^{\infty} y^a e^{-y} \, dy} \Big|_{a=0}$$

$$= -\frac{d}{da} \Gamma(a+1) \Big|_{a=0} = -\Gamma'(1)$$

Q 3

$$E(x^2) = \int_{-\infty}^{+\infty} x^2 \cdot e^{-x} e^{-e^{-x}} \, dx$$

$$= \int_{+\infty}^{0} (-\log y)^2 \cdot y \cdot e^{-y} \left(-\frac{dy}{y}\right)$$

$$= -\int_{+\infty}^{0} \boxed{\log^2 y} \, e^{-y} \, dy$$

$$= -\int_{+\infty}^{0} \boxed{\left.\frac{d^2 y^a}{d^2 a}\right|_{a=0}} e^{-y} \, dy \quad \text{by } (*)$$

$$= -\frac{d^2}{d^2 a} \int_{+\infty}^{0} y^a e^{-y} \, dy \Bigg|_{a=0}$$

$$= \frac{d^2}{d^2 a} \boxed{\int_0^{+\infty} y^a e^{-y} \, dy} \Bigg|_{a=0}$$

$$= \frac{d^2}{d^2 a} \Gamma(a+1) \Big|_{a=0} = \Gamma''(1)$$

$$\mathrm{Var}(X) = \Gamma''(1) - [\Gamma'(1)]^2.$$

Q7

$F(x) = 1 - e^{-x}$

Did in Lectures.

Q8

$$F(x) = \left[1-e^{-x}\right]^{\alpha}, \quad x > 0$$

Gumbel

$$\omega(F) = +\infty$$

$$\boxed{\begin{aligned} &F(x) = 1 \Rightarrow \left[1-e^{-x}\right]^{\alpha} = 1 \\ &\Rightarrow 1-e^{-x} = 1 \Rightarrow e^{-x} = 0 \Rightarrow \\ &-x = \log 0 = -\infty \Rightarrow x = +\infty \end{aligned}}$$

$$\lim_{t\uparrow\infty} \frac{1-F(\cancel{t}+\cancel{x}\gamma(t))}{1-F(t)} = \lim_{t\uparrow\infty} \frac{1-\left[1-e^{-\cancel{t}\cancel{x}\gamma(t)}\right]^{\alpha}}{1-\left[1-e^{-t}\right]^{\alpha}}$$

$$= \lim_{t\uparrow\infty} \frac{\cancel{1}-\left[\cancel{1}-\alpha e^{-\cancel{t}-\cancel{x}\gamma(t)}\right]}{\cancel{1}-\left[\cancel{1}-\alpha e^{-t}\right]}$$

$(1-z)^{\alpha} \simeq 1-\alpha z$
as $z \to 0$

$$= \lim_{t\uparrow\infty} \frac{\cancel{\alpha} e^{-t-x\gamma(t)}}{\cancel{\alpha} e^{-t}} = \lim_{t\uparrow\infty} e^{-x\gamma(t)}$$

$$= e^{-x} \quad \text{if} \quad \gamma(t) \equiv 1.$$

$\Rightarrow$ F belongs to Gumbel max domain

99

$$F(x) = x \quad , \quad 0 < x < 1$$

$$w(F) = 1$$

$$w(F) = 1 < \infty$$

$$\lim_{t \downarrow 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)} = \lim_{t \downarrow 0} \frac{\cancel{1} - (\cancel{1} - tx)}{\cancel{1} - (\cancel{1} - t)}$$

$$= \lim_{t \downarrow 0} \frac{tx}{t} = x$$

$\Rightarrow$ F belongs to the Weibull max domain.

Q18

$$F(x) = 1 - \left(\frac{K}{x}\right)^{\alpha}, \quad x \geq K$$

$$w(F) = +\infty$$

$$\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t \uparrow \infty} \frac{\cancel{1} - \left[\cancel{1} - \left(\frac{K}{tx}\right)^{\alpha}\right]}{\cancel{1} - \left[\cancel{1} - \left(\frac{K}{t}\right)^{\alpha}\right]}$$

$$= \lim_{t \uparrow \infty} \frac{\left(\frac{K}{tx}\right)^{\alpha}}{\left(\frac{K}{t}\right)^{\alpha}} = x^{-\alpha}$$

$\Rightarrow$ F belong to max domain of Fréchet.

LECTURE

4 OCTOBER

9:00-10:00AM

MATH3/4/68181

An Example where ETT does not hold

$$F(x) = 1 - \frac{1}{\log x}, \qquad x > e$$

$$\omega(F) = +\infty$$

$$\boxed{F(x) = 1 \Rightarrow 1 - \frac{1}{\log x} = 1 \Rightarrow -\frac{1}{\log x} = 0 \Rightarrow \log x = +\infty \Rightarrow x = +\infty}$$

Gumbel:

$$\frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \frac{1 - \left[1 - \frac{1}{\log(t + x\gamma(t))}\right]}{1 - \left[1 - \frac{1}{\log t}\right]}$$

$$= \frac{\log t}{\log(t + x\gamma(t))} = \frac{\log t}{\log t + \log\left(1 + \frac{x}{t}\gamma(t)\right)}$$

$$= \frac{1}{1 + \frac{\log\left(1 + \frac{x}{t}\gamma(t)\right)}{\log t}} \not\longrightarrow e^{-x} \quad \text{as } t \to \infty$$

$\Rightarrow$ (I) is not satisfied

Frechet :

$$\frac{1-F(tx)}{1-F(t)} = \frac{1-\left[1-\frac{1}{\log(tx)}\right]}{1-\left[1-\frac{1}{\log t}\right]}$$

$$= \frac{\frac{1}{\log(tx)}}{\frac{1}{\log t}} = \frac{\log t}{\log(tx)}$$

$$= \frac{\log t}{\log t + \log x} = \frac{1}{1 + \frac{\log x}{\log t}}$$

$$\xrightarrow[t \to \infty]{} 1$$

$\Rightarrow$ (II) is not satisfied

$(\overline{III})$ is not satisfied since $w(F) = +\infty$ is not finite.

$F(x) = 1 - \frac{1}{\log x}$ does not ~~belongs~~ belong to any of the three domains of attraction.

Q11 , Sheet 1

Show that F belongs to the same domain of attraction as G.

i) If G belongs to the Gumbel domain so does F

ii) If G belongs to the Frechet domain so does F

iii) If G belongs to the Weibull domain so does F.

L'Hopital's Rule

$$\boxed{\begin{gathered}\lim_{x \to a} \frac{g_1(x)}{g_2(x)} = \lim_{x \to a} \frac{g_1'(x)}{g_2'(x)} \\ g_1(a) = \pm\infty \\ g_2(a) = \pm\infty\end{gathered}}$$

(i) Suppose G belongs to the Gumbel domain. So, there exists $\gamma(t) > 0$ such that

$$\lim_{t \uparrow w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x}. \quad (*)$$

To show that F also belongs to the Gumbel domain

$$\lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t)) \rightarrow 0}{1 - F(t) \rightarrow 0}$$

LH Rule $\lim_{t \uparrow w(F)} \frac{-f(t + x\gamma(t))\,(1 + x\gamma'(t))}{-f(t)}$

$$= \lim_{t \uparrow w(F)} \frac{f(t + x\gamma(t))\,(1 + x\gamma'(t))}{f(t)}$$

$$= \lim_{t \uparrow w(F)} \frac{\cancel{K} g(t + x\gamma(t))\, G^{a-1}(t + x\gamma(t))\, [1 - G(t + x\gamma(t))]^{b-1} e^{-cG(t + x\gamma(t))} (1 + x\gamma'(t))}{\cancel{K} g(t)\, G^{a-1}(t)\, [1 - G(t)]^{b-1} e^{-cG(t)}}$$

$$= \lim_{t \uparrow w(F)} \frac{g(t + x\gamma(t))(1 + x\gamma'(t))}{g(t)} \cdot \left[\frac{G(t + x\gamma(t))}{G(t)}\right]^{a-1} \cdot \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b-1} \cdot e^{cG(t) - cG(t + x\gamma(t))}$$

$w(F) = w(G)$

$$\Downarrow$$

$$= \lim_{t \uparrow w(G)} \frac{g(t + x\gamma(t))(1 + x\gamma'(t))}{g(t)} \cdot \left[\frac{G(t + x\gamma(t)) \to 1}{G(t) \to 1}\right]^{a-1} \cdot \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b-1} \cdot e^{c\underbrace{G(t)}_{\to 1} - c\underbrace{G(t + x\gamma(t))}_{\to 1}}$$

$$= \lim_{t \uparrow w(G)} \frac{g(t + x\gamma(t))\,(1 + x\gamma'(t))}{g(t)} \cdot \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b-1}$$

L'H Rule applied in reverse

$$= \lim_{t \uparrow w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} \cdot \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b-1}$$

$$= \lim_{t \uparrow w(G)} \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b}$$

$\underline{\text{by}}$ (*) $\quad (e^{-x})^{b} = e^{-bx}$,

same type as e^{-x}

Hence F belongs to the Gumbel domain.

Extremal Types Thm

Suppose $X_1, X_2, \ldots, X_n$ are IID with CDF F. Let $M_n = \max(X_1, X_2, \ldots, X_n)$. If there exists $a_n > 0$ & $b_n \in \mathbb{R}$ such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \longrightarrow G(x)$$

as $n \longrightarrow \infty$ then $G(x)$ must be of the same type as

"Gumbel" $G(x) = e^{-e^{-x}}, \; -\infty < x < +\infty$

"Frechet" $G(x) = \begin{cases} 0, & x < 0 \\ e^{-x^{-\alpha}}, & x \geq 0 \end{cases}$

"Weibull" $G(x) = \begin{cases} e^{-(-x)^{\alpha}}, & x < 0 \\ 1, & x \geq 0 \end{cases}$

ETT has 3 limits

Not very convenient for statistical modeling.

Q: Is there a form that combines the 3 limits into one?

Answer: Yes. The form is known as the GEV (Generalized Extreme Value) distribution with CDF

$$G(x) = e^{-\left(1+\xi \cdot \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

where $-\infty < \mu < +\infty$ "location" parameter

$\sigma > 0$ "scale" parameter

$-\infty < \xi < +\infty$ "shape" parameter

$1 + \frac{\xi}{\sigma}(x-\mu) > 0$

$\boxed{\xi = 0}$

$$G(x) = \lim_{\xi \to 0} e^{-\left(1 + \xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= \lim_{\xi \to 0} e^{-\left(1 + \frac{\frac{x-\mu}{\sigma}}{\frac{1}{\xi}}\right)^{-\frac{1}{\xi}}}$$

$$= \lim_{a \to \infty} e^{-\left(1 + \frac{(x-\mu)/\sigma}{a}\right)^{-a}} \qquad \left[a = \frac{1}{\xi}\right]$$

$$= \lim_{a \to \infty} e^{-\left[\left(1 + \frac{(x-\mu)/\sigma}{a}\right)^{a}\right]^{-1}}$$

$$= \cancel{\lim} \; e^{-\left[e^{\frac{x-\mu}{\sigma}}\right]^{-1}} \qquad \boxed{\left(1 + \frac{y}{n}\right)^{n} \to e^{y}}$$

$$= e^{-e^{-\frac{x-\mu}{\sigma}}}$$

Gumbel CDF

Gumbel is the particular case of GEV when $\xi = 0$.

EXAMPLE CLASS

4 OCTOBER

10:00-11:00AM

MATH3/4/68181

Q1

$$\Lambda(x) = e^{-e^{-x}}$$

$$\Lambda'(x) = e^{-e^{-x}} (-1) e^{-x} (-1)$$

$$= e^{-e^{-x}} e^{-x}$$

$$\Phi_\alpha(x) = e^{-x^{-\alpha}} , \quad x > 0$$

$$\Phi_\alpha'(x) = e^{-x^{-\alpha}} (-1)(-\alpha) x^{-\alpha-1}$$

$$= \alpha e^{-x^{-\alpha}} x^{-\alpha-1}$$

$$\Psi_\alpha(x) = e^{-(-x)^\alpha} , \quad x < 0$$

$$\Psi_\alpha'(x) = e^{-(-x)^\alpha} (-1) \alpha (-x)^{\alpha-1} (-1)$$

$$= \alpha (-x)^{\alpha-1} e^{-(-x)^\alpha}$$

$$\left.\frac{d y^a}{da}\right|_{a=0} = \left. y^a \log y \right|_{a=0}$$

$$\Rightarrow \left.\frac{d y^a}{da}\right|_{a=0} = \log y \qquad (*)$$

$$\boxed{\begin{array}{l} \underline{\text{Gamma}}\ \ \underline{\text{Function}} \\ \Gamma(a) = \displaystyle\int_0^{+\infty} t^{a-1} e^{-t}\, dt \end{array}}$$

$$\frac{d y^a}{da} = y^a \log y$$

$$\left.\frac{d}{da}\left(\frac{d y^a}{da}\right)\right|_{a=0} = \left.\frac{d}{da}\left(y^a \log y\right)\right|_{a=0}$$

$$= \left.\left(\frac{d}{da} y^a\right) \log y \right|_{a=0}$$

$$= \left. y^a \log y \log y \right|_{a=0}$$

$$= \left. y^a \log^2 y \right|_{a=0}$$

$$\Rightarrow \left.\frac{d^2}{da^2} y^a \right|_{a=0} = \log^2 y \qquad (**)$$

Q2

$$E(X) = \int_{-\infty}^{\infty} x \cdot e^{-e^{-x}} e^{-x} \, dx$$

$$\boxed{\begin{array}{l} y = e^{-x} \quad \Rightarrow \quad x = -\log y \\ \Rightarrow \quad \frac{dx}{dy} = -\frac{1}{y} \end{array}}$$

$$= \int_{+\infty}^{0} (-\log y) \, e^{-y} \, y \left(-\frac{dy}{y}\right)$$

$$= -\int_{0}^{\infty} \log y \; e^{-y} \, dy$$

$$\underset{=}{(*)} \; -\int_{0}^{\infty} \left.\frac{d y^{a}}{da}\right|_{a=0} e^{-y} \, dy$$

$$= -\frac{d}{da} \left. \boxed{\int_{0}^{\infty} y^{a} e^{-y} \, dy} \right|_{a=0}$$

$$= -\frac{d}{da} \left. \Gamma(a+1) \right|_{a=0}$$

$$= -\Gamma'(1)$$

Q3

$$\operatorname{Var}(X) = E(X^2) - \left(E(X)\right)^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 e^{-e^{-x}} e^{-x}\, dx$$

$$= \int_{+\infty}^{0} (-\log y)^2 e^{-y}\, y \left(-\frac{dy}{y}\right)$$

$$= \int_0^{+\infty} \log^2 y \; e^{-y}\, dy$$

$$\overset{(**)}{=} \int_0^{+\infty} \frac{d^2}{da^2} y^a \Big|_{a=0} e^{-y}\, dy$$

$$= \frac{d^2}{da^2} \boxed{\int_0^{\infty} y^a e^{-y}\, dy} \Bigg|_{a=0}$$

$$= \frac{d^2}{da^2} \Gamma(a+1) \Big|_{a=0}$$

$$= \Gamma''(1)$$

$$\operatorname{Var}(X) = \Gamma''(1) - \left[\Gamma'(1)\right]^2$$

Q7

Did in Lectures

Q8

$$F(x) = \left[1 - e^{-x}\right]^{\alpha}, \quad x > 0$$

$$w(F) = +\infty$$

Gumbel

$$\lim_{t \uparrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{1 - \left[1 - e^{-t - x\gamma(t)}\right]^{\alpha}}{1 - \left[1 - e^{-t}\right]^{\alpha}}$$

$$\boxed{(1 - z)^{a} \approx 1 - az} \quad z \to 0$$

$$= \lim_{t \uparrow \infty} \frac{\cancel{1} - \left[\cancel{1} - \alpha e^{-t - x\gamma(t)}\right]}{\cancel{1} - \left[\cancel{1} - \alpha e^{-t}\right]}$$

$$= \lim_{t \uparrow \infty} \frac{\cancel{\alpha} e^{-t - x\gamma(t)}}{\cancel{\alpha} e^{-t}}$$

$$= \lim_{t \uparrow \infty} e^{-x\gamma(t)}$$

$$= e^{-x} \quad \text{if} \quad \gamma(t) \equiv 1 \quad \forall t.$$

$\Rightarrow$ F belongs to the Gumbel max domain.

Q9

Dis in lectures

Q10

$$F(x) = 1 - \left(\frac{K}{x}\right)^{\alpha}, \quad x \geq K$$

$$w(F) = +\infty$$

$$\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t \uparrow \infty} \frac{\cancel{1} - \left[\cancel{1} - \left(\frac{K}{tx}\right)^{\alpha}\right]}{\cancel{1} - \left[\cancel{1} - \left(\frac{K}{t}\right)^{\alpha}\right]}$$

$$= \lim_{t \uparrow \infty} \frac{\left(\frac{K}{tx}\right)^{\alpha}}{\left(\frac{K}{t}\right)^{\alpha}} = x^{-\alpha}$$

So, F belongs to Frechet domain of attraction.

$$\boxed{F(x) = 1 \Rightarrow 1 - \left(\frac{K}{x}\right)^{\alpha} = 1 \Rightarrow -\left(\frac{K}{x}\right)^{\alpha} = 0 \Rightarrow \frac{K}{x} = 0 \Rightarrow x = +\infty}$$

LECTURE

7 OCTOBER

9:00-10:00AM

MATH3/4/68181

Problem sheet for

{Monday 10 Oct 12–1 pm
{Tuesday 11 Oct 10–11 am

MATH3/4/68181: Extreme values and financial risk
Semester 1
Problem sheet 4

1. If $x_1, x_2, \ldots, x_n$ is a random sample from

$$f(x) = \sigma^{-1} \exp\left(-\frac{1}{\sigma}x\right) \exp\left\{-\exp\left(-\frac{x}{\sigma}\right)\right\}$$

find the mle of σ.

2. If $x_1, x_2, \ldots, x_n$ is a random sample from

$$f(x) = \lambda \sigma^{\lambda} x^{-\lambda-1} \exp\left(-\sigma^{\lambda} x^{-\lambda}\right)$$

find the mles of λ and σ.

3. If $x_1, x_2, \ldots, x_n$ is a random sample from

$$f(x) = \lambda \sigma^{-\lambda} x^{\lambda-1} \exp\left(-\sigma^{-\lambda} x^{\lambda}\right)$$

find the mles of λ and σ.

4. If $x_1, x_2, \ldots, x_n$ is a random sample from

$$f(x) = (1 - \lambda x)^{1/\lambda - 1}$$

find the mle of λ.

GEV distribution

CDF: $G(X) = e^{-\left(1+\xi \frac{X-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$

$-\infty < \xi < +\infty$ "shape" parameter

$\sigma > 0$ "scale" "

$-\infty < \mu < +\infty$ "location" "

GEV contains Gumbel, Frechet & Weibull as particular cases.

$\boxed{\xi = 0}$ $\lim_{\xi \to 0} e^{-\left(1+\xi \frac{X-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$

$= \lim_{\xi \to 0} e^{-\left(1+\frac{\frac{X-\mu}{\sigma}}{\frac{1}{\xi}}\right)^{-\frac{1}{\xi}}}$

$\boxed{a = \frac{1}{\xi}}$

$= \lim_{a \to \infty} e^{-\left(1+\frac{\frac{X-\mu}{\sigma}}{a}\right)^{-a}}$

$= \lim_{a \to \infty} e^{-\left[\left(1+\frac{\frac{X-\mu}{\sigma}}{a}\right)^{a}\right]^{-1}}$

$\boxed{\begin{array}{c}\left(1+\frac{z}{n}\right)^n \\ \downarrow \\ e^z\end{array}}$ $= \lim_{a \to \infty} e^{-\left[e^{\frac{X-\mu}{\sigma}}\right]^{-1}} = \boxed{e^{-e^{-\frac{X-\mu}{\sigma}}}}$

same type as $\boxed{e^{-e^{-X}}}$

$\Rightarrow$ GEV contains Gumbel as a particular case

$\boxed{\xi > 0}$

$$G(x) = e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= e^{-\left(\underbrace{\frac{\xi}{\sigma}}_{>0}x + \underbrace{1-\frac{\xi\mu}{\sigma}}_{\in\mathbb{R}}\right)^{-\frac{1}{\xi}}}$$

$$= \boxed{e^{-(ax+b)^{-\frac{1}{\xi}}}}$$

$a > 0$
$b \in \mathbb{R}$

same type as $\boxed{e^{-x^{-\frac{1}{\xi}}}}$

Fréchet CDF

GEV contains Fréchet as a particular case.

$\boxed{\xi < 0}$

$$G(x) = e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= e^{-\left(\frac{\xi x}{\sigma} + 1 - \frac{\xi\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= e^{-\left(-\underbrace{\frac{(-\xi)}{\sigma}}_{=a>0}x + \underbrace{\left(1-\frac{\xi\mu}{\sigma}\right)}_{=b\in\mathbb{R}}\right)^{-\frac{1}{\xi}}}$$

$$= e^{-(-ax+b)^{-\frac{1}{\xi}}}$$

same type as $\boxed{e^{-(-x)^{-\frac{1}{\xi}}}}$

Weibull CDF

GEV contains Weibull as a particular case.

CDF: $G(x) = e^{-\left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$

PDF: $$g(x) = \frac{1}{\sigma}\left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \cdot e^{-\left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

Domain: $-\infty < x < +\infty$ if $\xi \geq 0$

$-\infty < x < \mu - \frac{\sigma}{\xi}$ if $\xi < 0$

Moments

$\boxed{\xi > 0}$:

$$E(X^n) = \int_{-\infty}^{+\infty} x^n \cdot \frac{1}{\sigma}\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \cdot e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} dx$$

$$= \int_{-\infty}^{+\infty} \frac{\sigma^n}{\xi^n}\left(1+\xi\frac{x-\mu}{\sigma} - 1 + \frac{\xi\mu}{\sigma}\right)^n \cdot \frac{1}{\sigma}\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \cdot e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} dx$$

binom exp

$$= \frac{\sigma^{n-1}}{\xi^n}\sum_{k=0}^{n}\binom{n}{k}\left(-1+\frac{\xi\mu}{\sigma}\right)^{n-k} \int_{-\infty}^{+\infty}\left(1+\xi\frac{x-\mu}{\sigma}\right)^{k-\frac{1}{\xi}-1} e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} dx$$

$$\boxed{\begin{aligned} y &= \left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} \\ x &= \mu + \frac{\sigma}{\xi}\left(y^{-\xi}-1\right) \\ \frac{dx}{dy} &= -\sigma\, y^{-\xi-1} \end{aligned}}$$

$$= \frac{\sigma^{n-1}}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} \left(-1 + \frac{\xi \mu}{\sigma}\right)^{n-k} \int_{+\infty}^{0} \left(y^{-\xi}\right)^{k - \frac{1}{\xi} - 1} e^{-y} \left(-\sigma y^{-\xi - 1}\right) dy$$

$$= \frac{\sigma^{n-1+1}}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} \left(-1 + \frac{\xi \mu}{\sigma}\right)^{n-k} \boxed{\int_{0}^{+\infty} y^{-\xi k} e^{-y} \, dy}$$

$$= \frac{\sigma^n}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} \left(-1 + \frac{\xi \mu}{\sigma}\right)^{n-k} \Gamma(1 - \xi k)$$

Gamma Function

$\boxed{\xi < 0}$: <u>home</u> <u>work</u>

$\boxed{\xi = 0}$

$$E(X^n) = \int_{-\infty}^{+\infty} x^n \cdot \frac{1}{\sigma} e^{-\frac{X-\mu}{\sigma}} \cdot e^{-e^{-\frac{X-\mu}{\sigma}}} dx$$

$$\boxed{Y = \frac{X-\mu}{\sigma} \Rightarrow \begin{array}{l} X = \mu + \sigma y \\ dx = \sigma dy \end{array}}$$

$$= \int_{-\infty}^{+\infty} (\mu + \sigma y)^n e^{-Y} e^{-e^{-Y}} dy$$

$$= \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \sigma^k \int_{-\infty}^{+\infty} y^k e^{-Y} e^{-e^{-Y}} dy$$

$$\boxed{\begin{array}{l} Z = e^{-Y} \\ y = -\log z \\ dy = -\frac{dz}{z} \end{array}}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \sigma^k \int_{+\infty}^{0} (-\log z)^k z \cdot e^{-z} \cdot \left(-\frac{dz}{z}\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \sigma^k \int_0^{+\infty} (-1)^k \log^k z \, e^{-z} dz$$

$$\frac{d^k}{da^k} y^a \Big|_{a=0} = y^a \log^k y \Big|_{a=0}$$

$$\Rightarrow \boxed{\frac{d^k}{da^k} y^a \Big|_{a=0} = \log^k y}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \sigma^k (-1)^k$$

$$\cdot \frac{d^k}{da^k} \boxed{\int_0^{+\infty} z^a e^{-z} dz} \Big|_{a=0}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \sigma^k (-1)^k$$

$$\cdot \frac{d^k}{da^k} \Gamma(a+1) \Big|_{a=0}$$

~~MLE~~ ML estimation of μ, σ & ξ

Suppose $X_1, X_2, \ldots, X_n$ are IID from the GEV.

$$L(\mu,\sigma,\xi) = \prod_{i=1}^{n} \left\{ \frac{1}{\sigma} \cdot \left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} e^{-\left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} \right\}$$

$$= \frac{1}{\sigma^n} \left\{ \prod_{i=1}^{n} \left(1+\xi \frac{x_i-\mu}{\sigma}\right) \right\}^{-\frac{1}{\xi}-1} e^{-\sum_{i=1}^{n}\left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} \}$$

$$\log L = -n\log\sigma - \left(\frac{1}{\xi}+1\right)\sum_{i=1}^{n} \log\left(1+\xi \frac{x_i-\mu}{\sigma}\right) - \sum_{i=1}^{n}\left(1+\xi \frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$\frac{\partial \log L}{\partial \mu} = 0$$

$$\frac{\partial \log L}{\partial \sigma} = 0$$

$$\frac{\partial \log L}{\partial \xi} = 0$$

MLE equations for the GEV distribution

The MLEs of μ, σ and ξ are the simultaneous solutions of

$$\begin{aligned}\frac{\partial \log L}{\partial \mu} &= \frac{1+\xi}{\sigma}\sum_{i=1}^{n}\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-1} \\ &\quad -\frac{1}{\sigma}\sum_{i=1}^{n}\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \sigma} &= -\frac{n}{\sigma}+\frac{1+\xi}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-1} \\ &\quad -\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \xi} &= \frac{1}{\xi^2}\sum_{i=1}^{n}\log\left(1+\xi\frac{x_i-\mu}{\sigma}\right) \\ &\quad -\frac{1+\xi}{\xi\sigma}\sum_{i=1}^{n}(x_i-\mu)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-1} \\ &\quad -\frac{1}{\xi^2}\sum_{i=1}^{n}\log\left(1+\xi\frac{x_i-\mu}{\sigma}\right)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}} \\ &\quad +\frac{1}{\xi\sigma}\sum_{i=1}^{n}(x_i-\mu)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0.\end{aligned}$$

These eqns do not have a closed form solution.
R software — $\boxed{\text{fgev}(\circ)}$

T-year return level

$$G(x) = 1 - \frac{1}{T}$$

no of Years

$$\Rightarrow \quad e^{-\left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} = 1-\frac{1}{T}$$

$$\Rightarrow \quad \left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} = -\log\left(1-\frac{1}{T}\right)$$

$$\Rightarrow \quad 1+\xi \frac{x-\mu}{\sigma} = \left[-\log\left(1-\frac{1}{T}\right)\right]^{-\xi}$$

$$\Rightarrow \quad \boxed{x = \mu + \frac{\sigma}{\xi}\left[-\log\left(1-\frac{1}{T}\right)\right]^{-\xi}}$$

EXAMPLE CLASS

10 OCTOBER

12:00-13:00PM

MATH3/4/68181

Q1

$$L(\sigma) = \prod_{i=1}^{n} f(x_i)$$

$$= \prod_{i=1}^{n} \left[\frac{1}{\sigma} e^{-\frac{x_i}{\sigma}} e^{-e^{-\frac{x_i}{\sigma}}} \right]$$

$$= \frac{1}{\sigma^n} e^{-\frac{1}{\sigma}\sum_{i=1}^{n} x_i} e^{-\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}}$$

$$\log L(\sigma) = -n \log \sigma - \frac{1}{\sigma} \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}$$

$$\frac{d \log L}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^{n} x_i - \frac{1}{\sigma^2} \sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}}$$

The MLE of σ is the root of

$$\frac{d \log L}{d\sigma} = 0$$

$$\Longleftrightarrow \boxed{\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}} = n\sigma}$$

$$\left.\frac{d^2 \log L}{d\sigma^2}\right|_{\sigma=\hat{\sigma}} < 0$$

Q2

$$L(\lambda,\sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{\lambda} x_i^{-\lambda-1} e^{-\sigma^{\lambda} x_i^{-\lambda}} \right]$$

$$= \lambda^{n} \sigma^{n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{-\lambda-1} \cdot e^{-\sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda}}$$

$$\log L = n \log \lambda + n\lambda \log \sigma - (\lambda+1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n \log \sigma - \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda} \log\left(\frac{\sigma}{x_i}\right) \quad -(1)$$

$$\boxed{\frac{d y^{a}}{da} = y^{a} \log y}$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} - \sum_{i=1}^{n} \frac{\lambda \sigma^{\lambda-1}}{x_i^{\lambda}} \quad -(2)$$

$$(2) = 0 \Rightarrow \frac{n}{\sigma^{\lambda}} = \sum_{i=1}^{n} x_i^{-\lambda}$$

$$\Rightarrow \sigma = \left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \right]^{\frac{1}{\lambda}} \quad -(3)$$

Sub (3) into (1):

$$\frac{n}{\lambda} + \frac{n}{\lambda}\log\left[\frac{n}{\sum_{i=1}^{n} X_i^{-\lambda}}\right] - \sum_{i=1}^{n}\log X_i$$

$$- \left[\frac{n}{\sum_{i=1}^{n} X_i^{-\lambda}}\right]\sum_{i=1}^{n} X_i^{-\lambda}\log\sigma$$

$$+ \left[\frac{n}{\sum_{i=1}^{n} X_i^{-\lambda}}\right]\sum_{i=1}^{n} X_i^{-\lambda}\log X_i = 0 \qquad \text{— (4)}$$

(4) involves only λ.

The MLE $\hat{\lambda}$ is the root of (4).

$\hat{\sigma}$ follows from (3).

Q3

$$L(\lambda, \sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{-\lambda} x_i^{\lambda - 1} e^{-\left(\frac{x_i}{\sigma}\right)^{\lambda}} \right]$$

$$= \lambda^{n} \sigma^{-n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{\lambda - 1} e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}}$$

$$\log L = n \log \lambda - n\lambda \log \sigma + (\lambda - 1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma} \right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - n \log \sigma + \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma} \right)^{\lambda} \log \left(\frac{x_i}{\sigma} \right) = 0 \quad -(1)$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n\lambda}{\sigma} + \lambda \sum_{i=1}^{n} \frac{x_i^{\lambda}}{\sigma^{\lambda+1}} = 0 \quad -(2)$$

$$(2) \Rightarrow \quad \sigma = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right)^{\frac{1}{\lambda}}. \quad -(3)$$

Sub (3) into (1):

$$\frac{n}{\lambda} - \frac{n}{\lambda} \log \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right) + \sum_{i=1}^{n} \log x_i - \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right)^{-1} \sum_{i=1}^{n} x_i^{\lambda} \log x_i + \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right)^{-1} \log \sigma \sum_{i=1}^{n} x_i^{\lambda} = 0 \quad -(4)$$

(4) involves only λ

The MLE of λ is the root of (4).

The MLE of σ follows from (3).

Q4

$$L(\lambda) = \prod_{i=1}^{n} (1-\lambda x_i)^{\frac{1}{\lambda}-1}$$

$$= \left[\prod_{i=1}^{n} (1-\lambda x_i)\right]^{\frac{1}{\lambda}-1}$$

$$\log L = \left(\frac{1}{\lambda}-1\right) \sum_{i=1}^{n} \log(1-\lambda x_i)$$

$$\frac{d \log L}{d\lambda} = -\lambda^{-2} \sum_{i=1}^{n} \log(1-\lambda x_i) + \left(\frac{1}{\lambda}-1\right) \sum_{i=1}^{n} \frac{(-x_i)}{1-\lambda x_i} = 0$$

The MLE $\hat{\lambda}$ is the root of

$$\sum_{i=1}^{n} \log(1-\lambda x_i) = \lambda(\lambda-1) \sum_{i=1}^{n} \frac{x_i}{1-\lambda x_i}$$

$$\left.\frac{d^2 \log \lambda}{d\lambda^2}\right|_{\lambda=\hat{\lambda}} < 0$$

LECTURE

11 OCTOBER

9:00-10:00AM

MATH3/4/68181

$\boxed{\text{Last}}\ \boxed{\text{example}}\ \boxed{\text{on}}\ \boxed{\text{ETT}}$

$$f(x) = \frac{k}{x^2}, \quad 0 < a < x < b < \infty$$

$$F(x) = \int_a^x \frac{k}{y^2}\, dy$$

$$= k \left[-y^{-1}\right]_a^x$$

$$= k \left(\frac{1}{a} - \frac{1}{x}\right)$$

$$w(F) = b \qquad [\text{solve } F(x) = 1]$$

Gumbel:

$$\lim_{t \uparrow b} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \uparrow b} \frac{1 - k\left(\frac{1}{a} - \frac{1}{t + x\gamma(t)}\right)}{1 - k\left(\frac{1}{a} - \frac{1}{t}\right)}$$

$$= \lim_{t \uparrow b} \frac{1 - \frac{k}{a} + \frac{k}{t + x\gamma(t)}}{1 - \frac{k}{a} + \frac{k}{t}}$$

$$= \lim_{t \uparrow b} \frac{1 + \frac{k\left(1 - \frac{k}{a}\right)^{-1}}{t + x\gamma(t)}}{1 + \frac{k\left(1 - \frac{k}{a}\right)^{-1}}{t}} \neq e^{-x}$$

$\Rightarrow$ (I) is not satisfied

Frechet

$$w(F) = b \neq \infty$$

$\Rightarrow$ (II) is not satisfied

Weibull

$$w(F) = b < \infty$$

$$\lim_{t \downarrow 0} \frac{1 - F(b - tx)}{1 - F(b-t)}$$

$$= \lim_{t \downarrow 0} \frac{1 - k\left(\frac{1}{a} - \frac{1}{b-tx}\right)}{1 - k\left(\frac{1}{a} - \frac{1}{b-t}\right)}$$

$$= \lim_{t \downarrow 0} \frac{1 - \frac{k}{a} + \frac{k}{b-tx}}{1 - \frac{k}{a} + \frac{k}{b-t}}$$

$$= \lim_{t \downarrow 0} \frac{1 + \frac{k\left(1-\frac{k}{a}\right)^{-1}}{b - tx} \to 0}{1 + \frac{k\left(1-\frac{k}{a}\right)^{-1}}{b - t} \to 0} \neq x^{\alpha}$$

$\Rightarrow$ (III) is not satisfied

ETT does not hold for this F.

Q: Is there a quick way to say that ETT will not hold for a given F?

Answer: If F is the CDF of a discrete RV then ETT will not hold if

$$\lim_{k \to w(F)} \frac{Pr(X = k)}{1 - F(k-1)} \neq 0$$

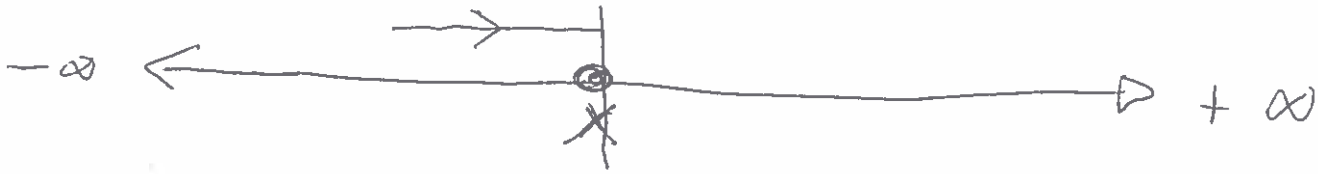
equivalently

$$\lim_{k \to w(F)} \frac{Pr(X = k)}{\sum_{j=k}^{\infty} P(X=j)} \neq 0$$

If F is the CDF of a continuous RV then ETT will not hold if

$$\lim_{x \to w(F)} \frac{f(x)}{1-F(x^-)} \neq 0$$

PDF CDF

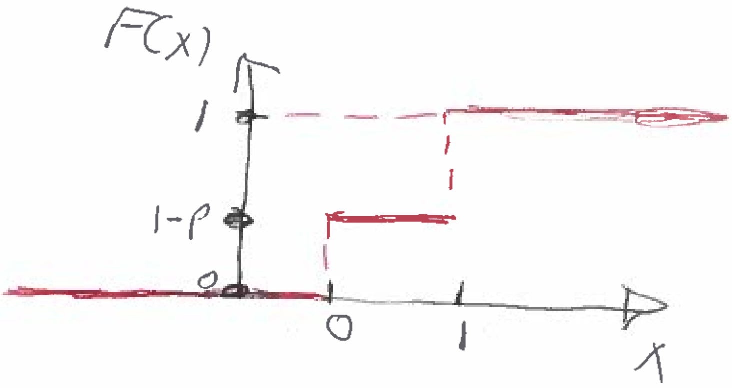


Ex1

$X \sim \underline{\text{Bernoulli}}\ (p)$

x	$P(X=x)$
1	p
0	$1-p$

x	$F(x)$
0	$1-p$
1	1



$\omega(F) = 1$

$$\lim_{k \to 1} \frac{P(X=k)}{1-F(k-1)}$$

$$= \frac{P(X=1)}{1-F(1-1)} = \frac{p}{1-(1-p)} = \frac{p}{p} = 1 \neq 0$$

$\Rightarrow$ ETT does not hold

Ex 2

$$\boxed{X \sim \text{Geom}(p)}$$

$$P(X=k) = p(1-p)^{k-1}, \quad k \geq 1$$

$$F(k) = 1-(1-p)^k$$

$$\omega(F) = +\infty \quad [\text{Solve } F(k) = 1]$$

$$\lim_{k \to \infty} \frac{p(1-p)^{k-1}}{\cancel{1} - [\cancel{1} - (1-p)^{k-1}]}$$

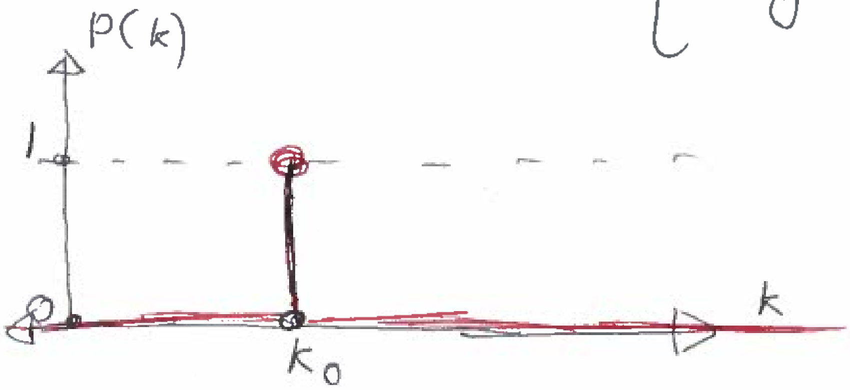
$$= \lim_{k \to \infty} \frac{p(1-p)^{k-1}}{(1-p)^{k-1}}$$

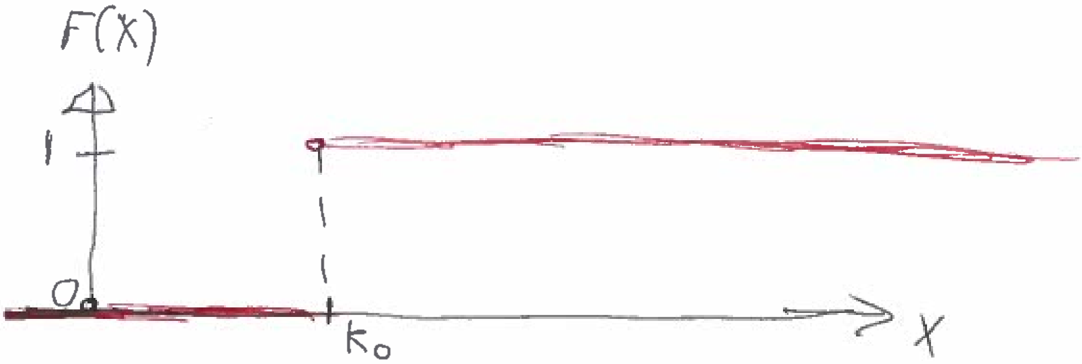
$$= p \neq 0$$

$\Rightarrow$ ETT does not hold.

Ex 3

$$p(k) = \begin{cases} 1 & \text{if} \quad k = k_0 \\ 0 & \text{if} \quad k \neq k_0 \end{cases}$$





$$\omega(F) = k_0$$

$$\lim_{k \to k_0} \frac{P(X = k)}{1 - F(k-1)} = \frac{P(X = k_0)}{1 - F(k_0 - 1)} = \frac{1}{1 - 0}$$
$$= 1 \neq 0$$

$\Rightarrow$ ETT does not hold

Ex 4

$$\boxed{X \sim \text{Binomial}(n, p)}$$

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

$$w(F) = n$$

$$\lim_{k \to n} \frac{P(X = k)}{1 - F(k-1)} = \frac{P(X = n)}{1 - F(n-1)}$$

$$= \frac{P(X = n)}{\boxed{1 - P(X \leq n-1)}} = \frac{P(X = n)}{P(X > n-1)}$$

$$= \frac{P(X = n)}{P(X = n)} = 1 \neq 0$$

$\Rightarrow$ ETT does not hold.

EXAMPLE CLASS

11 OCTOBER

10:00-11:00AM

MATH3/4/68181

Q1

$$L(\sigma) = \prod_{i=1}^{n} \left[\frac{1}{\sigma} e^{-\frac{x_i}{\sigma}} e^{-e^{-\frac{x_i}{\sigma}}} \right]$$

$$= \frac{1}{\sigma^n} e^{-\frac{1}{\sigma}\sum_{i=1}^{n} x_i} e^{-\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}}$$

$$\log L(\sigma) = -n\log\sigma - \frac{1}{\sigma}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}$$

$$\frac{d\log L}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{x_i}{\sigma^2} e^{-\frac{x_i}{\sigma}} = 0 \quad \text{— (1)}$$

$$(1) \times \sigma^2 \Rightarrow \boxed{-n\sigma = -\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}}} \quad \text{— (2)}$$

The MLE of σ is the root of (2).

$$\left. \frac{d^2 \log L}{d\sigma^2} \right|_{\sigma=\hat{\sigma}} < 0$$

Q2

$$L(\lambda, \sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{\lambda} x_i^{-\lambda-1} e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}} \right]$$

$$= \lambda^{n} \sigma^{n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{-\lambda-1} e^{-\sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda}}$$

$$\log L = n \log \lambda + n\lambda \log \sigma - (\lambda+1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n \log \sigma - \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda} \log\left(\frac{\sigma}{x_i}\right) = 0 \quad \text{— (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} - \lambda \sum_{i=1}^{n} \frac{\sigma^{\lambda-1}}{x_i^{\lambda}} = 0 \quad \text{— (2)}$$

$$(2) \Rightarrow \frac{n}{\sigma^{\lambda}} = \sum_{i=1}^{n} x_i^{-\lambda} \Rightarrow \sigma = \left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \right]^{\frac{1}{\lambda}} \quad \text{— (3)}$$

Sub (3) into (1):

$\log \sigma$

σ^{λ}

σ^{λ}

$$\frac{n}{\lambda} + \frac{n}{\lambda} \log \left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \right] - \sum_{i=1}^{n} \log x_i - \frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \cdot \frac{1}{\lambda} \log \left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \right] \cdot \sum_{i=1}^{n} x_i^{-\lambda}$$
$$+ \frac{n}{\sum_{i=1}^{n} x_1^{-\lambda}} \sum_{i=1}^{n} x_i^{-\lambda} \log x_i = 0 \quad \text{— (4)}$$

(4) involves only λ.

The MLE of λ is the root of (4).

The MLE of σ follows from (3).

$$\sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda} \log\left(\frac{\sigma}{x_i}\right)$$

$$= \sum_{i=1}^{n} \frac{\sigma^{\lambda}}{x_i^{\lambda}} \log\left(\frac{\sigma}{x_i}\right)$$

$$= \sigma^{\lambda} \sum_{i=1}^{n} x_i^{-\lambda} \left(\log \sigma - \log x_i\right)$$

$$= \sigma^{\lambda} \sum_{i=1}^{n} x_i^{-\lambda} \log \sigma - \sigma^{\lambda} \sum_{i=1}^{n} x_i^{-\lambda} \log x_i$$

$$= \sigma^{\lambda} \log \sigma \sum_{i=1}^{n} x_i^{-\lambda} - \sigma^{\lambda} \sum_{i=1}^{n} x_i^{-\lambda} \log x_i$$

Q3

$$L(\lambda, \sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{-\lambda} x_i^{\lambda-1} e^{-\left(\frac{x_i}{\sigma}\right)^{\lambda}} \right]$$

$$= \lambda^{n} \sigma^{-n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{\lambda-1} e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}}$$

$$\log L = n \log \lambda - n\lambda \log \sigma + (\lambda - 1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma} \right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - n \log \sigma + \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda} \log\left(\frac{x_i}{\sigma} \right) = 0 \quad \text{— (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n\lambda}{\sigma} + \lambda \sum_{i=1}^{n} \frac{x_i^{\lambda}}{\sigma^{\lambda+1}} = 0 \quad \text{— (2)}$$

$$(2) \Rightarrow \quad \sigma = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right)^{\frac{1}{\lambda}} \quad \text{— (3)}$$

Sub (3) into (1):

$$\frac{n}{\lambda} - \frac{n}{\lambda} \log\left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right] + \sum_{i=1}^{n} \log x_i - \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right]^{-1} \sum_{i=1}^{n} x_i^{\lambda} \log x_i + \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right]^{-1} \cdot \frac{1}{\lambda} \log\left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right] \sum_{i=1}^{n} x_i^{\lambda} = 0 \quad \text{— (4)}$$

MLE of λ is the root of (4)

MLE of σ follows from (3).

Q4

$$L(\lambda) = \prod_{i=1}^{n} \left[(1-\lambda x_i)^{\frac{1}{\lambda}-1} \right]$$

$$= \left[\prod_{i=1}^{n} (1-\lambda x_i) \right]^{\frac{1}{\lambda}-1}$$

$$\log L(\lambda) = \left(\frac{1}{\lambda}-1\right) \sum_{i=1}^{n} \log(1-\lambda x_i)$$

$$\frac{d \log L}{d\lambda} = -\frac{1}{\lambda^2} \sum_{i=1}^{n} \log(1-\lambda x_i)$$

$$+ \left(\frac{1}{\lambda}-1\right) \sum_{i=1}^{n} \frac{(-x_i)}{1-\lambda x_i} = 0 \quad -(1)$$

$$\Rightarrow \sum_{i=1}^{n} \log(1-\lambda x_i) = -\lambda(1-\lambda) \sum_{i=1}^{n} \frac{x_i}{1-\lambda x_i} \quad -(2)$$

The MLE of λ is the root of (2)

$$\left. \frac{d^2 \log L}{d\lambda^2} \right|_{\lambda=\hat{\lambda}} < 0$$

LECTURE

14 OCTOBER

9:00-10:00AM

MATH3/4/68181

Portfolio Theory

"Portfolio" is a collection of assets.

Let X_1 = Loss on asset 1

X_2 = " " " 2

$\vdots$

X_k = " " " k

Variables of interest

i) Total loss

$= X_1 + X_2 + \dots + X_k = S$

ii) Maximum loss

$= \max(X_1, X_2, \dots, X_k) = U$

iii) Minimum loss

$= \min(X_1, X_2, \dots, X_k) = V$

What are the distributions of these variables?

Scenarios

1) $X_1, X_2, \dots, X_k$ are IID RVs & k is fixed

2) $X_1, X_2, \dots, X_k$ are independent but not identical RVs & k is fixed

3) $X_1, X_2, \dots, X_k$ are dependent RVs & k is fixed

4) $X_1, X_2, \dots, X_k$ are IID RVs & k is a RV

5) $X_1, X_2, \dots, X_k$ are independent but not identical RVs & k is a RV

6) $X_1, X_2, \dots, X_k$ are dependent RVs & k is a RV

Scenario 1

total Loss (S')

CDF of X_1

$$F_S(s) = \underbrace{\int \cdots \int}_{\substack{k-1 \\ \text{integrals}}} F_1(s - X_2 - \cdots - X_k) \cdot f_2(X_2) \cdots f_k(X_k) \, dX_k \cdots dX_2$$

PDF of X_k

k-1 integrals PDF of X_2

$$f_S(s) = \overbrace{\int \cdots \int}^{} f_1(s - X_2 - \cdots - X_k) \cdot f_2(X_2) \cdots f_k(X_k) \cdot dX_k \cdots dX_2$$

PDF of X_1 PDF of X_2 PDF of X_k

$$E(S') = E(X_1) + E(X_2) + \cdots + E(X_k)$$
$$= k \cdot E(X_1)$$

$$Var(S') = Var(X_1) + Var(X_2) + \cdots + Var(X_k)$$
$$= k \cdot Var(X_1)$$

$$E[S'^m] = E[(X_1 + \cdots + X_k)^m]$$

$$= \sum_{m_1 + \cdots + m_k = m} \frac{m!}{m_1! \, m_2! \cdots m_k!} E(X_1^{m_1}) E(X_2^{m_2}) \cdots E(X_k^{m_k})$$

Maximum loss (U)

$$F_U(u) = P(U \leq u)$$
$$= P(\max(X_1, \dots, X_k) \leqslant u)$$
$$= P(X_1 \leqslant u, \dots, X_k \leqslant u)$$

indep

$$= P(X_1 \leqslant u) \cdots P(X_k \leqslant u)$$

identical

$$= F_1(u) \cdots F_k(u)$$
$$= F_1^k(u)$$

$$f_U(u) = k\, F_1^{k-1}(u)\, f_1(u)$$

$$E(U^m) = k \int_{-\infty}^{+\infty} u^m F_1^{k-1}(u)\, f_1(u)\, du$$

Minimum loss (V)

$$F_V(v) = P(V \leq v)$$
$$= 1 - P(V > v)$$
$$= 1 - P(\min(X_1, \ldots, X_k) > v)$$
$$= 1 - P(X_1 > v, \ldots, X_k > v)$$

indep

$$= 1 - P(X_1 > v) \ldots P(X_k > v)$$
$$= 1 - [1 - P(X_1 \leq v)] \ldots [1 - P(X_k \leq v)]$$
$$= 1 - [1 - F_1(v)] \ldots [1 - F_k(v)]$$

identical

$$= 1 - [1 - F_1(v)]^k$$

$$f_V(v) = k[1 - F_1(v)]^{k-1} f_1(v)$$

$$E[V^m] = k \int_{-\infty}^{+\infty} v^m [1 - F_1(v)]^{k-1} f_1(v)\,dv$$

Ex

$$X_i \sim N(\mu, \sigma^2) \quad \text{IID}$$

$$i = 1, 2, \dots, k$$

$$S = X_1 + \dots + X_k \sim N(k\mu, k\sigma^2)$$

$$f_S(s) = \frac{1}{\sqrt{2\pi}\sqrt{k}\,\sigma} e^{-\frac{(s-k\mu)^2}{2k\sigma^2}}$$

$$F_S(s) = \Phi\left(\frac{s-k\mu}{\sqrt{k}\,\sigma}\right)$$

CDF of $N(0,1)$

$$E(S) = k\mu$$

$$Var(S) = k\sigma^2$$

$\boxed{\text{Scenario 2}}$

Total loss (S')

$$F_{S'}(s) = \underbrace{\int \cdots \int}_{\substack{(k-1)\\ \text{integrals}}} F_1(s - x_2 - \cdots - x_k)\, f_2(x_2) \cdots f_k(x_k)\, dx_k \cdots dx_2$$

$$f_{S'}(s) = \underbrace{\int \cdots \int}_{\substack{(k-1)\\ \text{integrals}}} f_1(s - x_2 - \cdots - x_k)\, f_2(x_2) \cdots f_k(x_k)\, dx_k \cdots dx_2$$

$$E(S') = E(X_1) + \cdots E(X_k)$$

$$Var(S') = Var(X_1) + \cdots + Var(X_k)$$

eg

$$X_i \sim N(\mu_i, \sigma_i^2)$$

$$i = 1, 2, \ldots, k$$

$$S' = X_1 + \cdots + X_k$$

$$\sim N\left(\sum_{i=1}^{k} \mu_i, \sum_{i=1}^{k} \sigma_i^2\right)$$

$$f_{S'}(s) = \frac{1}{\sqrt{2\pi}\sqrt{\sum_{i=1}^{k} \sigma_i^2}} e^{-\frac{\left(s - \sum_{i=1}^{k} \mu_i\right)^{2}}{2\sum_{i=1}^{k} \sigma_i^2}}$$

$$F_S(s) = \Phi\left(\frac{s - \sum_{i=1}^{k} \mu_i}{\sqrt{\sum_{i=1}^{k} \sigma_i^2}}\right)$$

$$E(S') = \sum_{i=1}^{k} \mu_i$$

$$Var(S') = \sum_{i=1}^{k} \sigma_i^2$$

Maximum loss (U)

$$F_U(u) = P(U \leq u)$$
$$= P(\max(X_1, \ldots, X_k) \leq u)$$
$$= P(X_1 \leq u, \ldots, X_k \leq u)$$

indep

$$= P(X_1 \leq u) \cdots P(X_k \leq u)$$
$$= F_1(u) \cdots F_k(u)$$

$$f_U(u) = \sum_{m=1}^{k} f_m(u) \prod_{\substack{j=1 \\ j \neq m}}^{k} F_j(u)$$

$$E(U^m) = \sum_{m=1}^{k} \int_{-\infty}^{+\infty} u^m f_m(u) \prod_{\substack{j=1 \\ j \neq m}}^{k} F_j(u)\, du$$

Minimum loss (V)

$$F_V(v) = P(V \leq v)$$
$$= 1 - P(V > v)$$
$$= 1 - P(\min(X_1, \ldots, X_k) > v)$$
$$= 1 - P(X_1 > v, \ldots, X_k > v)$$

indep

$$= 1 - P(X_1 > v) \cdots P(X_k > v)$$
$$= 1 - [1 - P(X_1 \leq v)] \cdots [1 - P(X_k \leq v)]$$
$$= 1 - [1 - F_1(v)] \cdots [1 - F_k(v)]$$

$$f_V(v) = \sum_{m=1}^{k} f_m(v) \prod_{\substack{j=1 \\ j \neq m}}^{k} [1 - F_j(v)]$$

$$E(V^m) = \sum_{m=1}^{k} \int_{-\infty}^{+\infty} v^m f_m(v) \prod_{\substack{j=1 \\ j \neq m}}^{k} [1 - F_j(v)] \, dv$$

Scenario 3

Total loss (S)

$$F_S(s) = P(X_1 + \dots + X_k \le s)$$

$$= \underbrace{\int\int\cdots\int}_{\substack{k \text{ integrals} \\ X_1+X_2+\dots+X_k \le s}} f_{X_1, X_2, \dots, X_k}(x_1, x_2, \dots, x_k)\, dx_k \dots dx_2\, dx_1$$

Joint PDF of $(X_1, X_2, \dots, X_k)$

$$f_S(s) = \overbrace{\underbrace{\int\int\cdots\int}_{X_1+X_2+\dots+X_k = s}}^{\cancel{k \text{ integrals}}} f_{X_1, X_2, \dots, X_k}(x_1, x_2, \dots, x_k)\, dx_k \dots dx_2\, dx_1$$

$$E(S) = E(X_1) + E(X_2) + \dots + E(X_k)$$

$$Var(S) \neq Var(X_1) + Var(X_2) + \dots + Var(X_k)$$

Maximum loss (U)

$$F_U(u) = P(U \leq u)$$

$$= P(\max(X_1, \dots, X_k) \leq u)$$

$$= P(X_1 \leq u, \dots, X_k \leq u)$$

$$= \boxed{F_{X_1, X_2, \dots, X_k}}(u, u, \dots, u)$$

Joint CDF of $(X_1, X_2, \dots, X_k)$

$$f_U(u) = \frac{d F_U(u)}{du}$$

$$E(U^m) = \int_{-\infty}^{\infty} u^m f_U(u)\, du$$

EXAMPLE CLASS

17 OCTOBER

12:00-13:00PM

MATH3/4/68181

Q 4

$$p(k) = \frac{k^{-s}}{\zeta(s)}, \quad k \geq 1$$

$$\omega(F) = +\infty$$

$$\lim_{k \to \infty} \frac{P(X = k)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{p(k)}{\sum_{j=k}^{\infty} p(j)}$$

$$= \lim_{k \to \infty} \frac{\cancel{\frac{k^{-s}}{\zeta(s)}}}{\sum_{j=k}^{\infty} \cancel{\frac{j^{-s}}{\zeta(s)}}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{\sum_{j=k}^{\infty} j^{-s}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{\int_{k}^{\infty} x^{-s}\, dx}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{\left[\frac{x^{1-s}}{1-s}\right]_{k}^{\infty}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{0 - \frac{k^{1-s}}{1-s}} \quad \text{if } 1-s < 0$$

$$= \lim_{k \to \infty} \frac{s-1}{k}$$

$$= 0$$

$\Rightarrow$ ETT does hold.

<u>Homework</u>: which of (I) - (III) is satisfied?

$$\frac{d \log z}{dz} = \frac{1}{z}$$

$$\frac{d \log_2 z}{dz} = \frac{1}{(\log 2)\, z}$$

Q5

$$P(k) = -\log_2\left[1-(k+1)^{-2}\right], \quad k \geq 1$$

$$F(k) = 1 - \log_2\left[\frac{k+2}{k+1}\right], \quad k \geq 1$$

$$w(F) = +\infty$$

$$\lim_{k\to\infty} \frac{P(X=k)}{1-F(k-1)} = \lim \frac{-\log_2\left[1-(k+1)^{-2}\right]}{\cancel{1}-\left\{\cancel{1}-\log_2\left[\frac{k+1}{k}\right]\right\}}$$

$$= \lim_{k\to\infty} - \frac{\log_2\left[1-\frac{1}{(k+1)^2}\right]}{\log_2\left[\frac{k+1}{k}\right]}$$

$$= \lim_{k\to\infty} - \frac{\log_2\left[\frac{k^2+2k+\cancel{1}-\cancel{1}}{(k+1)^2}\right]}{\log_2\left[\frac{k+1}{k}\right]}$$

$$= \lim_{k\to\infty} - \frac{\log_2(k^2+2k) - 2\log(k+1)}{\log_2(k+1) - \log_2 k}$$

LH Rule

$$= \lim_{k\to\infty} - \frac{\frac{2k+2}{\cancel{(\log 2)}\, k^2+2k} - \frac{2}{\cancel{(\log 2)}\, k+1}}{\frac{1}{\cancel{(\log 2)}\, k+1} - \frac{1}{\cancel{(\log 2)}\, k}}$$

$$= \lim_{k\to\infty} - \frac{\frac{2(k+1)}{k(k+2)} - \frac{2}{k+1}}{-\frac{1}{k(k+1)}}$$

$$= \lim_{k \to \infty} \left[\frac{2(k+1)^2}{k+2} - 2k \right]$$

$$= \lim_{k \to \infty} 2 \left[\frac{k^2+2k+1 - k^2-2k}{k+2} \right]$$

$$= \lim_{k \to \infty} \frac{2}{k+2} = 0$$

$\Rightarrow$ ETT does hold.

<u>Homework</u>: Which of the conditions (I), (II) or (III) holds?

For any discrete RV,

$$\boxed{p(k) = P(X = k) = F(k) - F(k-1)}$$

Q7

$$F(x) = 1 - q^{(x+1)^a}, \quad x = 0, 1, \ldots$$

$$w(F) = +\infty \quad [\text{Solve } F(x) = 1]$$

$$\lim_{k \to \infty} \frac{p(k)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{F(k) - F(k-1)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{\cancel{1} - q^{(k+1)^a} - [\cancel{1} - q^{k^a}]}{\cancel{1} - [\cancel{1} - q^{k^a}]}$$

$$= \lim_{k \to \infty} \frac{q^{k^a} - q^{(k+1)^a}}{q^{k^a}}$$

$$= \lim_{k \to \infty} 1 - q^{(k+1)^a - k^a}$$

$$= \lim_{k \to \infty} 1 - q^{k^a(1+\frac{1}{k})^a - k^a}$$

$$= \lim_{k \to \infty} 1 - q^{k^a\left[(1+\frac{1}{k})^a - 1\right]}$$

$$= \lim_{k \to \infty} 1 - q^{k^a\left[\cancel{1} + \frac{a}{k} + \frac{a(a-1)}{2k^2} + \cdots - \cancel{1}\right]}$$

binomial expansion

$$= \lim_{k \to \infty} 1 - q^{k^a\left[\frac{a}{k} + \frac{a(a-1)}{2k^2} + \cdots\right]}$$

$$\approx \lim_{k \to \infty} 1 - q^{a k^{a-1}}$$

if $a = 1 \Rightarrow \lim = 1 - q$

if $a < 1 \Rightarrow \lim = 1 - 1 = 0$

if $a > 1 \Rightarrow \lim = 1 - 0 = 1$

ETT will hold only if $a<1$

In other cases ETT will not hold.

LECTURE

18 OCTOBER

9:00-10:00AM

MATH3/4/68181

Suppose (X, Y) is a random vector.

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$$

Joint CDF of (X, Y)

$$\bar{F}_{X,Y}(x, y) = P(X > x, Y > y)$$

Joint survivor function of (X, Y)

$$\begin{aligned} F_{X,Y}(x, y) = 1 &- \bar{F}_{X,Y}(-\infty, y) \\ &- \bar{F}_{X,Y}(x, -\infty) \\ &+ \bar{F}_{X,Y}(x, y) \end{aligned}$$

$$\begin{aligned} \bar{F}_{X,Y}(x, y) = 1 &- F_{X,Y}(x, \infty) \\ &- F_{X,Y}(\infty, y) \\ &+ F_{X,Y}(x, y) \end{aligned}$$

$$F_X(x) = P(X < x)$$ marginal CDF of X

$$= F_{X,Y}(x, \infty)$$

$$= 1 - \bar{F}_{X,Y}(x, -\infty)$$

$$F_Y(y) = P(Y < y)$$ marginal CDF of Y

$$= F_{X,Y}(\infty, y)$$

$$= 1 - \bar{F}_{X,Y}(-\infty, y)$$

Scenario 3

b) Maximum loss ($\bar{U}$)

$$F_U(u) = P(\max(X_1, \ldots, X_k) \leq u)$$

$$= P(X_1 \leq u, \ldots, X_k \leq u)$$

$$= F_{X_1, \ldots, X_k}(\underbrace{u, \ldots, u}_{k \text{ } u\text{'s}})$$

$$f_U(u) = \frac{dF_U(u)}{du}$$

$$E(U^m) = \int_{-\infty}^{+\infty} u^m f_U(u)\, du$$

c) <u>Minimum loss</u> (V)

$$F_V(v) = P(\min(X_1, \ldots, X_k) \leq v)$$

$$= 1 - P(\min(X_1, \ldots, X_k) > v)$$

$$= 1 - P(X_1 > v, \ldots, X_k > v)$$

$$= 1 - \bar{F}_{X_1, \ldots, X_k}\underbrace{(v, \ldots, v)}_{k \text{ } v\text{'s}}$$

$$f_V(v) = -\frac{d}{dv} \bar{F}_{X_1, \ldots, X_k}(v, \ldots, v)$$

$$E(V^m) = \int_{-\infty}^{\infty} v^m f_V(v)\, dv$$

Scenario 4

Total Loss (S')

$$F_{S'}(s) = P(X_1 + \dots + X_K \leqslant s)$$

Total Prob Rule $\rightarrow$

$$= \sum_{k=1}^{\infty} P(X_1 + \dots + X_K \leqslant s \mid K = k) \cdot P(K = k)$$

$$= \sum_{k=1}^{\infty} P(X_1 + \dots + X_k \leqslant s) \cdot P(K = k)$$

$$= \sum_{k=1}^{\infty} \left[\underbrace{\int \dots \int}_{k-1} F_1(s - x_2 - \dots - x_k) \cdot f_2(x_2) \dots f_k(x_k) \cdot dx_2 \dots dx_k \right] \cdot P(K = k)$$

$$f_{S'}(s) = \sum_{k=1}^{\infty} \left[\underbrace{\int \dots \int}_{k-1} f_1(s - x_2 - \dots - x_k) \cdot f_2(x_2) \dots f_k(x_k) \cdot dx_2 \dots dx_k \right] \cdot P(K = k)$$

$$E(S') = E(X_1 + \dots + X_K)$$

$$= \sum_{k=1}^{\infty} E(X_1 + \dots + X_K \mid K = k) \cdot P(K = k)$$

$$= \sum_{k=1}^{\infty} \left[E(X_1) + \dots + E(X_k) \right] P(K = k)$$

$$\Rightarrow \sum_{k=1}^{\infty} \left[k \cdot E(X) \right] \cdot P(K = k)$$

$$= E(X) \cdot \boxed{\sum_{k=1}^{\infty} k \cdot P(K = k)}$$

$$= E(X) \cdot E(K)$$

$$E(S'^2) = E\left[(X_1 + \dots + X_K)^2\right]$$

$$= \sum_{k=1}^{\infty} E\left[(X_1 + \dots + X_k)^2 \mid K = k\right] \cdot P(K=k)$$

$$= \sum_{k=1}^{\infty} E\left(\sum_{j=1}^{k} X_j^2 + \sum_{j \neq m} X_j X_m\right) \cdot P(K=k)$$

$$= \sum_{k=1}^{\infty} \left[\sum_{j=1}^{k} E(X_j^2) + \sum_{j \neq m} E(X_j) E(X_m)\right] \cdot P(K=k)$$

$$= \sum_{k=1}^{\infty} \left[k \cdot E(X^2) + (k^2 - k)(E(X))^2\right] \cdot P(K=k)$$

$$= \sum_{k=1}^{\infty} \left[k \cdot Var(X) + k^2 (E(X))^2\right] \cdot P(K=k)$$

$$= Var(X) \cdot \boxed{\sum_{k=1}^{\infty} k \cdot P(K=k)} + (E(X))^2 \cdot \boxed{\sum_{k=1}^{\infty} k^2 \cdot P(K=k)}$$

$$= Var(X) \cdot E(K) + (E(X))^2 \cdot E(K^2)$$

Prob Sheet 7

$X_1, \dots, X_\alpha$ IID Exp (λ).

$U = \max(X_1, \dots, X_\alpha)$

$$F_U(u) = P(\max(X_1, \dots, X_\alpha) \leq u)$$

$$= P(X_1 \leq u, \dots, X_\alpha \leq u)$$

indep

$$= P(X_1 \leq u) \dots P(X_\alpha \leq u)$$

$$= (1 - e^{-\lambda u}) \dots (1 - e^{-\lambda u})$$

$$= (1 - e^{-\lambda u})^\alpha$$

$$f_U(u) = \alpha \lambda e^{-\lambda u} (1 - e^{-\lambda u})^{\alpha - 1}$$

Beta Function

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} \, dt$$

$$E(u^n) = \int_0^\infty u^n \cdot \alpha\lambda e^{-\lambda u}\left(1-e^{-\lambda u}\right)^{\alpha-1} du$$

$$= \alpha\lambda \int_0^\infty u^n e^{-\lambda u}\left(1-e^{-\lambda u}\right)^{\alpha-1} du$$

$$\boxed{\begin{aligned} \text{Set} \quad y &= e^{-\lambda u} \\ u &= -\frac{1}{\lambda}\log y \\ \frac{du}{dy} &= -\frac{1}{\lambda y} \end{aligned}}$$

$$= \alpha\cancel{\lambda}\int_1^0 \left(-\frac{1}{\lambda}\log y\right)^n \cdot \cancel{y} \cdot (1-y)^{\alpha-1} \frac{dy}{(-\cancel{\lambda}\cancel{y})}$$

$$= \alpha \int_0^1 \left(-\frac{1}{\lambda}\right)^n (\log y)^n (1-y)^{\alpha-1} dy$$

$$= \alpha \int_0^1 \left(-\frac{1}{\lambda}\right)^n \left(\frac{d^n}{da^n} y^a \Big|_{a=0}\right)(1-y)^{\alpha-1} dy$$

$$= \frac{\alpha}{(-\lambda)^n}\frac{d^n}{da^n}\left[\int_0^1 y^a (1-y)^{\alpha-1} dy\right]\Bigg|_{a=0}$$

$$= \frac{\alpha}{(-\lambda)^n}\frac{d^n}{da^n} B(a+1, \alpha)\Big|_{a=0}$$

EXAMPLE CLASS

18 OCTOBER

10:00-11:00AM

MATH3/4/68181

Q4

$$p(k) = \frac{k^{-s}}{\zeta(s)}, \quad k \geq 1$$

$$\omega(F) = +\infty$$

$$\lim_{k \to \infty} \frac{p(k)}{1 - F(k-1)} = \lim_{k \to \infty} \frac{p(k)}{\sum_{j=k}^{\infty} p(j)}$$

$$= \lim_{k \to \infty} \frac{\frac{k^{-s}}{\cancel{\zeta(s)}}}{\sum_{j=k}^{\infty} \frac{j^{-s}}{\cancel{\zeta(s)}}} = \lim_{k \to \infty} \frac{k^{-s}}{\sum_{j=k}^{\infty} j^{-s}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{\int_k^{\infty} x^{-s}\,dx} = \lim_{k \to \infty} \frac{k^{-s}}{\left[\frac{x^{1-s}}{1-s}\right]_k^{\infty}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{0 - \frac{k^{1-s}}{1-s}} \qquad \text{if } s > 1$$

$$= \lim_{k \to \infty} \frac{s-1}{k} = 0$$

$\Rightarrow$ ETT must hold

Homework : which of (I), (II) or (III) is satisfied?

$$\frac{d \log z}{d z} = \frac{1}{z}$$

$$\frac{d \log_2 z}{d z} = \frac{1}{(\log 2) \cdot z}$$

$$
\begin{aligned}
1 - F(k-1) &= 1 - P(X \leq k-1) \\
&= P(X > k-1) \\
&= P(X \geq k) \\
&= \sum_{j=k}^{\infty} P(X = j) \\
&= \sum_{j=k}^{\infty} p(j)
\end{aligned}
$$

Q5

$$p(k) = -\log_2\left[1 - (k+1)^{-2}\right]$$

$$w(F) = +\infty$$

$$\boxed{\begin{aligned} F(k) &= \cancel{1} - \log_2\left[\frac{k+2}{k+1}\right] = \cancel{1} \\ \Rightarrow\ \log_2\left[\frac{k+2}{k+1}\right] &= 0 \\ \Rightarrow\ \frac{k+2}{k+1} &= 1 \\ \Rightarrow\ \frac{1+\frac{2}{k}}{1+\frac{1}{k}} &= 1 \\ \Rightarrow\ 1 + \frac{2}{k} &= 1 + \frac{1}{k} \\ \Rightarrow\ \frac{1}{k} &= 0 \\ \Rightarrow\ k &= +\infty \end{aligned}}$$

$$\lim_{k \to \infty} \frac{p(k)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{-\log_2\left[1 - (k+1)^{-2}\right]}{\cancel{1} - \left[\cancel{1} - \log_2\left[\frac{k+1}{k}\right]\right]}$$

$$= \lim_{k \to \infty} - \frac{\log_2\left[1 - (k+1)^{-2}\right]}{\log_2\left[\frac{k+1}{k}\right]}$$

$$= \lim_{k \to \infty} - \frac{\log_2\left[\frac{k^2+2k+\cancel{1}-\cancel{1}}{(k+1)^2}\right]}{\log_2(k+1) - \log_2 k}$$

$$= \lim_{k \to \infty} - \frac{\log_2(k^2+2k) - 2\log_2(k+1)}{\log_2(k+1) - \log_2 k}$$

LH Rule

$$= \lim_{k \to \infty} - \frac{\frac{2k+2}{\cancel{(\log 2)} \cdot (k^2+2k)} - \frac{2}{\cancel{(\log 2)}(k+1)}}{\frac{1}{\cancel{(\log 2)}(k+1)} - \frac{1}{\cancel{(\log 2)} \cdot k}}$$

$$= \lim_{k \to \infty} - \frac{\frac{2(k+1)}{k(k+2)} - \frac{2}{k+1}}{\frac{1}{k(k+1)}}$$

$$= \lim_{k \to \infty} - \left[\frac{2(k+1)^2}{k+2} - 2k\right]$$

$$= \lim_{k \to \infty} -2\,\frac{(k+1)^2 - k(k+2)}{k+2} = 0$$

$\Rightarrow$ ETT must hold.

<u>Home work</u>: which of (I), (II) or (III) holds?

For any discrete RV on integers,

$$\boxed{\begin{aligned} p(k) &= P(X=k) \\ &= F(k) - F(k-1) \end{aligned}}.$$

Q7

$$F(x) = 1 - q^{(x+1)^a}$$

$$F(x) = 1 \Rightarrow 1 - q^{(x+1)^a} = 1$$

$$\Rightarrow q^{(x+1)^a} = 0$$

$$\Rightarrow (x+1)^a = +\infty$$

$$\Rightarrow x = +\infty$$

$$\Rightarrow \omega(F) = +\infty$$

$$\lim_{k \to \infty} \frac{p(k)}{1 - F(k-1)} = \lim_{k \to \infty} \frac{F(k) - F(k-1)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{\cancel{1} - q^{(k+1)^a} - [\cancel{1} - q^{k^a}]}{\cancel{1} - [\cancel{1} - q^{k^a}]}$$

$$= \lim_{k \to \infty} \frac{q^{k^a} - q^{(k+1)^a}}{q^{k^a}}$$

$$= \lim_{k \to \infty} 1 - q^{(k+1)^a - k^a}$$

$$= \lim_{k \to \infty} 1 - q^{k^a \left(1 + \frac{1}{k}\right)^a - k^a}$$

$$= \lim_{k \to \infty} 1 - q^{k^a \left[\underbrace{\left(1 + \frac{1}{k}\right)^a}_{\text{Binomial Exp}} - 1\right]}$$

$$= \lim_{k \to \infty} 1 - q^{k^a\left[\cancel{1} + \frac{a}{k} + \frac{a(a-1)}{2k^2} + \cdots - \cancel{1}\right]}$$

$$\underbrace{\qquad\qquad\qquad}_{\text{Binomial Exp}}$$

$$= \lim_{k \to \infty} 1 - q^{k^a\left[\frac{a}{k} + \frac{a(a-1)}{2k^2} + \cdots\right]}$$

$$\approx \lim_{k \to \infty} 1 - q^{a k^{a-1}}$$

$\boxed{a = 1}$: $\lim = 1 - q$

$\boxed{a < 1}$: $\lim = 0$

$\boxed{a > 1}$: $\lim = 1 - 0 = 1$

ETT will not hold if $a = 1$ or $a > 1$

It will hold if $a < 1$

LECTURE

21 OCTOBER

9:00-10:00AM

MATH3/4/68181

Scenarios 1-4 for Math 38181

" 1-6 " Math 46181

Scenario 4

b) Maximum Loss (U)

$$F_U(u) = P[\max(X_1, \ldots, X_K) \leq u]$$

Total Prob Rule

$$\overset{\text{Total Prob Rule}}{=} \sum_{k=1}^{\infty} P[\max(X_1, \ldots, X_K) \leq u \mid K=k] P(K=k)$$

$$= \sum_{k=1}^{\infty} P[\max(X_1, \ldots, X_k) \leq u] \; P(K=k)$$

$$= \sum_{k=1}^{\infty} P[X_1 \leq u, \ldots, X_k \leq u] \; P(K=k)$$

indep

$$\overset{\text{indep}}{=} \sum_{k=1}^{\infty} P(X_1 \leq u) \cdots P(X_k \leq u) \cdot P(K=k)$$

identical

$$\overset{\text{identical}}{=} \sum_{k=1}^{\infty} F^k(u) \cdot P(K=k)$$

$$f_U(u) = \sum_{k=1}^{\infty} k\, F^{k-1}(u)\, f(u)\, P(K=k).$$

$$E(U^n) = \int_{-\infty}^{+\infty} u^n f_U(u)\, du$$

c) Minimum loss

$$F_V(v) = P[\min(X_1, \dots, X_K) < v]$$

$$= 1 - P[\min(X_1, \dots, X_K) > v]$$

$$= 1 - \sum_{k=1}^{\infty} P[\min(X_1, \dots, X_K) > v \mid K = k] \cdot P[K = k]$$

Total Prob Rule

$$= 1 - \sum_{k=1}^{\infty} P(X_1 > v, \dots, X_k > v) \cdot P[K = k]$$

indep

$$= 1 - \sum_{k=1}^{\infty} P(X_1 > v) \dots P(X_k > v) \cdot P(K = k)$$

$$= 1 - \sum_{k=1}^{\infty} (1 - P(X_1 \leq v)) \dots (1 - P(X_k \leq v)) P(K = k)$$

identical

$$= 1 - \sum_{k=1}^{\infty} [1 - F(v)]^k P(K = k)$$

$$f_V(v) = \sum_{k=0}^{\infty} k [1 - F(v)]^{k-1} f(v) P(K = k)$$

$$E(V^n) = \int_{-\infty}^{\infty} v^n f_V(v)\, dv$$

Financial Risk Measures

(hot topic!)

What is a financial risk measure?
It gives probabilities associated with a give loss.

Ex.

$$P(\text{Loss} > £1 \text{ million}) > 0.9$$

$\Rightarrow$ do not invest

$$P(\text{Loss} < £1000) < 10^{-20}$$

$\Rightarrow$ ok to invest

Math defn of a risk measure:

ρ : (class of RVs) $\longrightarrow$ $(0, \infty)$ is a risk measure if it satisfies

i) $\rho(0) = 0$ "normalised property"

ii) $\rho(X+c) = \rho(X) + c$ "translative property"

iii) $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$ "monotone property"

where c = const & X, Y are RVs representing loss.

Two most popular risk measures

Let X = loss with CDF F

1) Value at Risk (VaR) is defined by

$$\mathrm{VaR}_p(X) = \inf\{u : F(u) \geq p\}$$

due to J. P. Morgan in 1980s.

$\mathrm{VaR}_p(X)$ = " amount of loss exceeded with prob p "

2) Expected Shortfall (ES) is defined by

$$ES_p(X) = \frac{1}{p}\Big[E\left(X I\{X \leq \mathrm{VaR}_p(X)\}\right) + p\,\mathrm{VaR}_p(X) - \mathrm{VaR}_p(X) P\left(X \leq \mathrm{VaR}_p(X)\right)\Big]$$

$$I\{A\} = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false} \end{cases}$$

$ES_p(X)$ = " average loss given it has exceeded $\mathrm{VaR}_p(X)$ ".

Coherent risk measure is a (good) risk measure that satisfies (i)-(iii) and

(iv) $\rho(cX) = c\,\rho(X)$ "positive homogeneity"

(v) $\rho(X+Y) \leq \rho(X) + \rho(Y)$ "sub-additive"

where c = const & X, Y are RVs representing loss.

VaR & ES satisfy (i) - (iii)

$\Rightarrow$ they are risk measures.

VaR does not satisfy (v)

$\Rightarrow$ VaR is not a coherent risk measure

ES does satisfy (i) - (v)

$\Rightarrow$ ES is a coherent risk measure.

INVESTMENT & PENSIONS EUROPE

Is VaR to blame for the downturn?

IP Asia May 2009 **By Richard Newell**

Author and derivatives specialist Nassim Nicholas Taleb was recently quoted in a *New York Times* article entitled "Risk Mis-management". He made some valid points with regard to the usefulness of risk metrics at times of extreme market behaviour. But while VaR certainly has its laundry list of problems, Taleb takes VaR out of context by focusing on only one version of it; the Gaussian based parametric VaR, which he rightly points out is severely constrained by the dangerous assumption that asset returns follow a normal bell-shaped distribution.

In fact, he even goes so far as to state that VaR was highly responsible for the current financial crises. This is rather disturbing, as his claims seem to have gained a wider currency, thus detracting from the infinitely more important issues behind the crisis. If we look back in history, we can see quite clearly that most "blow-ups" were not due to poor allocation decisions based on an over-reliance on risk measurement and optimisation models, but were about leverage, unchecked greed, operational disaster and outright fraud.

While VaR is a requirement for a bank, most traders and fund managers would laugh if you asked them if they took VaR seriously. The reality, alarmingly, is that risk managers have hardly any clout when it comes to strong-arming a trader or liquidity. Risk manager warnings are often ignored or overridden as senior management tends to focus purely on profitability, not risk. This is not a risk model problem, but a corporate governance problem. Instead of bashing risk managers, we should be giving them more independence, capabilities and authority to identify and limit excessive risk taking.

Long Term Capital Management was leveraged 100 times at one point and Bear Stearns' credit hedge funds over 40 times. A simple cap on gross exposure would have helped to avoid the problems they encountered with leverage. Of course, this would have interfered with a strategy that depended heavily on leverage to 'boost' minuscule returns. Back in the 1990s, Nick Leeson at Barings, the Orange County debacle, events in Mexico and Korea - all of these events had excessive leverage in common. The problems that lie within VaR are its inability to fully capture leverage and liquidity risk. Good risk managers are fully aware of this shortcoming and, as a result, VaR is only one in a whole repertoire of tools, both quantitative and qualitative, that risk managers use to get a sense of the risks they are taking on.

Taleb gives the impression that risk managers are only managing risk according to Gaussian principles, where probabilities are assumed to be normally distributed. There is more to the story than he lets on. Interestingly enough, Taleb seems to be a big fan of Monte Carlo simulations (a method that does not need to assume normality in asset return distributions) as seen in his use of Monte Carlo in the book 'Fooled by Randomness'. Taleb suggests Monte Carlo simulators allow us to learn from the simulated future which is superior to learning from the past, because the past has a survivorship bias, and we also tend to denigrate the past by claiming misfortune had by others will not happen to us. Most sophisticated risk managers use Monte Carlo very much in the same way he does.

VaR: The number that killed us

By Pablo Triana

December 1, 2010 • Reprints

FROM THE ARCHIVES

On Sept. 10, 2009 former trader and bestselling author Nassim Taleb did something that he very seldom does: he wore a tie. Taleb has oftentimes publicly expressed his distaste for the blood-constraining artifacts, as well as for those who tend to don them, so the Lebanese-American let the world know that was a very special day for him by betraying a sacred personal disposition.

So what prompted the composer of "The Black Swan" to button his shirt all the way up on that fall date? He had been invited to a very solemn venue by very distinguished hosts. And that was an invitation that Taleb had every intention of accepting. In fact, he had been waiting and expecting it for more than a decade. The raison d'etre of the event for which his company was now being required had been close to Taleb's heart for most of his professional and intellectual life. It represented a central theme in his actions and ideas, close to an obsession. He had through the years incessantly warned as to the havoc that might be wreaked should others massively act in a manner counter to his convictions. Such concerns typically went unheeded (to the detriment, it turned out, of society), but now he was being offered a pulpit that seemed irresistible. This time, the world would have no option but to listen attentively.

As Taleb entered the Rayburn Building of the U.S. House of Representatives on Capitol Hill that September morning, he must have felt vindication. As he approached the sober room where several men and women awaited the start of the House Committee on Science and Technology's hearing on the responsibility of mathematical model Value at Risk (VaR) for the terrible economic and financial crisis that had caused so much misery, Taleb probably reflected proudly on all those times when, indefatigably and in the face of harsh opposition, he alerted us of the lethal threat to the system posed by the widespread use of VaR in finance. Now that the damage wrought by VaR seemed so inescapably obvious that lawmakers had been motivated into investigating the device, Taleb no longer seemed like a lone wolf howling at the moon.

What is so wrong about VaR, and why was Taleb so concerned about its impact? More importantly, why should VaR be held responsible for the crisis? VaR is a number that purports to estimate future losses derived from a portfolio of financial assets, and presents two major problems: 1) it is doomed to being a very wrong estimate, because of its analytical foundations and the realities of real-life markets; 2) in spite of such (well-known) deficiencies, it has for the past two decades become an ubiquitously influential force in the financial world, capable of directing decision-making inside the most important banks. In other words, by letting trading activity be guided by VaR, we have essentially exposed our economic fate to a deeply flawed mechanism. Such flawedness, as was the case not only in this crisis but also before, can yield untold malaise.

One dimension in a 3D world

VaR is an untrustworthy measure of future market risk for one main reason: it is calculated by looking at the past. The upcoming risk of a trading asset (a stock, bond or derivative) is essentially assumed to mirror its behavior over the historical time period arbitrarily selected for the calculation (one year, five years, etc). If such past happened to be placid (no big

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Euro	1.10000	+0.00065	(+0.06%)
Gold	1269.5	-0.4	(-0.03%)
Oil	51.12	-0.48	(-0.93%)
Gas	3.190	+0.020	(+0.63%)
Corn	3582	+06	(+0.21%)

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BULLETIN

The Role Value At Risk (VaR) Played in the 2008 Financial Crisis

ARTICL

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GAUR MEHR

@g_m

Gaurav Mehra is a recent gradua from the Univer

What is Value At Risk?

In the aftermath of the 2008 financial crisis, a myriad of factors leading to the calamity were extensively examined by various public and private entities. It became apparent that some factors had played more of a role than others. Some of these critical factors included the secured subprime mortgages from Fannie Mae and

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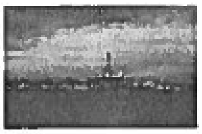

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VaR and its Role in the Credit Crisis

by Mark Kirkland, VP Treasury, Bombardier Transportation

The causes of the credit crisis of 2009 will be discussed by many for numerous years to come, although probably for fewer years than we now think. People have a unique ability to forget, perhaps black out, the worst episodes. I have sat down on a number of occasions and tried to think, what were the possible causes of the crisis? An inherent weakness in accounting of results, large numbers of over the counter derivatives with large fair values, weak governance by regulatory bodies or even that bankers were paid too much? In the end, I believe that none of the above was a key contributor to the crisis. In my mind there are two unrelated causes.

> It is now clear that very few shareholders of banks understood the risks thtat some banks were in fact taking.

The first is the mode of compensation in the financial industry. Not the amounts. Most bankers receive a kind of option pay out. If the firm makes a large profit (based on the mark to market of future uncertain cash flows), the employees receive large cash bonuses. If the firm makes a loss, in the worst case, staff may receive no bonus. Clearly, for a betting man, this gives carte blanche to load up the company with significant risk. Since most bonuses are not discussed with the owners of the company (the shareholders) but set by a compensation committee, often chaired by senior employees, there is a tendency to overpay since this justifies the compensation of the very people making the decisions. I will not dwell on this cause much longer – except to stress that the whole model encourages large risk taking.

The second is the point of this article. Risk was and still is, very badly understood, managed and reported. It is now clear that very few shareholders of banks understood the risks that some banks were in fact taking. In part, this is because disclosure of risk is unclear. A more fundamental issue, however, is that it appears that some of the banks did not fully comprehend the risk and actually outsourced much of their risk assessment to the rating agencies and then used flawed measures such as Value at Risk (VaR) not only to manage risk but also to report to management and shareholders alike.

Key Points

- The author distinguishes two chief causes of the financial crisis:
  - the financial industry's compensation structure, which encourages risk taking
  - reliance on flawed measures of risk
- The pros and cons of VaR
- A massive understatement of [illegible] which treated structured products as AAA/A[illegible] models which do not allow for correlations [illegible] positive to negative

A recipe for disaster?

Consider first the structured products themselves. Collateralised loan obligations (CLO), collateralised debt obligations (CDO) and even collateralised mortgage obligations (CMO) were all highly structured to maximise yield while ensuring that the most senior tranches would be rated AAA/ Aaa by the rating agencies. Bankers followed the formulae given by the rating agencies, which, coincidentally, were paid to help structure the products.

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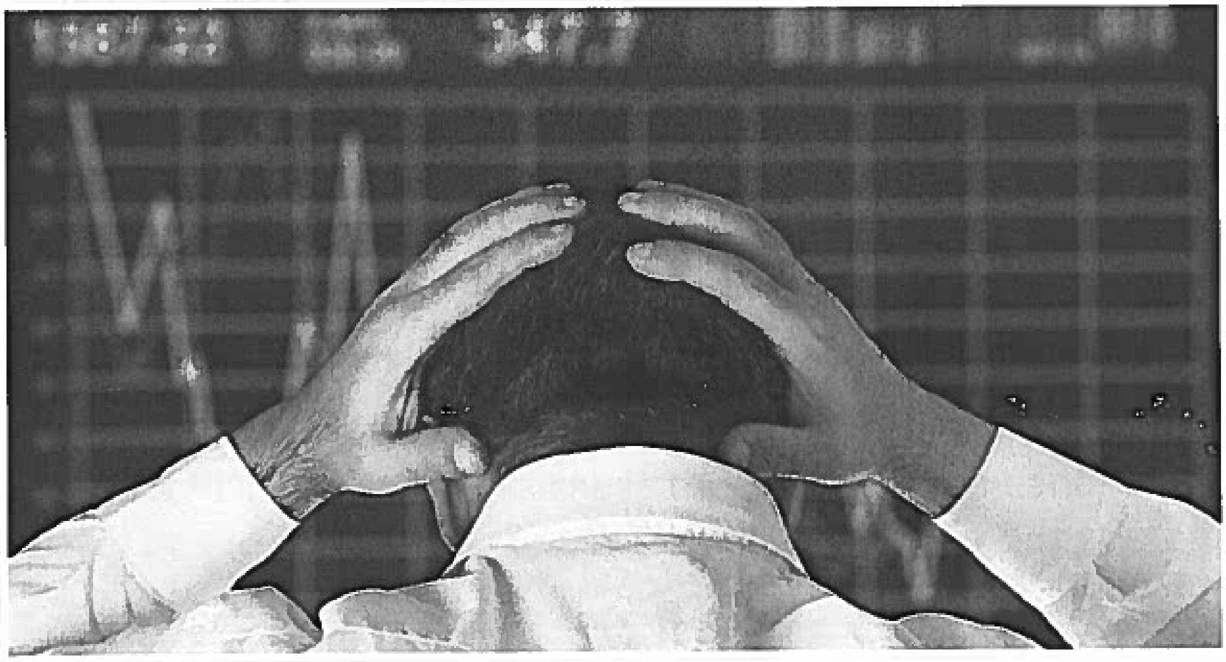
7

News - Did Value at Risk cause the crisis it was meant to avert?



News

Did Value at Risk cause the crisis it was meant to avert?

12 May 2016

What were the causes of the crisis of 2008? We show that managing risk using the procedure recommended by Basel II, which is called *Value at Risk*, may have played a central role. We make a very simple model for the banking system that captures the key elements of risk management under Value at Risk. Providing the banks' only take modest risks, the financial system remains stable. But if they take higher risks, or if the banking sector gets larger, the market begins to spontaneously oscillate, in a way that resembles the period leading up to and including the Global Financial Crisis. For about 10 - 15 years prices and leverage slowly rise while volatility slowly falls, then prices and leverage suddenly crash and volatility

Suppose X = loss is an absolutely continuous RV. In this case,

$$VaR_p(X) = F^{-1}(p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_p(u)\,du$$

eg $X \sim N(\mu, \sigma^2)$.

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

CDF of $N(0,1)$

$$F(x) = p$$

$$\Rightarrow \Phi\left(\frac{x-\mu}{\sigma}\right) = p$$

$$\Rightarrow \frac{x-\mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow x = \mu + \sigma\Phi^{-1}(p)$$

$$\Rightarrow \boxed{VaR_p(X) = \mu + \sigma\Phi^{-1}(p)}$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_p(u)\,du$$

$$= \frac{1}{p}\int_0^p \left[\mu + \sigma\Phi^{-1}(u)\right] du$$

$$\boxed{= \mu + \frac{\sigma}{p}\int_0^p \Phi^{-1}(u)\,du}$$

eg 2

$$F(x) = x^{\alpha}, \quad 0 < x < 1$$

$$F(x) = p$$

$$\Rightarrow \quad x^{\alpha} = p$$

$$\Rightarrow \quad x = p^{\frac{1}{\alpha}}$$

$$\Rightarrow \quad VaR_p(X) = p^{\frac{1}{\alpha}}$$

$$ES_p(X) = \frac{1}{p}\int_0^p u^{\frac{1}{\alpha}}\,du$$

$$= \frac{1}{p}\left[\frac{u^{\frac{1}{\alpha}+1}}{\frac{1}{\alpha}+1}\right]_0^p$$

$$= \frac{p^{\frac{1}{\alpha}}}{\frac{1}{\alpha}+1}.$$

Properties of VaR

i) $\mathrm{VaR}_p(X+c) = \mathrm{VaR}_p(X) + c$

"translative property"

ii) $\mathrm{VaR}_p(cX) = c \cdot \mathrm{VaR}_p(X)$

"positive homogenity"

iii) $\mathrm{VaR}_p(X) = -\mathrm{VaR}_{1-p}(-X)$

iv) $X \geq 0 \Rightarrow \mathrm{VaR}_p(X) \geq 0$

v) $X \geq Y \Rightarrow \mathrm{VaR}_p(X) \geq \mathrm{VaR}_p(Y)$.

"monotone property"

Home work: prove (i) - (v).

EXAMPLE CLASS

24 OCTOBER

12:00-13:00PM

MATH3/4/68181

Q1

$$F(x) = 1 - e^{-\lambda x}$$

$$F(x) = p$$

$$\Rightarrow \quad 1 - e^{-\lambda x} = p$$

$$\Rightarrow \quad e^{-\lambda x} = 1 - p$$

$$\Rightarrow \quad -\lambda x = \log(1-p)$$

$$\Rightarrow \quad x = -\frac{1}{\lambda} \log(1-p) = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p} \int_0^p VaR_u(X)\, du$$

$$= \frac{(-1)}{\lambda p} \int_0^p \log(1-u)\, du$$

by parts

$$= -\frac{1}{\lambda p} \left\{ \Big[u \cdot \log(1-u) \Big]_0^p + \int_0^p \frac{u}{1-u}\, du \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1-p) - 0 + \int_0^p \frac{u-1+1}{1-u}\, du \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1-p) + \int_0^p \left(-1 + \frac{1}{1-u} \right) du \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1-p) + \Big[-u - \log(1-u) \Big]_0^p \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1-p) - p - \log(1-p) - 0 \right\}$$

P3

$$F(x) = \frac{x-a}{b-a}$$

$$F(x) = p$$

$$\Rightarrow \frac{x-a}{b-a} = p \Rightarrow x = a + (b-a)\cdot p$$
$$= VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_p(u)\,du$$

$$= \frac{1}{p}\int_0^p \left[a + (b-a)\cdot u\right] du$$

$$= \frac{1}{p}\left[a\cdot u + \frac{(b-a)\,u^2}{2}\right]_0^p$$

$$= a + \frac{(b-a)\cdot p}{2}$$

Q4

$$F(x) = 1 - \left(\frac{K}{x}\right)^a = p$$

$$\Rightarrow \left(\frac{K}{x}\right)^a = 1 - p$$

$$\Rightarrow \frac{K}{\cancel{x}} = (1-p)^{\frac{1}{a}}$$

$$\Rightarrow X = K(1-p)^{-\frac{1}{a}} = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p K(1-u)^{-\frac{1}{a}}\,du$$

$$= \frac{K}{p}\left[\frac{(1-u)^{1-\frac{1}{a}}}{(-1)\left(1-\frac{1}{a}\right)}\right]_0^p$$

$$= \frac{Ka}{p(1-a)}\left[(1-u)^{1-\frac{1}{a}}\right]_0^p$$

$$= \frac{Ka}{p(1-a)}\left[(1-p)^{1-\frac{1}{a}} - 1\right]$$

Q6

$$F(x) = \left[1 + \left(\frac{x}{a}\right)^{-b}\right]^{-1} = p$$

$$\Rightarrow \quad 1 + \left(\frac{x}{a}\right)^{-b} = \frac{1}{p}$$

$$\Rightarrow \quad \left(\frac{x}{a}\right)^{-b} = \frac{1}{p} - 1 = \frac{1-p}{p}$$

$$\Rightarrow \quad \frac{x}{a} = \left(\frac{1-p}{p}\right)^{-\frac{1}{b}}$$

$$\Rightarrow \quad x = a\left(\frac{1-p}{p}\right)^{-\frac{1}{b}} = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p a\left(\frac{1-u}{u}\right)^{-\frac{1}{b}} du$$

$$= \frac{a}{p}\int_0^p u^{\frac{1}{b}}(1-u)^{-\frac{1}{b}} du$$

$$B_X(a,b) = \int_0^X t^{a-1}(1-t)^{b-1}\,dt$$

Incomplete Beta Function

$$= \frac{a}{p} B_p\left(\frac{1}{b}+1,\ 1-\frac{1}{b}\right)$$

Q7

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = p$$

$$\Rightarrow \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = 1 - p$$

$$\Rightarrow 1 + \frac{x}{\lambda} = (1-p)^{-\frac{1}{\alpha}}$$

$$\Rightarrow x = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right] = VaR_p(x)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \lambda \cdot \left[(1-u)^{-\frac{1}{\alpha}} - 1\right] du$$

$$= \frac{\lambda}{p}\left[\frac{(1-u)^{1-\frac{1}{\alpha}}}{(-1)\left(1-\frac{1}{\alpha}\right)} - u\right]_0^p$$

$$= \frac{\lambda}{p}\left[\frac{(1-p)^{1-\frac{1}{\alpha}}}{\frac{1}{\alpha}-1} - p - \frac{1}{\frac{1}{\alpha}-1} + 0\right]$$

$$= \frac{\lambda\alpha\left[(1-p)^{1-\frac{1}{\alpha}} - 1\right]}{(1-\alpha)\,p} - \lambda$$

Q8

$$F(x) = e^{-\left(\frac{\sigma}{x}\right)^{\alpha}} = p$$

$$\Rightarrow \left(\frac{\sigma}{x}\right)^{\alpha} = -\log p$$

$$\Rightarrow \frac{\sigma}{x} = (-\log p)^{\frac{1}{\alpha}}$$

$$\Rightarrow x = \sigma(-\log p)^{-\frac{1}{\alpha}} = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \sigma \cdot (-\log u)^{-\frac{1}{\alpha}}\, du$$

$$= \frac{\sigma}{p}\int_0^p (-\log u)^{-\frac{1}{\alpha}}\, du$$

$$\boxed{\begin{aligned} Y = -\log u \Rightarrow\ & u = e^{-Y} \\ \Rightarrow\ & \frac{du}{dY} = -e^{-Y} \end{aligned}}$$

$$= \frac{\sigma}{p}\int_{+\infty}^{-\log p} Y^{-\frac{1}{\alpha}}\left(-e^{-Y}\right) dY$$

$$= \frac{\sigma}{p}\int_{-\log p}^{+\infty} Y^{-\frac{1}{\alpha}} e^{-Y}\, dY$$

$$\boxed{\Gamma(a,x) = \int_x^{+\infty} Y^{a-1} e^{-Y}\, dY}$$

Complementary Incomplete gamma function

$$= \frac{\sigma}{p}\,\Gamma\left(-\log p,\ 1-\frac{1}{\alpha}\right)$$

LECTURE

25 OCTOBER

9:00-10:00AM

MATH3/4/68181

$\boxed{\text{Proof of (i)}}$ Assume X is abs. cont. RV

$$\mathrm{VaR}_p(X+c) = \mathrm{VaR}_p(X) + c$$

$$\Leftrightarrow \quad F_{X+c}^{-1}(p) = F_X^{-1}(p) + c$$

$$\Leftrightarrow \quad F_{X+c}^{-1}(p) - c = F_X^{-1}(p)$$

$$\Leftrightarrow \quad F_X\left(F_{X+c}^{-1}(p) - c\right) = F_X\left(F_X^{-1}(p)\right)$$

$$\Leftrightarrow \quad F_X\left(F_{X+c}^{-1}(p) - c\right) = p$$

$$\Leftrightarrow \quad P\left(X \leq F_{X+c}^{-1}(p) - c\right) = p$$

$$\Leftrightarrow \quad P\left(X + c \leq F_{X+c}^{-1}(p)\right) = p$$

$$\Leftrightarrow \quad F_{X+c}\left(F_{X+c}^{-1}(p)\right) = p$$

$$\Leftrightarrow \quad p = p$$

Result is proved.

(iii)

$$\mathrm{VaR}_p(X) = -\mathrm{VaR}_{1-p}(-X)$$

$$\iff F_X^{-1}(p) = -F_{-X}^{-1}(1-p)$$

$$\iff F_X(F_X^{-1}(p)) = F_X(-F_{-X}^{-1}(1-p))$$

$$\iff p = F_X(-F_{-X}^{-1}(1-p))$$

$$\iff p = P(X \leq -F_{-X}^{-1}(1-p))$$

$$\iff p = P(-X \geq F_{-X}^{-1}(1-p))$$

$$\iff p = 1 - P(-X \leq F_{-X}^{-1}(1-p))$$

$$\iff p = 1 - F_{-X}(F_{-X}^{-1}(1-p))$$

$$= 1 - (1-p) = p$$

The result is proved.

Estimation methods for VaR

i) Parametric estimation methods

ii) Non-parametric " "

iii) Semi-parametric " "

Math 38181

Math 4/68181

Parametric Estimation Methods

- X = Loss

a) **Normal distn**

$X \sim N(\mu, \sigma^2)$

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

CDF of $N(0,1)$

$$VaR_p(X) = \mu + \sigma \Phi^{-1}(p)$$

Suppose $X_1, X_2, \ldots, X_n$ is a random sample on X. The MLEs of μ and σ are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2}$$

The MLE of $VaR_p(X)$ is

$$\widehat{VaR_p}(X) = \hat{\mu} + \hat{\sigma}\, \Phi^{-1}(p)$$

$$\boxed{\text{An estimator } \hat{\theta} \text{ is } \underline{\text{unbiased}} \text{ for } \theta \text{ if } E(\hat{\theta}) = \theta}$$

Is $\widehat{VaR}_p(X)$ unbiased for $VaR_p(X)$?

$$E[\widehat{VaR}_p(X)]$$

$$= E[\hat{\mu} + \hat{\sigma}\,\Phi^{-1}(p)]$$

$$= E(\hat{\mu}) + E(\hat{\sigma})\,\Phi^{-1}(p)$$

$$= E(\bar{X}) + E\left[\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}\right]\Phi^{-1}(p)$$

$$= \mu + E\left[\sigma\sqrt{\frac{\cancel{(n-1)}\,\chi^2_{n-1}}{n}}\right]\Phi^{-1}(p)$$

$$\boxed{\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \bar{X})^2 \sim \chi^2_{n-1}}$$

Math 20802

$$= \mu + \sigma\sqrt{\frac{\cancel{n-1}}{n}}\,E\left[\sqrt{\chi^2_{n-1}}\right]\cdot\Phi^{-1}(p)$$

Home work

$$\neq \mu + \sigma\,\Phi^{-1}(p) = VaR_p(X)$$

$\Longrightarrow$ $\widehat{VaR}_p(X)$ is biased

b) <u>Variance - Covariance method</u>

X_i = Loss for asset i,
$i = 1, 2, \ldots, k$

k = no of assets

T = Weighted Loss $= \sum_{i=1}^{k} w_i X_i$

weights

Suppose $X_i \sim N(\mu_i, \sigma_i^2)$ are indep RVs.

$$T \sim N\left(\sum_{i=1}^{k} w_i \mu_i, \sum_{i=1}^{k} w_i^2 \sigma_i^2\right)$$

$$VaR_p(T) = \sum_{i=1}^{k} w_i \mu_i + \sqrt{\sum_{i=1}^{k} w_i^2 \sigma_i^2}\, \Phi^{-1}(p)$$

Suppose $X_{i,1}, X_{i,2}, \ldots, X_{i,n}$ is a random sample on X_i. Let

$$\bar{X}_i = \frac{1}{n} \sum_{j=1}^{n} X_{i,j}$$

$$s_i = \sqrt{\frac{1}{n} \sum_{j=1}^{n} \left(X_{i,j} - \bar{X}_i \right)^2}$$

The MLEs of μ_i & σ_i are $\bar{X}_i$ & s_i respectively. So, the MLE of $VaR_p(T)$ is

$$\widehat{VaR_p(T)} = \sum_{i=1}^{n} w_i \bar{X}_i + \sqrt{\sum_{i=1}^{n} w_i^2 s_i^2} \cdot \Phi^{-1}(p)$$

Home work : Show that $\widehat{VaR}_p(T)$ is a biased estimator of $VaR_p(T)$.

c) <u>Weibull distribution</u>

X has the CDF $F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}}$,

$x > 0$

$$F(x) = p$$

$$\Rightarrow 1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}} = p$$

$$\Rightarrow e^{-\left(\frac{x}{\theta}\right)^{\beta}} = 1 - p$$

$$\Rightarrow \left(\frac{x}{\theta}\right)^{\beta} = -\log(1-p)$$

$$\Rightarrow \frac{x}{\theta} = \left[-\log(1-p)\right]^{\frac{1}{\beta}}$$

$$\Rightarrow VaR_p(X) = \theta\left[-\log(1-p)\right]^{\frac{1}{\beta}}$$

Suppose $X_1, X_2, \ldots, X_n$ is a random sample on X. The MLEs of θ and β are given by

$$\left(\frac{\bar{X}}{S}\right)^2 = \frac{\Gamma^2\left(1+\frac{1}{\beta}\right)}{\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)} \quad - (1)$$

and $$\hat{\theta} = \frac{\bar{X}}{\Gamma\left(1+\frac{1}{\hat{\beta}}\right)} \quad - (2)$$

$\hat{\beta}$ is the root of (1)

Sub into (2) to get $\hat{\theta}$.

So, the MLE of VaR is

$$\widehat{VaR}_p(X) = \hat{\theta}\left[-\log(1-p)\right]^{\frac{1}{\hat{\beta}}}.$$

EXAMPLE CLASS

25 OCTOBER

10:00-11:00AM

MATH3/4/68181

Q1

$$F(x) = 1 - e^{-\lambda x}$$

$$1 - e^{-\lambda x} = p$$

$$\Rightarrow \quad e^{-\lambda x} = 1 - p$$

$$\Rightarrow \quad -\lambda x = \log(1-p)$$

$$\Rightarrow \quad x = -\frac{1}{\lambda} \log(1-p)$$

$$\Rightarrow \quad VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_p(u)\,du$$

$$= \frac{1}{p}\int_0^p \left(-\frac{1}{\lambda}\log(1-u)\right) du$$

$$= -\frac{1}{\lambda p}\int_0^p \log(1-u)\,du$$

by parts

$$= -\frac{1}{\lambda p}\left\{\left[u \cdot \log(1-u)\right]_0^p + \int_0^p \frac{u}{1-u}\,du\right\}$$

$$= -\frac{1}{\lambda p}\left\{p \cdot \log(1-p) - 0 + \int_0^p \frac{u-1+1}{1-u}\,du\right\}$$

$$= -\frac{1}{\lambda p}\left\{p \cdot \log(1-p) + \int_0^p \left(-1 + \frac{1}{1-u}\right) du\right\}$$

$$= -\frac{1}{\lambda p}\left\{p \cdot \log(1-p) + \left[-u - \log(1-u)\right]_0^p\right\}$$

$$= -\frac{1}{\lambda p}\left\{p \cdot \log(1-p) - p - \log(1-p) - 0\right\}$$

Q3

$$F(x) = \frac{x-a}{b-a}$$

$$\frac{x-a}{b-a} = p$$

$$\Rightarrow \quad x = a + (b-a)\,p$$

$$\Rightarrow \quad VaR_p(X) = a + (b-a)\,p$$

$$ES_p(X) = \frac{1}{p}\int_0^p \left[a + (b-a)\,u\right] du$$

$$= \frac{1}{p}\cdot\left[a u + (b-a)\cdot\frac{u^2}{2}\right]_0^p$$

$$= a + (b-a)\cdot\frac{p}{2}$$

Q4

$$F(x) = 1 - \left(\frac{K}{x}\right)^{a}$$

$$1 - \left(\frac{K}{x}\right)^{a} = p$$

$$\Rightarrow \left(\frac{K}{x}\right)^{a} = 1-p$$

$$\Rightarrow \frac{K}{x} = (1-p)^{\frac{1}{a}}$$

$$\Rightarrow x = K(1-p)^{-\frac{1}{a}} = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p K \cdot (1-u)^{-\frac{1}{a}} du$$

$$= \frac{K}{p}\int_0^p (1-u)^{-\frac{1}{a}} du$$

$$= \frac{K}{p}\left[\frac{(1-u)^{1-\frac{1}{a}}}{(-1)\left(1-\frac{1}{a}\right)}\right]_0^p$$

$$= \frac{Ka\left[(1-p)^{1-\frac{1}{a}} - 1\right]}{p(1-a)}$$

Q6

$$F(x) = \left[1 + \left(\frac{x}{a}\right)^{-b}\right]^{-1} = p$$

$$\Rightarrow \quad 1 + \left(\frac{x}{a}\right)^{-b} = \frac{1}{p}$$

$$\Rightarrow \quad \left(\frac{x}{a}\right)^{-b} = \frac{1}{p} - 1 = \frac{1-p}{p}$$

$$\Rightarrow \quad \frac{x}{a} = \left(\frac{1-p}{p}\right)^{-\frac{1}{b}}$$

$$\Rightarrow \quad x = a\left(\frac{1-p}{p}\right)^{-\frac{1}{b}} = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p a \cdot \left(\frac{1-u}{u}\right)^{-\frac{1}{b}} du$$

$$= \frac{a}{p}\int_0^p u^{\frac{1}{b}} (1-u)^{-\frac{1}{b}} du$$

$$\boxed{B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\, dt \quad \text{Incomplete Beta Function}}$$

$$= \frac{a}{p} \cdot B_p\left(1+\frac{1}{b}, 1-\frac{1}{b}\right).$$

Q7

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = p$$

$$\Rightarrow \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = 1 - p$$

$$\Rightarrow 1 + \frac{x}{\lambda} = (1-p)^{-\frac{1}{\alpha}}$$

$$\Rightarrow x = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right] = VaR_p$$

$$ES_p(X) = \frac{1}{p}\int_0^p \lambda\left[(1-u)^{-\frac{1}{\alpha}} - 1\right] du$$

$$= \frac{\lambda}{p}\int_0^p \left[(1-u)^{-\frac{1}{\alpha}} - 1\right] du$$

$$= \frac{\lambda}{p}\left[\frac{(1-u)^{1-\frac{1}{\alpha}}}{(-1)\left(1-\frac{1}{\alpha}\right)} - u\right]_0^p$$

$$= \frac{\lambda}{p}\left[\frac{(1-p)^{1-\frac{1}{\alpha}}}{\frac{1}{\alpha} - 1} - p - \frac{1}{\frac{1}{\alpha} - 1}\right]$$

Q8

$$F(x) = e^{-\left(\frac{\sigma}{x}\right)^{\alpha}} = p$$

$$\Rightarrow \quad -\left(\frac{\sigma}{x}\right)^{\alpha} = \log p$$

$$\Rightarrow \quad \left(\frac{\sigma}{x}\right)^{\alpha} = -\log p$$

$$\Rightarrow \quad \frac{\sigma}{x} = (-\log p)^{\frac{1}{\alpha}}$$

$$\Rightarrow \quad x = \sigma(-\log p)^{-\frac{1}{\alpha}}$$

$$= VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \sigma \cdot (-\log u)^{-\frac{1}{\alpha}}\, du$$

$$= \frac{\sigma}{p}\cdot\int_0^p (-\log u)^{-\frac{1}{\alpha}}\, du$$

$$\boxed{y = -\log u \Rightarrow u = e^{-y} \Rightarrow \frac{du}{dy} = -e^{-y}}$$

$$= \frac{\sigma}{p}\cdot\int_{+\infty}^{-\log p} y^{-\frac{1}{\alpha}}\left(-e^{-y}\right) dy$$

$$= \frac{\sigma}{p}\cdot\int_{-\log p}^{+\infty} y^{-\frac{1}{\alpha}}\, e^{-y}\, dy$$

$$\boxed{\Gamma(a,x) = \int_x^{+\infty} y^{a-1}\, e^{-y}\, dy}$$ Comp Incomplete Gamma Function

$$= \frac{\sigma}{p}\cdot\Gamma\left(1-\frac{1}{\alpha}, -\log p\right)$$

LECTURE

28 OCTOBER

9:00-10:00AM

MATH3/4/68181

Reading Week
(Mon 31st Oct. - Fri 4th Nov)

- No classes
- Mon 31st Oct - 11am - 5pm
- Tues 1st Nov - away
- Wed 2nd Nov - away
- Thurs 3rd Nov - 11am - 5pm
- Fri 4th Nov - 11am - 5pm

No appointments needed
Office ATB 2.223

Estimation methods for VaR

i) Parametric estimation methods

ii) Non-parametric " "

iii) Semi-parametric " "

Non-parametric estimation methods for VaR

a) Historical method

Data: $x_1, x_2, \ldots, x_n$

Ordered data: $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$

The historical estimator for VaR is

$$\widehat{VaR}_p(X) = x_{(i)} \quad \text{if } p \in \left(\frac{i-1}{n}, \frac{i}{n}\right]$$

eg

Data: 7 8 2 1 -1

$n = 5$

Ordered data: -1 1 2 7 8

$$\widehat{VaR}_{0.2}(X) = x_{(1)} = -1$$

$$\widehat{VaR}_{0.9}(X) = x_{(5)} = 8$$

Basel committee uses the historical method for estimating VaR.

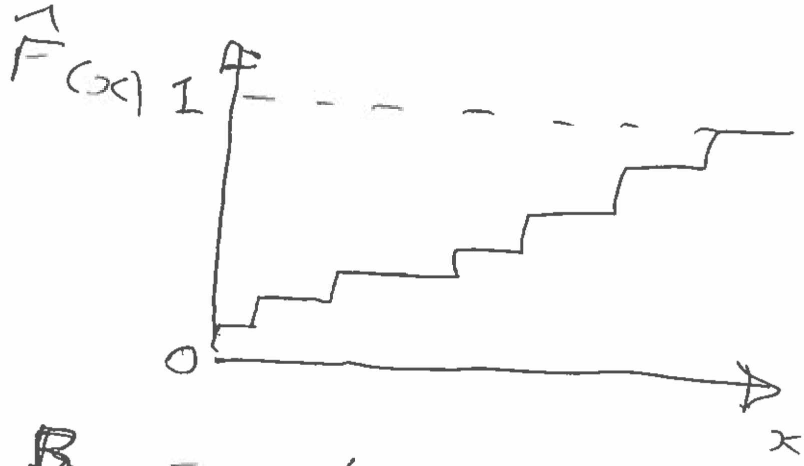
b) <u>Bootstrap method</u>

(Efron, Stanford Univ)

Data: $x_1, x_2, \ldots, x_n$

$$\hat{F}(x) = \frac{1}{n} \sum_{j=1}^{n} I\{x_j \le x\}$$

"empirical CDF"



- simulate B samples each of size n from $\hat{F}$
- $\widehat{VaR}_p^{(1)}$ = historical estimator for the 1st sample

  $\widehat{VaR}_p^{(2)}$ = " for the 2nd sample

  ⋮

  $\widehat{VaR}_p^{(B)}$ = " for the B^{th} sample
- $\widehat{VaR}_p = \text{mean}\left(VaR_p^{(1)}, \ldots, VaR_p^{(B)}\right)$

  $= \text{median}(\; " \;, \ldots, \; " \;)$

c) Jackknife method

Data: $x_1, x_2, \dots, x_n$

- $\widehat{VaR}_p^{(1)}$ = historical estimator for $x_2, \dots, x_n$

  $\widehat{VaR}_p^{(2)}$ = historical estimator for $x_1, x_3, \dots, x_n$

  $\widehat{VaR}_p^{(3)}$ = historical estimator for $x_1, x_2, x_4, \dots, x_n$

  $\vdots$

  $\widehat{VaR}_p^{(n)}$ = historical estimator for $x_1, x_2, \dots, x_{n-1}$

- $\widehat{VaR}_p$ = mean$(VaR_p^{(1)}, \dots, VaR_p^{(n)})$

  = median(" , ... , ")

d) Kernel method

Data : $x_1, x_2, \ldots, x_n$

$$\hat{F}(x) = \frac{1}{n} \sum_{j=1}^{n} G\left(\frac{x - x_j}{h}\right) \quad (*)$$

"band width"

where

$$G(x) = \int_{-\infty}^{x} K(u)\, du$$

Kernel estimator of CDF

"Kernel function"

eg $\quad K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

How to estimate VaR?

- Solve

$$\hat{F}(x) = p$$

$$\Leftrightarrow \frac{1}{n}\sum_{j=1}^{n} G\left(\frac{x - x_j}{h}\right) = p$$

The root is $\widehat{VaR}_p$.

- estimate $\widehat{VaR}$ by

$$\frac{\sum_{i=1}^{n} \hat{F}\left(\frac{i-\frac{1}{2}}{n} - p\right) x_{(i)}}{\sum_{i=1}^{n} \hat{F}\left(\frac{i-\frac{1}{2}}{n} - p\right)}$$

where $\hat{F}(\cdot)$ is given by $(*)$.

e) Jadhav and Ramanathan's method

Data : $x_1, x_2, \ldots, x_n$

Let
$$i = \left[np + \tfrac{1}{2}\right]$$
$$j = [np]$$
$$k = [(n+1)p]$$
$$g = np - j$$
$$h = (n+1)p - k$$
$$\cancel{n = p+1}$$

(where $[y]$ is the largest integer less than or equal to y)

$$\widehat{VaR}_p = (1-g)\,x_{(j)} + g\,x_{(j+1)}$$

$$\widehat{VaR}_p = x_{(j+1)}$$

$$\widehat{VaR}_p = (1-h)\,x_{(k)} + h\,x_{(k+1)}$$

$$\widehat{VaR}_p = \begin{cases} x_{(j)} & g < \frac{1}{2} \\ x_{(j+1)} & g \geq \frac{1}{2} \end{cases}$$

$$\widehat{VaR}_p = \begin{cases} x_{(j)} & g = 0 \\ x_{(j+1)} & g > 0 \end{cases}$$

Fri 11 Nov 9:00 - 10:00

Revision Class for the Test

X = Loss with CDF F

$$VaR_p(X) = \inf\{u : F(u) \geq p\}$$

$$ES_p(X) = \frac{1}{p}\Big[E(X I\{X \leq VaR_p(X)\}) + p\, VaR_p(X) - VaR_p(X) \Pr(X \leq VaR_p(X))\Big]$$

If X is absolutely continous RV

$$VaR_p(X) = F^{-1}(p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\, dt$$

Mathematical Properties of VaR

a). $VaR_p(X+c) = VaR_p(X) + c$

"translation equivalent"

b). $VaR_p(cX) = c\,VaR_p(X), \quad c > 0$

"positively homogeneous"

c) $VaR_p(X) = -VaR_{1-p}(-X)$

d). $X \geq 0 \implies VaR_p(X) \geq 0$

e) $X \geq Y \implies VaR_p(X) \geq VaR_p(Y)$

"monotonicity"

Mathematical properties of ES

a) $X \geq Y \implies ES_p(X) \geq ES_p(Y)$

b) $X \geq 0 \implies ES_p(X) \geq 0$

c) $ES_p(cX) = c \cdot ES_p(X)$, $c > 0$

d) $ES_p(X + c) = ES_p(X) + c$

e) $ES_p(X + Y) \leq ES_p(X) + ES_p(Y)$

II) Parametric methods

i) $X \sim N(\mu, \sigma^2)$

$$F^{-1}(p) = \mu + \sigma \Phi^{-1}(p)$$

$$ES_p(X) = \frac{1}{p} \int_0^p \left[\mu + \sigma \Phi^{-1}(t)\right] dt$$

$$= \mu + \frac{\sigma}{p} \int_0^p \Phi^{-1}(t)\, dt$$

Suppose $X_1, \ldots, X_n$ is a random sample on X. The MLEs of μ and σ are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X},$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2}.$$

So, the MLE of $ES_p(X)$ is

$$\hat{\mu} + \frac{\hat{\sigma}}{p} \int_0^p \Phi^{-1}(\cancel{t})\, dt$$

$$\boxed{= \bar{X} + \frac{1}{p} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2} \int_0^p \Phi^{-1}(t)\, dt}$$

ii) $X \sim LN(\mu, \sigma^2)$

$$F^{-1}(p) = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$ES_p(X) = \frac{1}{p}\int_0^p e^{\mu + \sigma\Phi^{-1}(t)}\,dt$$

$$= \frac{e^{\mu}}{p}\int_0^p e^{\sigma\Phi^{-1}(t)}\,dt$$

Suppose $X_1, X_2, \dots, X_n$ is a random sample on X. Then the MLEs of μ & σ are

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} \log X_i$$

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (\log X_i - \hat{\mu})^2}$$

So, the MLE of $ES_p(X)$ is

$$\boxed{\frac{e^{\hat{\mu}}}{p}\int_0^p e^{\hat{\sigma}\Phi^{-1}(t)}\,dt}$$

iii) $X \sim \text{Uni}[a, b]$

$$F^{-1}(p) = a + (b-a)p$$

$$ES_p(X) = \frac{1}{p}\int_0^p [a + (b-a)t]\,dt$$

$$= \frac{1}{p}\left[at + \frac{(b-a)t^2}{2}\right]_0^p$$

$$= a + \frac{(b-a)p}{2}$$

Suppose $X_1, \ldots, X_n$ is a random sample on X. Then the MLEs of a & b are

$$\hat{a} = \min(X_1, \ldots, X_n)$$

$$\hat{b} = \max(X_1, \ldots, X_n).$$

So, the MLE of $ES_p(X)$ is

$$\boxed{\min(X_1, \ldots, X_n) + \frac{p}{2}\left[\max(X_1, \ldots, X_n) - \min(X_1, \ldots, X_n)\right]}$$

Estimation methods for ES

- Parametric methods

- Nonparametric methods

2) Non-parametric methods for ES

i) Historical method

Data : $x_1, x_2, \ldots, x_n$

Order the data: $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$

$\uparrow$ smallest $\qquad\qquad$ $\uparrow$ largest

$$\boxed{\widehat{ES}_p(x) = \frac{\sum_{i=[np]}^{n} x_{(i)}}{n - [np]}}$$

eg

$-3 \quad 3 \quad 0 \quad -5 \quad 2$

$$\begin{array}{ccccc} -5 & -3 & 0 & 2 & 3 \\ \| & \| & \| & \| & \| \\ x_{(1)} & x_{(2)} & x_{(3)} & x_{(4)} & x_{(5)} \end{array}$$

$$\widehat{ES}_{0.6}(x) = \frac{0 + 2 + 3}{5 - 3} = \frac{5}{2} = 2.5$$

There are many variations of HM:

ii) due to Yamai & Yoshiba (2002)

$$\widehat{ES}_p(X) = \frac{1}{n(\alpha-\beta)} \sum_{i=[n\beta]}^{[n\alpha]} X_{(i)}$$

α, β fixed consts depending on p.

iii) due to Inui & Kijima (2005)

$$\widehat{ES}_p(X) = \begin{cases} -\bar{X}_{k:n} & \text{if } n(1-p) \text{ is an integer} \\ -p\bar{X}_{k:n} - (1-p)\bar{X}_{k+1:n} & \text{if } n(1-p) \text{ is not an integer} \end{cases}$$

where $\bar{X}_{k:n} = \frac{1}{k}\left[X_{(1)} + \cdots + X_{(k)}\right]$.

iv) due to Chen (2008)

$$\widehat{ES}_p(X) = \frac{1}{1+[np]} \sum_{i=1}^{n} X_i I\{X_i \geq X_{([n(1-p)]+1)}\}$$

v) due to Peraochi & Tanase (2008)

$$\widehat{ES}_p(X) = \frac{1}{np} \sum_{i=1}^{[np]} X_{(i)} + \left(1 - \frac{[np]}{np}\right) X_{([np]+1)}$$

Sheet 2, Q7

$$F(x) = 1 - q^{(x+1)^\alpha}$$

$$\lim_{k\to\infty} \frac{P(X=k)}{1-F(k-1)} = 0 \iff \text{ETT holds}$$

$$\boxed{P(X=k) = F(k) - F(k-1)}$$

TRUE for any discrete RV

$$\Rightarrow P(X=k) = 1 - q^{(k+1)^\alpha} - \left(1 - q^{k^\alpha}\right)$$

$$= q^{k^\alpha} - q^{(k+1)^\alpha}$$

$$= q^{k^\alpha}\left[1 - q^{(k+1)^\alpha - k^\alpha}\right]$$

$$\Rightarrow \lim_{k\to\infty} \frac{P(X=k)}{1-F(k-1)} = \lim_{k\to\infty} \frac{q^{k^\alpha}\left[1 - q^{(k+1)^\alpha - k^\alpha}\right]}{1-\left(1-q^{k^\alpha}\right)}$$

$$= \lim_{k\to\infty} \frac{q^{k^\alpha}\left[1 - q^{(k+1)^\alpha - k^\alpha}\right]}{q^{k^\alpha}}$$

$$= \lim_{k\to\infty}\left[1 - q^{(k+1)^\alpha - k^\alpha}\right]$$

$$= 0 \quad \text{if } \alpha < 1$$

$$(k+1)^{\alpha} - k^{\alpha}$$

$$= k^{\alpha}\left(1+\frac{1}{k}\right)^{\alpha} - k^{\alpha}$$

$$= k^{\alpha}\left[\left(1+\frac{1}{k}\right)^{\alpha} - 1\right]$$

$$= k^{\alpha}\left[\sum_{j=0}^{\infty}\binom{\alpha}{j}\left(\frac{1}{k}\right)^{j} - 1\right]$$

$$= k^{\alpha}\left[\cancel{1} + \frac{\alpha}{k} + \frac{\alpha(\alpha-1)}{2}\cdot\frac{1}{k^2} + \cdots - \cancel{1}\right]$$

$$= \alpha k^{\alpha-1} + \frac{\alpha(\alpha-1)}{2}\cdot k^{\alpha-2} + \cdots,$$

Suppose $\alpha < 1$

$\downarrow 0 \qquad \downarrow 0 \qquad \ldots$

$$\Rightarrow (k+1)^{\alpha} - k^{\alpha} \to 0 \text{ as } k \to \infty$$

Sheet 2, Q4

$$p(k) = \frac{k^{-s}}{\zeta(s)}$$

$$1 - F(k-1) = \sum_{i=k}^{\infty} p(i)$$

$$= \sum_{i=k}^{\infty} \frac{i^{-s}}{\zeta(s)}$$

$$= \frac{1}{\zeta(s)} \sum_{i=k}^{\infty} i^{-s}$$

$$\approx \frac{1}{\zeta(s)} \int_k^{\infty} x^{-s}\, dx$$

$$= \frac{1}{\zeta(s)} \left[\frac{x^{1-s}}{1-s} \right]_k^{\infty}$$

$$= \frac{1}{\zeta(s)} \left[0 - \frac{k^{1-s}}{1-s} \right], \quad s > 1$$

$$= \frac{1}{\zeta(s)} \frac{k^{1-s}}{s-1}$$

$$\lim \frac{p(k)}{1-F(k-1)} = \lim \frac{s-1}{k} = 0 \Rightarrow \text{ETT holds}$$

$$F(x) = 1 - \left[1 - e^{-\frac{\lambda}{x}}\right]^{\alpha} \quad , \; w(F) = \infty$$

$$\text{I}: \lim_{t\to\infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t\to\infty} \frac{\left[1 - e^{-\frac{\lambda}{t + x\gamma(t)}}\right]^{\alpha}}{\left[1 - e^{-\frac{\lambda}{t}}\right]^{\alpha}}$$

$$= \lim_{t\to\infty} \left[\frac{1 - e^{-\frac{\lambda}{t+x\gamma(t)}}}{1 - e^{-\frac{\lambda}{t}}}\right]^{\alpha}$$

$\boxed{e^{-x} \approx 1 - x \text{ if } x \text{ is small}}$

$$\approx \lim_{t\to\infty} \left[\frac{\cancel{1} - \left(\cancel{1} - \frac{\cancel{\lambda}}{t + x\gamma(t)}\right)}{\cancel{1} - \left(\cancel{1} - \frac{\cancel{\lambda}}{t}\right)}\right]^{\alpha}$$

$$= \lim_{t\to\infty} \left[\frac{t}{t + x\gamma(t)}\right]^{\alpha}$$

$$= \lim_{t\to\infty} \left[\frac{1}{1 + x\frac{\gamma(t)}{t}}\right]^{\alpha} \neq e^{-x}$$

$\Rightarrow$ Cond (I) not satisfied

(II)

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{\cancel{1} - \left\{\cancel{1} - \left[1 - e^{-\frac{\lambda}{tx}}\right]^{\alpha}\right\}}{\cancel{1} - \left\{\cancel{1} - \left[1 - e^{-\frac{\lambda}{t}}\right]^{\alpha}\right\}}$$

$$= \lim_{t \to \infty} \left[\frac{1 - e^{-\frac{\lambda}{tx}}}{1 - e^{-\frac{\lambda}{t}}}\right]^{\alpha}$$

$$\boxed{e^{-x} \approx x \quad \text{for small } x}$$

$$\approx \lim_{t \to \infty} \left[\frac{\cancel{1} - (\cancel{1} - \frac{\lambda}{tx})}{\cancel{1} - (\cancel{1} - \frac{\lambda}{t})}\right]^{\alpha}$$

$$= \lim_{t \to \infty} \left(\frac{1}{x}\right)^{\alpha} = x^{-\alpha} \Rightarrow$$

Cond $\underline{\overline{II}}$ satisfied.

Non-parametric Estimation of ES

i) Historical method

ii) variant ~~due~~ to Yamai & Yoshiba (2002)

iii) variant due to Inui & Kijima (2005)

iv) variant due to Chen (2008)

v) variant due to Peracchi & Tanase (2008)

vi) Kernel method

$$\widehat{ES}_p = \frac{1}{np} \sum_{i=1}^{n} X_i \, A_h\left(\hat{q}(p) - X_i\right)$$

where

$$\hat{q}(p) = \sum_{i=1}^{n} \left[\int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(t-p)\,dt \right] X_{(i)},$$

"kernel"

$$K_h(u) = \frac{1}{h} K\left(\frac{u}{h}\right),$$

"band width"

$$A(x) = \int_{-\infty}^{x} K(u)\,du, \quad A_h(u) = A\left(\frac{u}{h}\right)$$

vii) Bootstrap estimator

Data : $X_1, X_2, \dots, X_n$

- construct the empirical CDF

$$\hat{F}(t) = \frac{1}{n} \sum_{i=1}^{n} I\{X_i \le t\}$$

- ~~con~~ simulate B samples each of size n from $\hat{F}$

- use the historical method to estimate ES for the B samples, resulting in $\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \dots, ES_p^{(B)}$

- $\widehat{ES}_p = \text{mean}\left(\widehat{ES}_p^{(1)}, ES_p^{(2)}, \dots, ES_p^{(B)}\right)$.

Bootstrap estimator $\widehat{ES}_p$.

viii) Jackknife estimator

Data : $X_1, X_2, \ldots, X_n$

- estimate ES by the historical method for $X_2, X_3, \ldots, X_n$, resulting in $\widehat{ES}_p^{(1)}$

- estimate ES by the historical method for $X_1, X_3, \ldots, X_n$, resulting in $\widehat{ES}_p^{(2)}$

$$\vdots$$

- estimate ES by the historical method for $X_1, X_2, \ldots, X_{n-1}$, resulting in $\widehat{ES}_p^{(n)}$.

- $\widehat{ES}_p = \text{mean}\left(\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \ldots, ES_p^{(n)}\right)$

Jackknife estimator of ES.

ix) Richardson's method

(Prof at School of Math, Uni. of Manchester long time ago.)

Data: $X_1, \dots, X_n$

a) Compute the empirical cdf $\hat{F}(\cdot)$

b) Simulate $Y_1, \dots, Y_N$ from $\hat{F}(\cdot)$

c) estimate ES by historical method for $Y_1, \dots, Y_N$

d) Repeat b) and c) say 1000 times resulting in $\widehat{ES}_{N,1}, \widehat{ES}_{N,2}, \dots, \widehat{ES}_{N,1000}$.

e) Set

$$m_N = \frac{1}{1000} \sum_{i=1}^{1000} \widehat{ES}_{N,i}$$

f) set $s_n = m_{N_n}$ for $n = 1, 2, \dots, k+1$ for some $k, N_1, N_2, \dots, N_{k+1}$

[e.g. $k = 2$, $N_1 = 100$, $N_2 = 200$, $N_3 = 300$]

eg $k = 2$, $N_1 = 100$, $N_2 = 200$, $N_3 = 300$

$$\widehat{ES}_v = \frac{\begin{vmatrix} m_{100} & m_{200} & m_{300} \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{1}{4} & \frac{1}{9} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{1}{4} & \frac{1}{9} \end{vmatrix}}$$

$$= \frac{m_{100} \cdot \left(\frac{1}{18} - \frac{1}{12}\right) - m_{200}\left(\frac{1}{9} - \frac{1}{3}\right) + m_{300} \cdot \left(\frac{1}{4} - \frac{1}{2}\right)}{\frac{1}{18} - \frac{1}{12} - \left(\frac{1}{9} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{2}\right)}$$

g)

$$\hat{ES}_p = \frac{\begin{vmatrix} s_1 & s_2 & \cdots & s_{k+1} \\ 1 & \frac{1}{2} & \cdots & \frac{1}{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1^k & \left(\frac{1}{2}\right)^k & \cdots & \left(\frac{1}{k+1}\right)^k \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \frac{1}{2} & \cdots & \frac{1}{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1^k & \left(\frac{1}{2}\right)^k & \cdots & \left(\frac{1}{k+1}\right)^k \end{vmatrix}}$$

Richardson's estimator

$w(F) = \sup\{x : F(x) < 1\}$

if F is abs cont CDF

then take $w(F)$ as the x

s.t. $F(x) = 1$.

$$F(x) = 1 - q^{(x+1)^a} = 1$$

$$\Rightarrow q^{(x+1)^a} = 0 \quad , \quad 0 < q < 1$$

$$\Rightarrow (x+1)^a = \infty$$

$$\Rightarrow x+1 = \infty$$

$$\Rightarrow x = \infty$$

$$\Rightarrow w(F) = \infty$$

EXAMPLE CLASS

7 NOVEMBER

12:00-13:00PM

MATH3/4/68181

Q1

$X_1, X_2, \ldots, X_n$ IID $Exp(\lambda)$

$$VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

$$ES_p(X) = -\frac{p\cdot\log(1-p) - p - \log(1-p)}{p\lambda}$$

Find the MLE of λ.

$$L(\lambda) = \prod_{i=1}^{n}\left[\lambda e^{-\lambda x_i}\right]$$

$$= \lambda^n e^{-\lambda\sum_{i=1}^{n} x_i}$$

$$\log L(\lambda) = n\log\lambda - \lambda\sum_{i=1}^{n} x_i$$

$$\frac{d\log L}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{X}}$$

$$\frac{d^2\log L}{d\lambda^2} = -\frac{n}{\lambda^2} < 0$$

$\Rightarrow \hat{\lambda} = \frac{1}{\bar{X}}$ is an MLE

$$\Rightarrow \widehat{VaR}_p(X) = -\bar{X}\cdot\log(1-p)$$

$$\widehat{ES}_p(X) = -\bar{X}\,\frac{p\cdot\log(1-p) - p - \log(1-p)}{p}$$

Q2

$X_1, X_2, \ldots, X_n$ IID $f(x) = a x^{a-1}$

$$VaR_p(X) = p^{\frac{1}{a}}$$

$$ES_p(X) = \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1}$$

$$L(a) = \prod_{i=1}^{n} \left[a x_i^{a-1} \right] = a^n \left(\prod_{i=1}^{n} x_i \right)^{a-1}$$

$$\log L = n \log a + (a-1) \sum_{i=1}^{n} \log x_i$$

$$\frac{d \log L}{da} = \frac{n}{a} + \sum_{i=1}^{n} \log x_i = 0$$

$$\hat{a} = - \frac{n}{\sum_{i=1}^{n} \log x_i}$$

$$\frac{d^2 \log L}{da^2} = -\frac{n}{a^2} < 0$$

$$\hat{a} = - \frac{n}{\sum_{i=1}^{n} \log x_i} \text{ is an MLE}$$

$$\widehat{VaR}_p(X) = p^{-\frac{\sum_{i=1}^{n} \log x_i}{n}}$$

$$\widehat{ES}_p(X) = \frac{p^{-\frac{\sum_{i=1}^{n} \log x_i}{n}}}{-\frac{\sum_{i=1}^{n} \log x_i}{n} + 1}$$

Q3

$X_1, X_2, \ldots, X_n$ IID $N(\mu, \sigma^2)$

$$VaR_p(X) = \mu + \sigma\,\Phi^{-1}(p)$$

$$ES_p(X) = \mu + \frac{\sigma}{p} \cdot \int_0^p \Phi^{-1}(t)\,dt$$

Math 20802

$$\begin{cases} \hat{\mu} = \bar{X} \\ \hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} \end{cases}$$

$$\widehat{VaR}_p(X) = \bar{X} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} \cdot \Phi^{-1}(p)$$

$$\widehat{ES}_p(X) = \bar{X} + \frac{1}{p} \cdot \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} \cdot \int_0^p \Phi^{-1}(t)\,dt$$

Q4

$X_1, X_2, \dots, X_n$ IID $LN(\mu, \sigma^2)$

$$VaR_p(X) = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$ES_p(X) = \frac{e^{\mu}}{p} \cdot \int_0^p e^{\sigma \Phi^{-1}(t)}\, dt$$

Math2082

$\Rightarrow$ $X_1, X_2, \dots, X_n$ IID $LN(\mu, \sigma^2)$

$\Rightarrow$ $\log X_1, \log X_2, \dots, \log X_n$ IID $N(\mu, \sigma^2)$

$$\Rightarrow \hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \log X_i$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\log X_i - \hat{\mu})^2}$$

$$\Rightarrow \widehat{VaR_p(X)} = e^{\frac{1}{n}\sum_{i=1}^{n} \log X_i} \cdot e^{\sqrt{\frac{1}{n}\sum_{i=1}^{n} (\log X_i - \hat{\mu})^2}\, \Phi^{-1}(p)}$$

$$\widehat{ES}_p(X) = \frac{e^{\frac{1}{n}\sum_{i=1}^{n} \log X_i}}{p} \int_0^p e^{\sqrt{\frac{1}{n}\sum_{i=1}^{n} (\log X_i - \hat{\mu})^2}\, \Phi^{-1}(t)}\, dt$$

LECTURE

8 NOVEMBER

9:00-10:00AM

MATH3/4/68181

In- Class Test

Tues 15 Nov

Math	38181	9:00- 10:00	AM	Uni Pla B
Math	4/68181	9:00- 10:30	AM	Sch Ruth

Expected Shortfall

- 2^{nd} most popular risk measure due to Artzner et al (1997)
- ES is a coherent risk measure (VaR is not a coherent risk measure)
- X = loss the ES is defined by

$$ES_p(X) = \frac{1}{p}\Big[E\left(X\, I\{X \le VaR_p(X)\}\right) + p \cdot VaR_p(X) - VaR_p(X) \cdot P(X \le VaR_p(X))\Big]$$

where $I\{\cdot\}$ denotes the indicator function

- If X is absolutely continuous

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

Properties of ES

i) $X > Y \Rightarrow ES_p(X) \geq ES_p(Y)$

ii) $ES_p(cX) = c \cdot ES_p(X)$

iii) $ES_p(X+c) = ES_p(X) + c$

iv) $ES_p(X+Y) \leq ES_p(X) + ES_p(Y)$

where X, Y are RVs and c is a constant.

Proof of (ii) Assume X is absolutely continuous. Then

$$
\begin{aligned}
ES_p(cX) &= \frac{1}{p}\int_0^p \mathrm{VaR}_t(cX)\,dt \\
&= \frac{1}{p}\int_0^p c\cdot \mathrm{VaR}_t(X)\,dt \\
&= c\cdot \frac{1}{p}\int_0^p \mathrm{VaR}_t(X)\,dt \\
&= c\cdot ES_p(X).
\end{aligned}
$$

Proof of (iii)

$$
\begin{aligned}
ES_p(X+c) &= \frac{1}{p}\int_0^p \mathrm{VaR}_t(X+c)\,dt \\
&= \frac{1}{p}\int_0^p \left[\mathrm{VaR}_t(X)+c\right]dt \\
&= \frac{1}{p}\left[\int_0^p \mathrm{VaR}_p(X)\,dt + c\cdot p\right] \\
&= ES_p(X) + c
\end{aligned}
$$

Estimation methods for ES

i) Parametric estimation methods

ii) Non-parametric " "

iii) Semi-parametric " "

→ Math 38181

→ Math 46181

Parametric Estimation Methods

a) Normal distribution

$$X \sim N(\mu, \sigma^2)$$

$$ES_p(X) = \mu + \frac{\sigma}{p} \cdot \int_0^p \Phi^{-1}(t)\, dt$$

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$.

The MLEs of μ & σ are

$$\hat{\mu} = \overline{X}$$

Math 20802

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2}$$

The MLE for $ES_p(X)$ is

$$\widehat{ES_p}(X) = \overline{X} + \frac{1}{p} \cdot \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2} \cdot \int_0^p \Phi^{-1}(t)\, dt$$

$\widehat{ES}_p(X)$ is a <u>biased</u> estimator of $ES_p(X)$.

$$E\left[\widehat{ES}_p(X)\right] = E[\bar{X}] + \frac{1}{p} \cdot E\left[\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}\right] \cdot \int_0^p \Phi^{-1}(t)\,dt$$

$$= \mu + \frac{1}{p} E\left[\sigma\sqrt{\frac{\chi^2_{n-1}}{n}}\right] \cdot \int_0^p \Phi^{-1}(t)\,dt$$

because $\boxed{\sum_{i=1}^{n}(X_i - \bar{X})^2 \sim \sigma^2 \chi^2_{n-1}}$

$$= \mu + \frac{\sigma}{p} E\left[\sqrt{\frac{\chi^2_{n-1}}{n}}\right] \cdot \int_0^p \Phi^{-1}(t)\,dt$$

$$\neq \mu + \frac{\sigma}{p} \cdot \int_0^p \Phi^{-1}(t)\,dt$$

$$= ES_p(X)$$

$\boxed{\text{Home}}$ $\boxed{\text{work}}$

b) <u>Generalized Pareto distribution</u>

X has the CDF

$$F(x) = 1 - q\left(1 + \xi\,\frac{x-u}{\sigma}\right)^{-\frac{1}{\xi}}$$

where $q = P(X > u)$

$u =$ threshold

Set $F(x) = p$

$$\Rightarrow 1 - q\left(1 + \xi\,\frac{x-u}{\sigma}\right)^{-\frac{1}{\xi}} = p$$

$$\Rightarrow \left(1 + \xi\,\frac{x-u}{\sigma}\right)^{-\frac{1}{\xi}} = \frac{1-p}{q}$$

$$\Rightarrow 1 + \xi\,\frac{x-u}{\sigma} = \left(\frac{1-p}{q}\right)^{-\xi}$$

$$\Rightarrow x = u + \frac{\sigma}{\xi}\left[\left(\frac{1-p}{q}\right)^{-\xi} - 1\right]$$

$$= VaR_p(X)$$

$$\Rightarrow ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= u - \frac{\sigma}{\xi} + \frac{\sigma q^{\xi}}{p\xi}\int_0^p (1-t)^{-\xi}\,dt$$

$$= u - \frac{\sigma}{\xi} + \frac{\sigma q^{\xi}}{p\xi}\,\frac{(1-p)^{1-\xi} - 1}{\xi - 1}$$

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the GP.

Let $\hat{\sigma}$ & $\hat{\xi}$ denote the MLEs of σ & ξ. See notes earlier on how to get these.

The MLE for $ES_p(X)$ is

$$\widehat{ES_p(X)} = u - \frac{\hat{\sigma}}{\hat{\xi}} + \frac{\hat{\sigma}\, q^{\hat{\xi}}}{p\hat{\xi}} \frac{(1-p)^{1-\hat{\xi}} - 1}{\hat{\xi} - 1}$$

c) <u>GEV distribution</u>

X has the CDF

$$F(x) = e^{-\left(1+\xi \cdot \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$-\infty < \mu < +\infty$$
$$\sigma > 0$$
$$-\infty < \xi < +\infty$$

Set $F(x) = p$

$$x = \mu + \frac{\sigma}{\xi}\left[(-\log p)^{-\xi} - 1\right]$$
$$= VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\, dt$$

$$= \mu - \frac{\sigma}{\xi} + \frac{\sigma}{p\xi}\int_0^p (-\log t)^{-\xi}\, dt$$

If $X_1, X_2, \ldots, X_n$ is a random sample from the GEV the MLEs $\hat{\mu}$, $\hat{\sigma}$ & $\hat{\xi}$ can be obtained (see notes earlier).
The MLE of $ES_p(X)$ is

$$\widehat{ES_p(X)} = \hat{\mu} - \frac{\hat{\sigma}}{\hat{\xi}} + \frac{\hat{\sigma}}{p\hat{\xi}}\int_0^p (-\log t)^{-\hat{\xi}}\, dt$$

EXAMPLE CLASS

8 NOVEMBER

10:00-11:00AM

MATH3/4/68181

Q1

$$X \sim Exp(\lambda)$$

$$VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

$$ES_p(X) = -\frac{p\cdot\log(1-p) - p - \log(1-p)}{p\lambda}$$

$$L(\lambda) = \prod_{i=1}^{n}\left[\lambda e^{-\lambda x_i}\right] = \lambda^n e^{-\lambda\sum_{i=1}^{n} x_i}$$

$$\log L = n\log\lambda - \lambda\sum_{i=1}^{n} x_i$$

$$\frac{d\log L}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0 \Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{X}}$$

$$\frac{d^2\log L}{d\lambda^2} = -\frac{n}{\lambda^2} < 0 \Rightarrow \hat{\lambda} = \frac{1}{\bar{X}} \text{ is an MLE}$$

$\Rightarrow$ The MLEs of VaR & ES are

$$\widehat{VaR}_p(X) = -\bar{X}\cdot\log(1-p)$$

$$\widehat{ES}_p(X) = -\bar{X}\cdot\frac{p\cdot\log(1-p) - p - \log(1-p)}{p}$$

Q2

$$VaR_p(X) = p^{\frac{1}{a}}$$

$$ES_p(X) = \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1}$$

$$L(a) = \prod_{i=1}^{n}\left[a\, x_i^{a-1}\right] = a^n\left(\prod_{i=1}^{n} x_i\right)^{a-1}$$

$$\log L = n\log a + (a-1)\sum_{i=1}^{n}\log x_i$$

$$\frac{d\log L}{da} = \frac{n}{a} + \sum_{i=1}^{n}\log x_i = 0 \Rightarrow \hat{a} = -\frac{n}{\sum_{i=1}^{n}\log x_i}$$

$$\frac{d^2\log L}{da^2} = -\frac{n}{a^2} < 0 \Rightarrow \hat{a} = -\frac{n}{\sum_{i=1}^{n}\log x_i} \text{ is an MLE}$$

The MLEs of VaR & ES are

$$\widehat{VaR}_p(X) = p^{-\frac{\sum_{i=1}^{n}\log x_i}{n}}$$

$$\widehat{ES}_p(X) = \frac{p^{-\frac{\sum_{i=1}^{n}\log x_i}{n}}}{-\frac{\sum\log x_i}{n}+1}$$

$$= \frac{p^{-\frac{\sum_{i=1}^{n}\log x_i}{n}}}{-\frac{\sum_{i=1}^{n}\log x_i}{n}+1}$$

Q3 $X \sim N(\mu, \sigma^2)$

$$VaR_p(X) = \mu + \sigma\,\Phi^{-1}(p)$$

$$ES_p(X) = \mu + \frac{\sigma}{p} \cdot \int_0^p \Phi^{-1}(t)\,dt$$

If $X_1, X_2, \ldots, X_n$ IID $N(\mu, \sigma^2)$

Math 2082 notes

$$\hat{\mu} = \bar{X}$$

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

The MLEs of VaR & ES are

$$\widehat{VaR}_p(X) = \bar{X} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} \cdot \Phi^{-1}(p)$$

$$\widehat{ES}_p(X) = \bar{X} + \frac{1}{p} \cdot \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} \cdot \int_0^p \Phi^{-1}(t)\,dt$$

Q4 $X \sim LN(\mu, \sigma^2)$

$$\mathrm{VaR}_p(X) = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$\mathrm{ES}_p(X) = \frac{e^{\mu}}{p} \cdot \int_0^p e^{\sigma \Phi^{-1}(t)} \, dt$$

$X_1, X_2, \ldots, X_n$ IID $LN(\mu, \sigma^2)$

$\Rightarrow \log X_1, \log X_2, \ldots, \log X_n$ IID $N(\mu, \sigma^2)$ Math20802

$$\Rightarrow \quad \hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \log X_i$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\log X_i - \hat{\mu})^2}$$

$\Rightarrow$ MLEs of VaR & ES are

$$\widehat{\mathrm{VaR}}_p(X) = e^{\hat{\mu} + \hat{\sigma} \Phi^{-1}(p)}$$

$$\widehat{\mathrm{ES}}_p(X) = \frac{e^{\hat{\mu}}}{p} \int_0^p e^{\hat{\sigma} \Phi^{-1}(t)} \, dt$$

P5, Q6

Use the Indicator function approach to find the MLEs.

(Math20802)

LECTURE

11 NOVEMBER

9:00-10:00AM

MATH3/4/68181

Week 8

Mon	14	Nov	12 – 1 (Zo Th A)	Revision class
Tues	15	Nov	9 – 11	In-class test
Thurs	17	Nov	12 – 1	Lecture (only 4/6)
Fri	18	Nov	9 – 10	Lecture (3/4/6)

Estimation Methods for ES

- Parametric methods
- Non-parametric "
- Semi-parametric "

Non parametric estimation methods

a) Historical method

Let X_t = return at time t

Let $X_{(1)} \leqslant X_{(2)} \leqslant \cdots \leqslant X_{(n)}$ be the ordered returns. The historical estimator of ES is

$$\widehat{ES}_p(X) = \frac{1}{[np]} \sum_{i=0}^{[np]} X_{(i)}$$

where $[x]$ denotes the largest integer $\leqslant x$.

b) Kernel method

Let X_t = return at time t

Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the ordered returns. The kernel estimator of ES is

$$\widehat{ES}_p = \frac{1}{np} \sum_{i=1}^{n} X_i A_h(\hat{q}_p - X_i)$$

where

$$\hat{q}_p = \sum_{i=1}^{n} \left[\int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(t-p)\, dt \right] X_{(i)}$$

$$A_h(u) = \int_{-\infty}^{\frac{u}{h}} K(t)\, dt$$

$$K_h(u) = \frac{1}{h} \cdot K\left(\frac{u}{h}\right)$$

h = bandwidth

$K(\cdot)$ = kernel function

eg $K(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$

$$F(x) = \frac{\left[1 - e^{-2x}\right]^2}{0.5 + 0.5\left[1 - e^{-2x}\right]^2}$$

$$\omega(F) = +\infty$$

$$\lim_{t \uparrow \infty} \frac{1 - F(t + x \cdot \gamma(t))}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{1 - \dfrac{\left[1 - e^{-2(t + x\gamma(t))}\right]^2}{0.5 + 0.5\left[1 - e^{-2(t + x\gamma(t))}\right]^2}}{1 - \dfrac{\left[1 - e^{-2t}\right]^2}{0.5 + 0.5\left[1 - e^{-2t}\right]^2}}$$

1

1

$$= \lim_{t \uparrow \infty} \frac{1 - \left[1 - e^{-2(t + x\gamma(t))}\right]^2}{1 - \left[1 - e^{-2t}\right]^2}$$

$$= \lim_{t \uparrow \infty} \frac{\cancel{1} - \left[\cancel{1} - \cancel{2} \cdot e^{-2(t + x\gamma(t))}\right]}{\cancel{1} - \left[\cancel{1} - \cancel{2} \cdot e^{-2t}\right]}$$

$$(1-z)^{\alpha} \approx 1 - \alpha z$$

$$= \lim_{t \uparrow \infty} \frac{e^{-2(t + x\gamma(t))}}{e^{-2t}}$$

$$= \lim_{t \uparrow \infty} e^{-2x\gamma(t)} = e^{-x} \quad \text{if } \gamma(t) = \frac{1}{2}$$

$$F(x) = \Phi^2(x)$$

$$w(F) = +\infty$$

$$\lim_{t \uparrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \lim_{t \uparrow \infty} \frac{1 - \Phi^2(t + x\gamma(t))}{1 - \Phi^2(t)}$$

$$\overset{LH}{=} \lim_{t \uparrow \infty} \frac{\cancel{-}\cancel{2} \cdot \boxed{\Phi(t + x\gamma(t))}^{\nearrow 1} \, \phi(t + x\gamma(t)) \, (1 + x\gamma'(t))}{\cancel{-}\cancel{2} \, \underbrace{\boxed{\Phi(t)}}_{\searrow 1} \cdot \phi(t)}$$

$$= \lim_{t \uparrow \infty} \frac{\phi(t + x\gamma(t)) \, (1 + x\gamma'(t))}{\phi(t)}$$

$$= \lim_{t \uparrow \infty} \frac{\cancel{\frac{1}{\sqrt{2\pi}}} \, e^{-\frac{(t + x\gamma(t))^2}{2}}}{\cancel{\frac{1}{\sqrt{2\pi}}} \, e^{-\frac{t^2}{2}}} \cdot (1 + x\gamma'(t))$$

$$= \lim_{t \uparrow \infty} e^{\frac{t^2 - (t + x\gamma(t))^2}{2}} \cdot (1 + x\gamma'(t))$$

$$= \lim_{t \uparrow \infty} e^{-\frac{2xt\gamma(t) + x^2\gamma^2(t)}{2}} \cdot (1 + x\gamma'(t))$$

$$= \lim_{t \uparrow \infty} e^{-xt\gamma(t) - \frac{x^2\gamma^2(t)}{2}} \cdot (1 + x\gamma'(t))$$

choose $\gamma(t) = \frac{1}{t}$

$$= \lim_{t \to \infty} e^{-x - \frac{x^2}{2t^2}} \cdot \left(1 + x\left(-\frac{1}{t^2}\right)\right)$$

$$\frac{x^2}{2t^2} \to 0 \qquad x\left(-\frac{1}{t^2}\right) \to 0$$

$$= e^{-x}$$

$$p(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

$$w(F) = +\infty$$

$$\lim_{k\to\infty} \frac{P(X=k)}{1-F(k-1)} = \lim_{k\to\infty} \frac{P(X=k)}{\sum_{j=k}^{\infty} P(X=j)}$$

$$= \lim_{k\to\infty} \frac{\frac{\cancel{e^{-\lambda}}\lambda^k}{k!}}{\sum_{j=k}^{\infty} \frac{\cancel{e^{-\lambda}}\lambda^j}{j!}} = \lim_{k\to\infty} \frac{\frac{\lambda^k}{k!}}{\sum_{j=k}^{\infty} \frac{\lambda^j}{j!}}$$

$$= \lim_{k\to\infty} \frac{1}{\sum_{j=k}^{\infty} \frac{\lambda^{j-k}\, k!}{j!}} \qquad \text{———} \qquad (*)$$

$$\frac{k!}{j!} = \frac{\cancel{1}\cdot\cancel{2}\cdots\cancel{k}}{\cancel{1}\cdot\cancel{2}\cdots\cancel{k}\cdots j} = \frac{1}{(k+1)(k+2)\cdots j}$$

$$= \frac{1}{\underbrace{(k+1)}_{\geq k}\underbrace{(k+2)}_{\geq k}\cdots\underbrace{(k+j-k)}_{\geq k}} \qquad \geq k^{j-k}$$

$$\leq \frac{1}{k^{j-k}}$$

$$(\ast) \geq \lim_{k\to\infty} \frac{1}{\sum_{j=k}^{\infty} \frac{\lambda^{j-k}}{k^{j-k}}}$$

$$= \lim_{k\to\infty} \frac{1}{\sum_{j=k}^{\infty} \left(\frac{\lambda}{k}\right)^{j-k}}$$

$$= \lim_{k\to\infty} \frac{1}{\sum_{m=0}^{\infty} \left(\frac{\lambda}{k}\right)^{m}} \qquad \boxed{m = j-k}$$

$$= \lim_{k\to\infty} \frac{1}{\frac{1}{1-\frac{\lambda}{k}}} \qquad \boxed{\sum_{m=0}^{\infty} r^m = \frac{1}{1-r}}$$

$$\frac{\lambda}{k} \to 0$$

$$= 1$$

$$\Rightarrow \lim_{k\to\infty} \frac{P(X=k)}{1-F(k-1)} \geq 1$$

$\Rightarrow$ EIT cannot hold.

$$f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < +\infty$$

$$w(F) = +\infty$$

$$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$\overset{LH}{=} \lim_{t \to \infty} \frac{-f(t + x\gamma(t)) \cdot (1 + x\gamma'(t))}{-f(t)}$$

$$= \lim_{t \to \infty} \frac{\cancel{\tfrac{1}{2}} e^{-|t + x\gamma(t)|} (1 + x\gamma'(t))}{\cancel{\tfrac{1}{2}} e^{-|t|}}$$

$\vdots$

$$= e^{-x}$$

EXAMPLE CLASS

14 NOVEMBER

12:00-13:00PM

MATH3/4/68181

Revision for In-Class Test

$$F(x) = 1 - e^{1-(1+\lambda x)^{\alpha}}$$

Set $F(x) = 1 \Rightarrow x = +\infty \Rightarrow \omega(F) = +\infty$

$$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \lim_{t \to \infty} \frac{\cancel{1} - \{\cancel{1} - e^{1-(1+\lambda t+\lambda x\gamma(t))^{\alpha}}\}}{\cancel{1} - \{\cancel{1} - e^{1-(1+\lambda t)^{\alpha}}\}}$$

$$= \lim_{t \to \infty} e^{(1+\lambda t)^{\alpha} - (1+\lambda t+\lambda x\gamma(t))^{\alpha}}$$

$$= \lim_{t \to \infty} e^{(1+\lambda t)^{\alpha}\left[1 - \left(1 + \frac{\lambda x\gamma(t)}{1+\lambda t}\right)^{\alpha}\right]}$$

$$= \lim_{t \to \infty} e^{(1+\lambda t)^{\alpha}\left[\cancel{1} - \left(\cancel{1} + \alpha \cdot \frac{\lambda x\gamma(t)}{1+\lambda t}\right)\right]}$$

$$\boxed{(1+z)^{\alpha} \approx 1 + \alpha z}$$

$$= \lim_{t \to \infty} e^{-(1+\lambda t)^{\alpha-1} \alpha\lambda x \gamma(t)}$$

$$= e^{-x} \quad \text{if} \quad \gamma(t) = (1+\lambda t)^{-\alpha+1} \cdot \frac{1}{\alpha\lambda}.$$

$\Rightarrow$ F belongs to Gumbel domain.

$$F(x) = 1 - \frac{(0.5)^2 e^{-4x}}{\left[1 - 0.5\, e^{-2x}\right]^2}$$

$$F(x) = 1 \Rightarrow x = +\infty \Rightarrow w(F) = +\infty$$

$$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \lim_{t \to \infty} \frac{\left\{ \cancel{1} - \cancel{1} + \frac{(0.5)^2 e^{-4(t + x\gamma(t))}}{\left[1 - 0.5\, e^{-2(t + x\gamma(t))}\right]^2} \right\}}{\left\{ \cancel{1} - \cancel{1} + \frac{(0.5)^2 e^{-4t}}{\left[1 - 0.5\, e^{-2t}\right]^2} \right\}}$$

0

0

$$= \lim_{t \to \infty} \frac{\cancel{(0.5)^2} e^{-4(t + x\gamma(t))}}{\cancel{(0.5)^2} e^{-4t}}$$

$$= \lim_{t \to \infty} e^{-4x\gamma(t)}$$

$$= e^{-x} \quad \text{if} \quad \gamma(t) = \frac{1}{4}.$$

$$F(x) = \{1 - [1 - G^{\theta}(x)]^4\}^{\alpha}$$

i) G belongs to Gumbel.

$$\lim_{t \to w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to w(F)} \frac{1 - \{1 - [1 - G^{\theta}(t + x\gamma(t))]^4\}^{\alpha}}{1 - \{1 - [1 - G^{\theta}(t)]^4\}^{\alpha}}$$

$$= \lim_{t \to w(G)} \frac{\cancel{1} - \{\cancel{1} - \cancel{\alpha} \cdot [1 - G^{\theta}(t + x\gamma(t))]^4\}}{\cancel{1} - \{\cancel{1} - \cancel{\alpha}[1 - G^{\theta}(t)]^4\}}$$

$$\boxed{1} - z]^{\alpha} \approx 1 - \alpha z$$

$$= \lim_{t \to w(G)} \left[\frac{1 - G^{\theta}(t + x\gamma(t))}{1 - G^{\theta}(t)}\right]^4$$

$$= \lim_{t \to w(G)} \left\{\frac{1 - [1 - (1 - G(t + x\gamma(t)))]^{\theta}}{1 - [1 - (1 - G(t))]^{\theta}}\right\}^4$$

$$= \lim_{t \to w(G)} \left\{\frac{\cancel{1} - [\cancel{1} - \cancel{\theta} \cdot (1 - G(t + x\gamma(t)))]}{\cancel{1} - [\cancel{1} - \cancel{\theta}(1 - G(t))]}\right\}^4$$

$$= \lim_{t \to w(G)} \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^4 = e^{-4x}$$

Show F belongs to the same domain as G.

i) F belongs to Gumbel domain
$\Rightarrow$ G " " " "

ii) F belongs to Fréchet domain
$\Rightarrow$ G " " " "

iii) F belongs to Weibull domain
$\Rightarrow$ G " " " "

$$F(x) = 1 - q^{(x+1)^a}, \quad \begin{array}{l} 0 < q < 1 \\ a > 1 \\ x = 0, 1, \dots \end{array}$$

$$F(x) = 1 \Rightarrow x = +\infty \Rightarrow \omega(F) = +\infty$$

$$\lim_{k\to\infty} \frac{P(X=k)}{1-F(k-1)} = \lim_{k\to\infty} \frac{F(k) - F(k-1)}{1 - F(k-1)}$$

$$= \lim_{k\to\infty} \frac{\cancel{1} - q^{(k+1)^a} - [\cancel{1} - q^{k^a}]}{\cancel{1} - [\cancel{1} - q^{k^a}]}$$

$$= \lim_{k\to\infty} \frac{-q^{(k+1)^a} + q^{k^a}}{q^{k^a}} = \lim_{k\to\infty} \left[-q^{(k+1)^a - k^a} + 1\right]$$

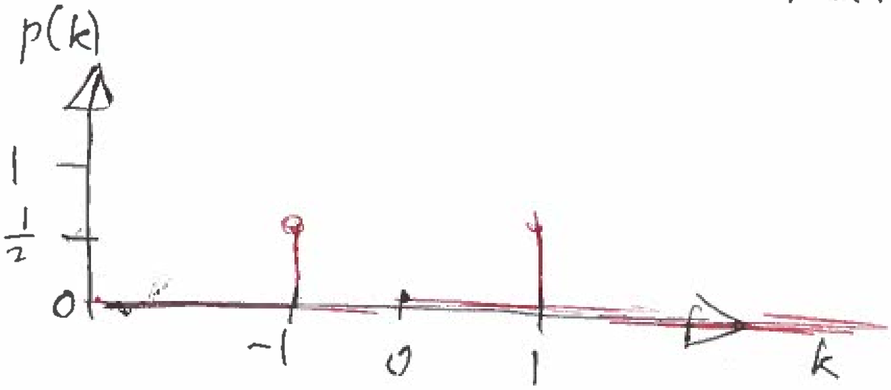
$$= \lim_{k\to\infty} \left[-q^{\left[\underbrace{\left(1+\frac{1}{k}\right)^a}_{\text{Bin exp}} - 1\right] k^a} + 1\right]$$

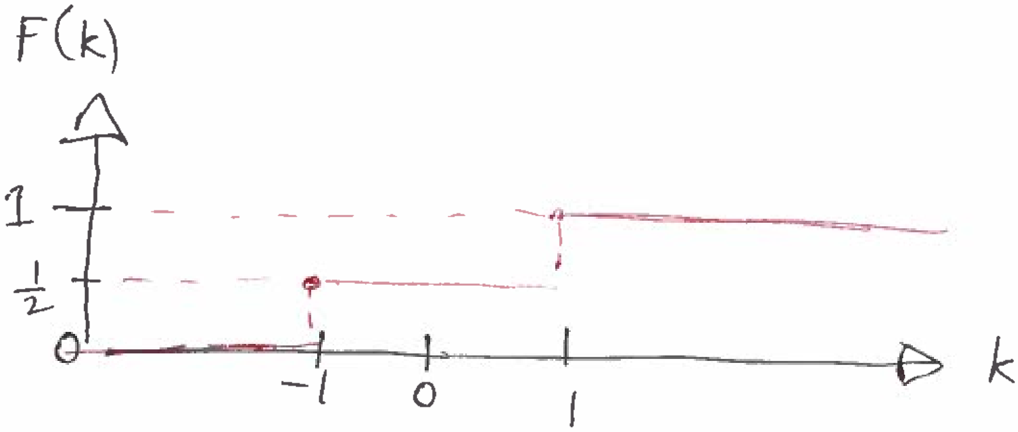
$$= \lim_{k\to\infty} \left[-q^{\left[\cancel{1} + \underbrace{\frac{a}{k} + \frac{a(a-1)}{2k^2} + \dots}\, - \cancel{1}\right] k^a} + 1\right]$$

$$= \lim_{k\to\infty} \left[-q^{\overset{\to +\infty}{a k^{a-1}}} + 1\right] = 1 \neq 0$$

$\Rightarrow$ ETT does not hold.

$$p(k) = \begin{cases} \frac{1}{2} & k = -1, 1 \\ 0 & \text{otherwise} \end{cases}$$





$$w(F) = +1$$

$$\lim_{k \to w(F)} \frac{P(X=k)}{1 - F(k-1)} = \frac{P(X=1)}{1 - F(1-1)} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \neq ($$

ETT does not hold.

LECTURE

18 NOVEMBER

9:00-10:00AM

MATH3/4/68181

In-Class Test Feedback

Q1 (i) the statement of ETT should be complete

(ii) ✓

(iii) e^{-8x} for Gumbel

$x^{-8\beta}$ for Frechet

$x^{8\beta}$ for Weibull

Q2 (i)

(ii)

(iii)

(iv)

(v)

$\Bigg\}$ details of working + state the domain name

Q1 (iii)

$$F(x) = 1 - \left\{ 1 - \left\{ 1 - \left[1 - G(x) \right]^2 \right\}^3 \right\}^4$$

(i) G belongs to Gumbel domain

$\Rightarrow$ F " " " "

(ii) G belongs to Frechet domain

$\Rightarrow$ F " " " "

(iii) G belongs to Weibull domain

$\Rightarrow$ F " " " "

(ii) Assume G belongs to Gumbel domain. That is

$$\lim_{t \uparrow w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x}$$

$$\lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \uparrow w(F)} \frac{\{1 - \{1 - [1 - G(t + x\gamma(t))]^2\}^3\}^4}{\{1 - \{1 - [1 - G(t)]^2\}^3\}^4}$$

$$= \lim_{t \uparrow w(F)} \left[\frac{1 - \{1 - [1 - G(t + x\gamma(t))]^2\}^3}{1 - \{1 - [1 - G(t)]^2\}^3} \right]^4$$

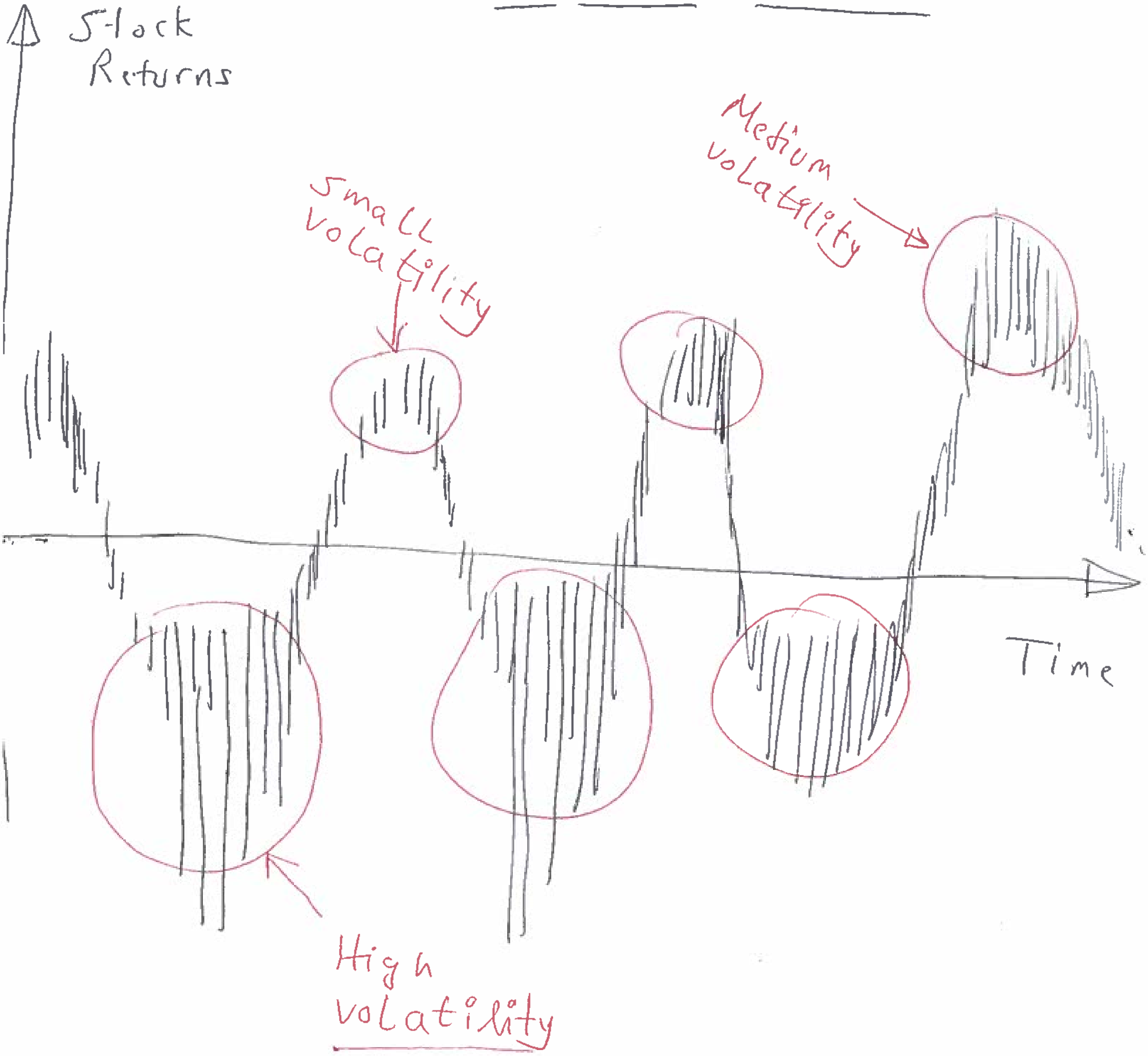
$$= \lim_{t \uparrow w(F)} \left[\frac{\cancel{1} - \{\cancel{1} - \cancel{3}[1 - G(t + x\gamma(t))]^2\}}{\cancel{1} - \{\cancel{1} - \cancel{3}[1 - G(t)]^2\}} \right]^4$$

$$\boxed{(1 - z)^a \approx 1 - az}$$

$$= \lim_{t \uparrow w(G)} \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)} \right]^8$$

$$= e^{-8x}$$

Models for Stock Returns



X = Stock Return at time t

V = Volatility at time t

Assume that both X & V are RVs.

Suppose $X|V$ has PDF $f_{X|V}$

Suppose too V has PDF g.

Then the PDF of X is

$$\boxed{f_X(x) = \int_0^{\infty} f_{X|V}(x|v)\, g(v)\, dv}$$

Total Prob Rule

The corresponding CDF is

$$\boxed{F_X(x) = \int_0^{\infty} F_{X|V}(x|v)\, g(v)\, dv}$$

where $F_{X|V}$ is the CDF of $X|V$.
The nth moment of X is

$$E(X^n) = E\left[E(X^n|V)\right].$$

In particular,

$$E(X) = E\left[E(X|V)\right],$$

$$Var(X) = E\left[E(X^2|V)\right] - \left\{E\left[E(X|V)\right]\right\}^2$$

$X =$ Observable

$V =$ Not observable

eg

$$X \sim N(0, \sigma^2)$$

$\sigma \sim$ Frechet PDF $\boxed{\dfrac{2}{\sigma^3} e^{-\frac{1}{\sigma^2}}}, \sigma > 0$

What is the distribution of X?

$$f_X(x) = \int_0^\infty \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}} \cdot \underbrace{\frac{2}{\sigma^3} e^{-\frac{1}{\sigma^2}}} d\sigma$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sigma^4} e^{-\left(\frac{x^2}{2}+1\right)\frac{1}{\sigma^2}} d\sigma$$

Set $Y = \left(\frac{x^2}{2}+1\right)\frac{1}{\sigma^2}$

$$\sigma^2 = \left(\frac{x^2}{2}+1\right)\frac{1}{y}$$

$$\sigma = \sqrt{\frac{x^2}{2}+1}\,\frac{1}{\sqrt{y}}$$

$$\frac{d\sigma}{dy} = \sqrt{\frac{x^2}{2}+1}\left(-\frac{1}{2}\right) y^{-\frac{3}{2}}$$

$$= \frac{\cancel{2}}{\sqrt{2\pi}} \int_{\infty}^{0} \frac{y^2}{\left(\frac{x^2}{2}+1\right)^2} \cdot e^{-y} \sqrt{\frac{x^2}{2}+1} \cdot \left(\cancel{-}\frac{1}{\cancel{2}}\right) y^{-\frac{3}{2}}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{x^2}{2}+1\right)^{-\frac{3}{2}} \int_{0}^{\infty} y^{\frac{1}{2}} e^{-y}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{x^2}{2}+1\right)^{-\frac{3}{2}} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{x^2}{2}+1\right)^{-\frac{3}{2}} \frac{1}{2} \cdot \sqrt{\pi}$$

$$\left[\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{1}{2\sqrt{2}} \left(\frac{x^2}{2}+1\right)^{-\frac{3}{2}}$$

EXAMPLE CLASS

21 NOVEMBER

12:00-13:00PM

MATH3/4/68181

Q1

X = Stock Returns

$X \sim Exp(\lambda)$

λ = a RV

$\lambda \sim Exp(a)$

$$f_X(x) = \int_0^\infty \boxed{f_{X|V}(x|v)} \, \boxed{g(v)} \, dv$$

cond PDF given V — PDF of V

$$= \int_0^\infty \lambda e^{-\lambda x} \cdot \cancel{a} e^{-a\lambda} \, d\lambda$$

$$= a \int_0^\infty \lambda e^{-(a+x)\lambda} \, d\lambda$$

$$\boxed{\begin{aligned} \text{Set} \quad y &= (a+x)\lambda \\ \lambda &= \frac{y}{a+x} \\ d\lambda &= \frac{dy}{a+x} \end{aligned}}$$

$$= a \int_0^\infty \frac{y}{a+x} \cdot e^{-y} \frac{dy}{a+x}$$

$$= \frac{a}{(a+x)^2} \int_0^\infty y e^{-y} \, dy = \boxed{\frac{a}{(a+x)^2}}$$

Suppose $X_1, X_2, \ldots, X_n$ (a random sample) on X. The likelihood of a is

$$L(a) = \prod_{i=1}^{n} \frac{a}{(a+x_i)^2}$$

$$\log L = n \log a - 2 \sum_{i=1}^{m} \log(a + x_i)$$

$$\frac{d \log L}{da} = \frac{n}{a} - 2 \sum_{i=1}^{n} \frac{1}{a+x_i}$$

The MLE of a is the root of

$$\frac{n}{a} = 2 \sum_{i=1}^{n} \frac{1}{a+x_i} .$$

Q2

$$\lambda \sim \text{Uni}[a,b]$$

$$f_X(x) = \int_a^b \lambda e^{-\lambda x} \cdot \frac{1}{b-a} \, d\lambda$$

$$= \frac{1}{b-a} \int_a^b \lambda e^{-\lambda x} \, d\lambda$$

$$= \frac{1}{b-a} \left\{ \left[\lambda \cdot \frac{e^{-\lambda x}}{(-x)} \right]_a^b + \frac{1}{x} \int_a^b e^{-\lambda x} \, d\lambda \right\}$$

$$= \frac{1}{b-a} \left\{ -\frac{b e^{-bx}}{x} + \frac{a e^{-ax}}{x} + \frac{1}{x} \left[\frac{e^{-\lambda x}}{(-x)} \right]_a^b \right\}$$

$$= \frac{1}{b-a} \left\{ -\frac{b e^{-bx}}{x} + \frac{a e^{-ax}}{x} - \frac{e^{-ba} - e^{-ax}}{x^2} \right\}$$

$$L(a,b) = \prod_{i=1}^{n} f_X(x_i)$$

Q3

λ has PDF $a\lambda^{a-1}$, $0<\lambda<1$

$$f_X(x) = \int_0^1 \lambda e^{-\lambda x} \cdot a\lambda^{a-1} \, d\lambda$$

$$= a\int_0^1 \lambda^a e^{-\lambda x} \, d\lambda$$

$$\boxed{\begin{aligned} \text{Set} \quad y &= \lambda x \\ \cancel{\lambda} &= \frac{y}{x} \\ d\lambda &= \frac{dy}{x} \end{aligned}}$$

$$= a\int_0^x \left(\frac{y}{x}\right)^a e^{-y} \frac{dy}{x}$$

$$= \frac{a}{x^{a+1}} \int_0^x y^a e^{-y} \, dy$$

$$= \frac{a}{x^{a+1}} \gamma(a+1, x)$$

$$\boxed{\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} \, dt}$$

Incomplete gamma function.

Q4

λ has PDF $\dfrac{aK^a}{\lambda^{a+1}}$, $\lambda > K$

$$f_X(x) = \int_K^\infty \lambda e^{-\lambda x} \cdot \frac{aK^a}{\lambda^{a+1}} \, d\lambda$$

$$= aK^a \int_K^\infty \lambda^{-a} e^{-\lambda x} \, d\lambda$$

$$\boxed{\begin{aligned} y &= \lambda x \\ \lambda &= \frac{y}{x} \\ d\lambda &= \frac{dy}{x} \end{aligned}}$$

$$= aK^a \int_{xK}^\infty \left(\frac{y}{x}\right)^{-a} e^{-y} \, \frac{dy}{x}$$

$$= aK^a x^{a-1} \int_{xK}^\infty y^{-a} e^{-y} \, dy$$

$$= aK^a x^{a-1} \, \Gamma(1-a, xK)$$

$$\boxed{\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} \, dt}$$

Complementary Incomp Gamma Functio

LECTURE

22 NOVEMBER

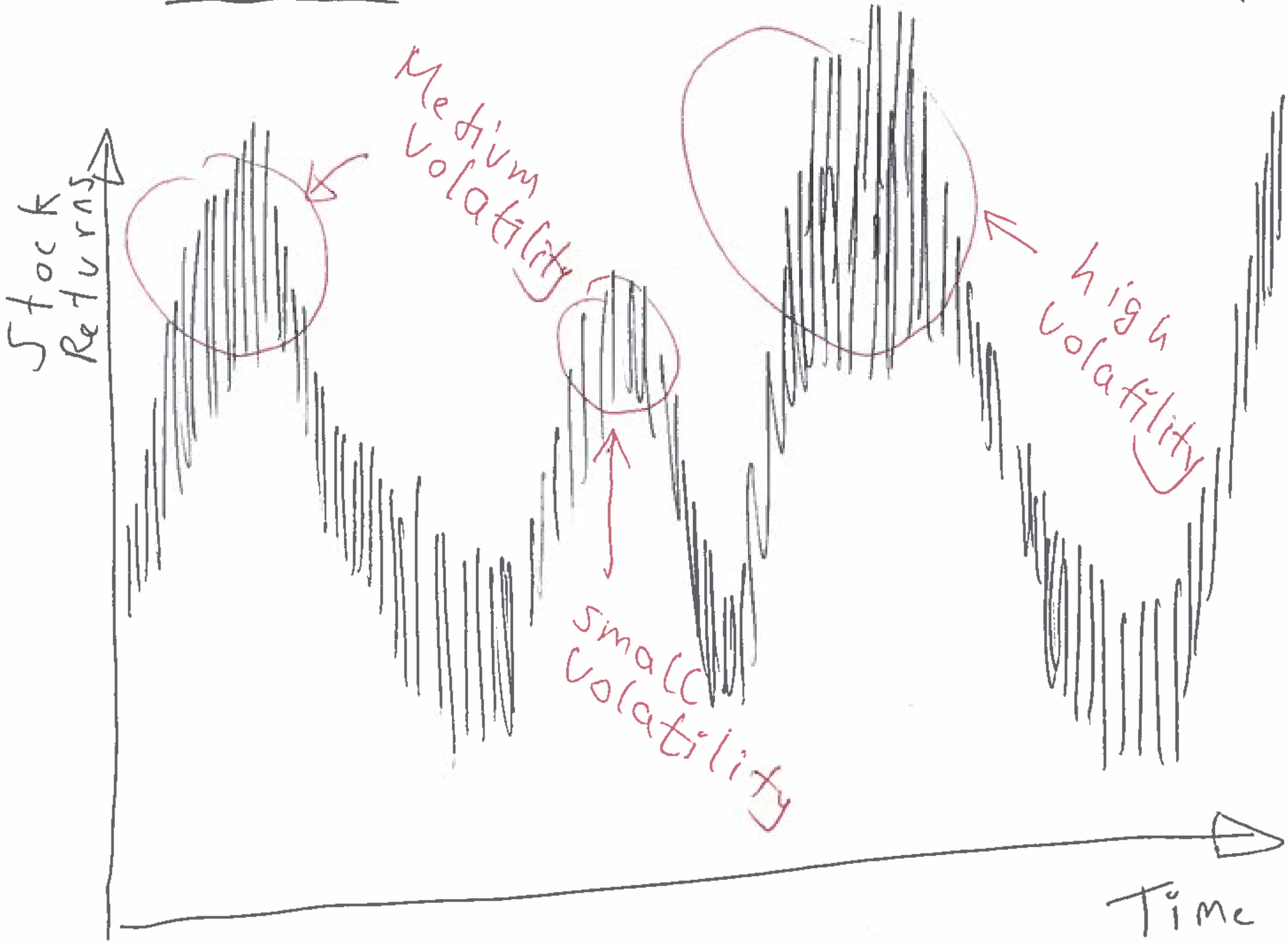
9:00-10:00AM

MATH3/4/68181

Bonus Question

- 200 students
- Bonus Q will be different
- Work independently
- Date 5th Dec Mon
- Deadline 23rd Dec Fri
- the bonus Q will be ~~sent~~ emailed to you as soon as you complete the UEQ.

Models for Stock Returns



X = Stock Returns at time t

V = volatility at time t

Both X and V as RVs.

X = Observable RV

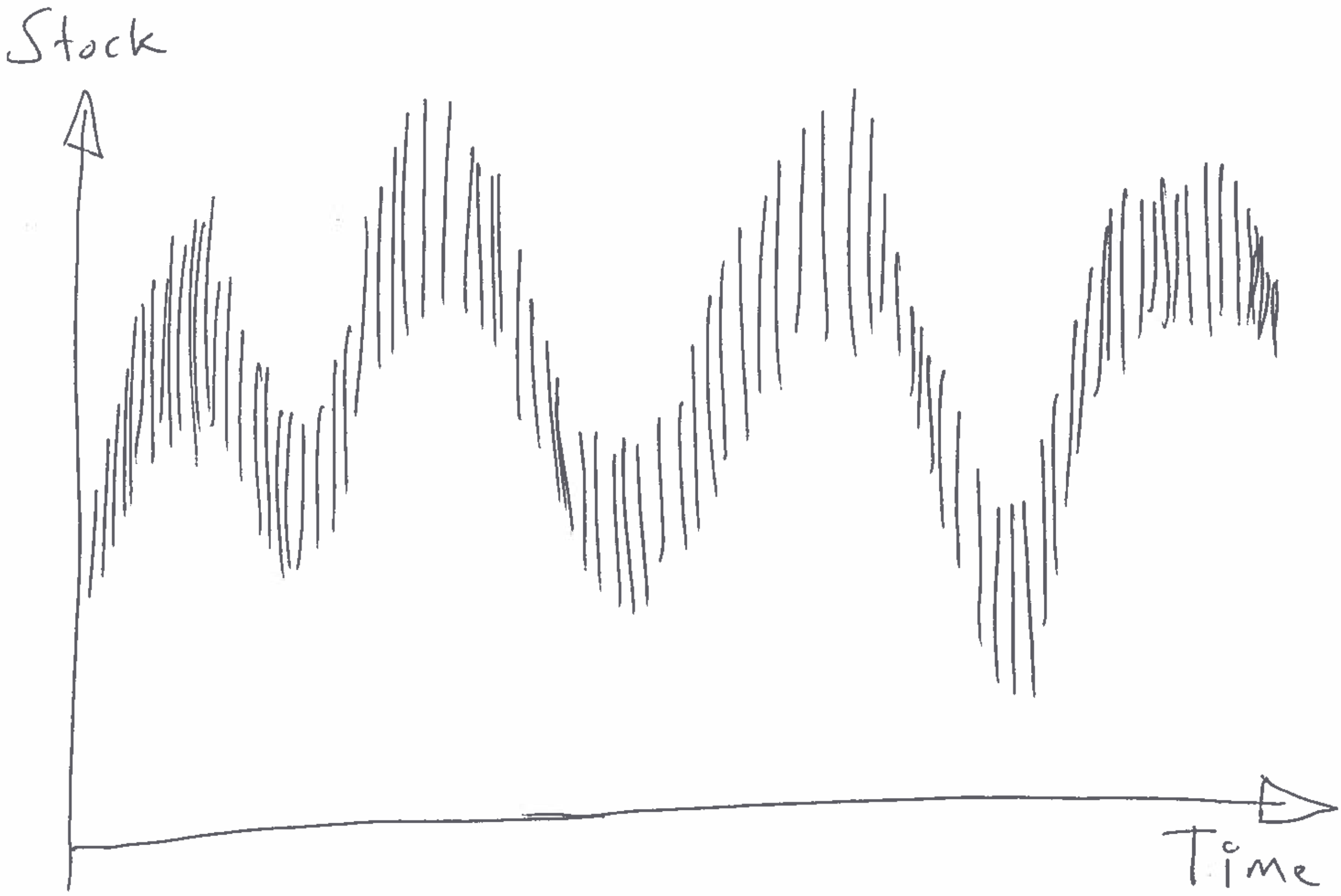
V = Not an observable RV

$$f_X(x) = \int_0^\infty \boxed{f_{X|V}(x|v)}\,\boxed{g(v)}\,dv$$

Cond PDF of X given V

PDF of V

Models for Stocks II



Let X_t = Stock at time t

$$X_t = (X_t - X_{t-1}) + (X_{t-1} - X_{t-2}) + \cdots + (X_1 - X_0) + X_0$$

$$\Rightarrow X_t - X_0 = \overbrace{(X_t - X_{t-1})}^{Z_t} + \overbrace{(X_{t-1} - X_{t-2})}^{Z_{t-1}} + \cdots + \underbrace{(X_1 - X_0)}_{Z_1}$$

$$= Z_t + Z_{t-1} + \cdots + Z_1$$

$$= \sum_{i=1}^{t} Z_i$$

Suppose X_0 is a fixed value

$$X_t = X_0 + \sum_{i=1}^{t} Z_i$$

How to forecast X_t for large t?

$$E(X_t - X_0) = \sum_{i=1}^{t} E(Z_i)$$

$$E\left[(X_t - X_0)^2\right] = E\left[\left(\sum_{i=1}^{t} Z_i\right)^2\right]$$

$$= \sum_{i=1}^{t} E\left[(Z_i)^2\right] + \sum_{i \neq j} E\left[Z_i Z_j\right]$$

$$E\left[(X_t - X_0)^3\right] = E\left[\left(\sum_{i=1}^{t} Z_i\right)^3\right]$$

$$= \sum_{i=1}^{t} E(Z_i^3) + \sum_{\substack{\text{all } (i,j,k) \\ \text{are distinct but two}}} E(Z_i Z_j Z_k)$$

$$+ \sum_{\text{all distinct}} E(Z_i Z_j Z_k)$$

Assume $Z_1, Z_2, \dots, Z_t$ are IID.

$$E(X_t - X_0) = t \cdot E(Z)$$

$$E\left[(X_t - X_0)^2\right] = t \cdot E(Z^2) + t(t-1)\left(E(Z)\right)^2$$

$$E\left[(X_t - X_0)^3\right] = t \cdot E(Z^3) + 3t(t-1)E(Z^2)E(Z) + t(t-1)(t-2)\left(E(Z)\right)^3$$

$$\begin{aligned}
Var(X_t - X_0) &= E\left[(X_t - X_0)^2\right] - \left(E(X_t - X_0)\right)^2 \\
&= t \cdot E(Z^2) + t(t-1)\left(E(Z)\right)^2 - t^2\left(E(Z)\right)^2 \\
&= t \cdot E(Z^2) - t \cdot \left(E(Z)\right)^2 \\
&= t \cdot Var(Z).
\end{aligned}$$

eg

Suppose Z_i are indep $N(\mu_i, \sigma_i^2)$ RVs.

$$X_t - X_0 = \sum_{i=1}^{t} Z_i$$

$$\sim N\left(\sum_{i=1}^{t} \mu_i , \sum_{i=1}^{t} \sigma_i^2\right)$$

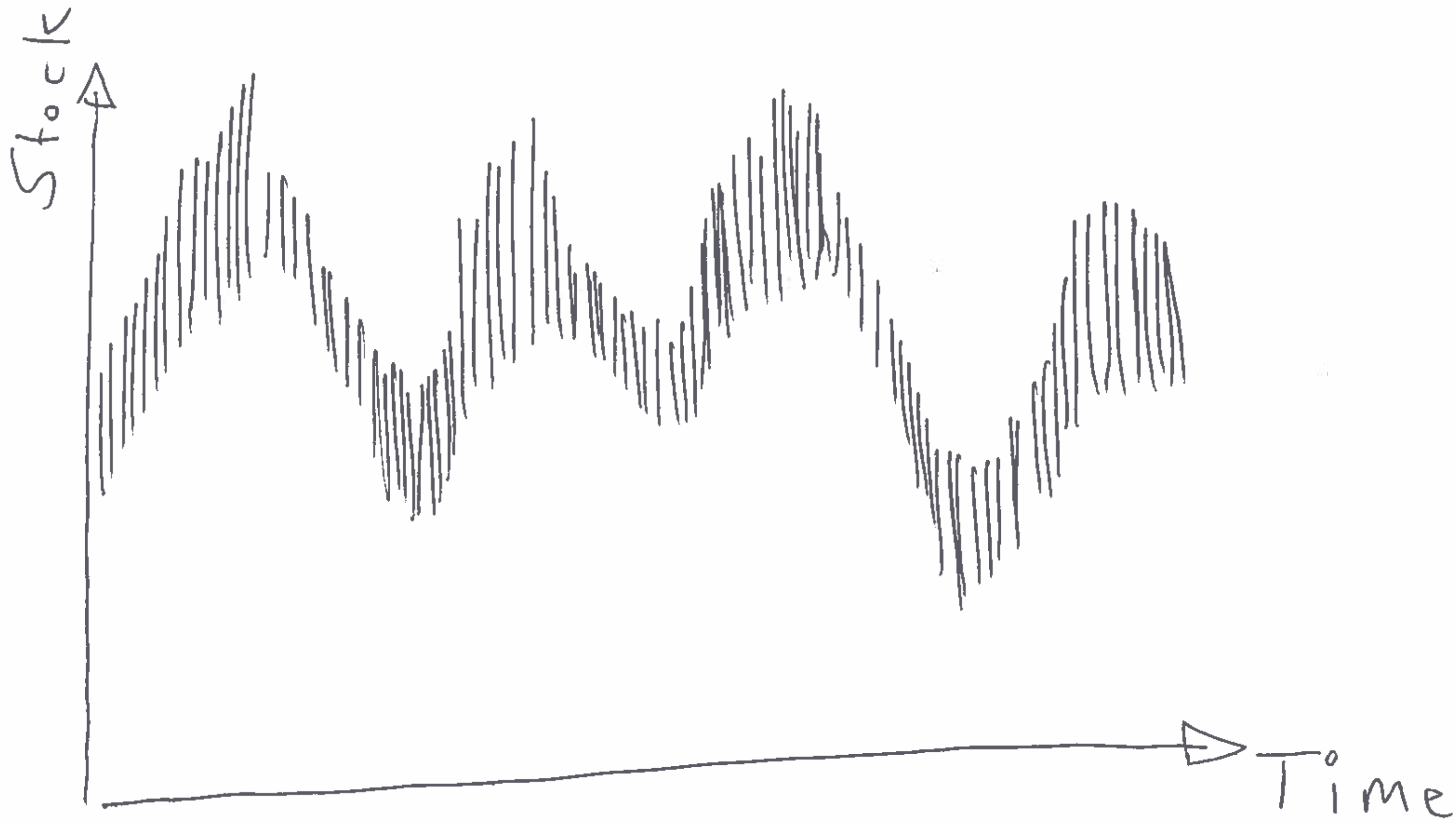
$$E(X_t - X_0) = \sum_{i=1}^{t} \mu_i$$

$$E\left[(X_t - X_0)^2\right] = \left(\sum_{i=1}^{t} \mu_i\right)^2 + \sum_{i=1}^{t} \sigma_i^2$$

$$E\left[(X_t - X_0)^3\right] = ?$$

$$E\left[(X_t - X_0)^4\right] = ?$$

Models for Stocks III



Let X_t = Stock value at time t

$$X_t = \left(\frac{X_t}{X_{t-1}}\right)\cdot\left(\frac{X_{t-1}}{X_{t-2}}\right)\cdots\left(\frac{X_2}{X_1}\right)\left(\frac{X_1}{X_0}\right)\cdot X_0$$

$$\Rightarrow \frac{X_t}{X_0} = \underbrace{\left(\frac{X_t}{X_{t-1}}\right)}_{Z_t}\cdot\underbrace{\left(\frac{X_{t-1}}{X_{t-2}}\right)}_{Z_{t-1}}\cdots\underbrace{\left(\frac{X_2}{X_1}\right)}_{Z_2}\cdot\underbrace{\left(\frac{X_1}{X_0}\right)}_{Z_1}$$

$$= Z_t \cdot Z_{t-1} \cdots Z_2 Z_1$$

$$= \prod_{i=1}^{t} Z_i$$

Suppose X_0 is a fixed value.

$$E\left(\frac{X_t}{X_0}\right) = E\left(\prod_{i=1}^{t} Z_i\right)$$

$$E\left[\left(\frac{X_t}{X_0}\right)^2\right] = E\left(\prod_{i=1}^{t} Z_i^2\right)$$

In general,

$$E\left[\left(\frac{X_t}{X_0}\right)^n\right] = E\left(\prod_{i=1}^{t} Z_i^n\right)$$

If Z_i are indep RVs then

$$= \prod_{i=1}^{t} E\left(Z_i^n\right)$$

EXAMPLE CLASS

22 NOVEMBER

10:00-11:00AM

MATH3/4/68181

Q1

$X =$ Stock Returns

$X \mid \lambda \sim \mathrm{Exp}(\lambda)$

volatility

$\lambda \sim \mathrm{Exp}(a)$

$$f_X(x) = \int_0^\infty f_{X|\lambda}(x|\lambda)\; g(\lambda)\; d\lambda$$

$$= \int_0^\infty \lambda e^{-\lambda x}\; a e^{-a\lambda}\; d\lambda$$

$$= a\int_0^\infty \lambda e^{-(x+a)\lambda}\; d\lambda$$

$$\boxed{\begin{aligned} \text{Set} \quad y &= (x+a)\lambda \\ \lambda &= \frac{y}{x+a} \\ d\lambda &= \frac{dy}{x+a} \end{aligned}}$$

$$= a\int_0^\infty \frac{y}{x+a}\, e^{-y}\, \frac{dy}{x+a}$$

$$= \frac{a}{(x+a)^2}\int_0^\infty y e^{-y}\, dy = \frac{a}{(x+a)^2}.$$

Suppose $X_1, X_2, \ldots, X_n$ is a random sample on X. The likelihood of a is

$$L(a) = \prod_{i=1}^{n} \frac{a}{(x_i + a)^2} = \frac{a^n}{\prod_{i=1}^{n} (x_i + a)^2}$$

$$\log L = n \log a - 2 \sum_{i=1}^{n} \log (x_i + a)$$

$$\frac{d \log L}{da} = \frac{n}{a} - 2 \sum_{i=1}^{n} \frac{1}{x_i + a}$$

The MLE of a is the root of

$$\frac{n}{a} = 2 \sum_{i=1}^{n} \frac{1}{x_i + a}$$

Q2 $\lambda \sim \text{Uni}[a, b]$

$$f_X(x) = \int_a^b \lambda e^{-\lambda x} \cdot \frac{1}{b-a} \cdot d\lambda$$

$$= \frac{1}{b-a} \int_a^b \lambda e^{-\lambda x} \, d\lambda$$

Parts

$$\overset{\downarrow}{=} \frac{1}{b-a} \left\{ \left[\lambda \cdot \frac{e^{-\lambda x}}{(-x)} \right]_a^b + \frac{1}{x} \int_a^b e^{-\lambda x} d\lambda \right\}$$

$$= \frac{1}{b-a} \left\{ -\frac{b e^{-bx}}{x} + \frac{a e^{-ax}}{x} + \frac{1}{x} \left[\frac{e^{-\lambda x}}{(-x)} \right]_a^b \right\}$$

$$= \frac{1}{b-a} \left\{ -\frac{b e^{-bx}}{x} + \frac{a e^{-ax}}{x} - \frac{e^{-bx} - e^{-ax}}{x^2} \right\}$$

$$L(a, b) = \prod_{i=1}^{n} f_X(x_i)$$

Q3 λ has PDF $a\lambda^{a-1}, \; 0<\lambda<1$

$$f_X(x) = \int f_{X|\lambda}(x|\lambda)\, g(\lambda)\, d\lambda$$

$$= \int_0^1 \lambda e^{-\lambda x} \cdot a\lambda^{a-1}\, d\lambda$$

$$= a\int_0^1 \lambda^a e^{-\lambda x}\, d\lambda$$

$$\boxed{\text{Set } y = \lambda x \Rightarrow \lambda = \frac{y}{x} \Rightarrow d\lambda = \frac{dy}{x}}$$

$$= a\int_0^x \left(\frac{y}{x}\right)^a e^{-y}\, \frac{dy}{x}$$

$$= a x^{-a-1} \int_0^x y^a e^{-y}\, dy$$

$$= a x^{-a-1}\, \gamma(a+1, x)$$

$$\boxed{\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t}\, dt}$$

Incomplete Gamma Function

Please complete the estimation part by yourself.

Q4

λ has PDF $\dfrac{a K^a}{\lambda^{a+1}}, \quad \lambda > K$

$$f_X(x) = \int_K^\infty \lambda e^{-\lambda x} \cdot \frac{a K^a}{\lambda^{a+1}} \, d\lambda$$

$$= a K^a \int_K^\infty \frac{1}{\lambda^a} e^{-\lambda x} \, d\lambda$$

$$\boxed{\text{Set } y = \lambda x \Rightarrow \lambda = \frac{y}{x} \Rightarrow d\lambda = \frac{dy}{x}}$$

$$= a K^a \int_{Kx}^\infty \frac{x^a}{y^a} e^{-y} \frac{dy}{x}$$

$$= a K^a x^{a-1} \int_{Kx}^\infty y^{-a} e^{-y} \, dy$$

$$= a K^a x^{a-1} \, \Gamma(1-a, Kx)$$

$$\boxed{\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} \, dt}$$

Complementary Incomplete Gamma Fn

Please complete estimation part.

LECTURE

24 NOVEMBER

12:00-13:00PM

MATH4/68181

Copulas

What is a copula?

A copula is a function from $[0,1] \times [0,1]$ to $[0,1]$ that satisfies certain conditions. Usually denoted by C.

Ways to construct copulas:

1) Suppose (X, Y) with joint CDF $F_{X,Y}$. Then

$$C(u,v) = F_{X,Y}\left(F_X^{-1}(u), F_Y^{-1}(v)\right)$$

is a copula, where F_X denotes the CDF of X & F_Y denotes the CDF of Y. Every joint CDF has a corresponding copula.

2) Suppose (X, Y) with marginal CDFs F_X, F_Y. Then

$$F_{X,Y}(x,y) = C(F_X(x), F_Y(y)) \quad (*)$$

is a valid joint CDF of (X, Y).

Every copula can be used to define a joint CDF of (X, Y).

3) If $C(u, v) = uv$ then $(*)$ reduces to

$$F_{X,Y}(x,y) = F_X(x)\, F_Y(y),$$

implying that X & Y are completely independent. $C(u,v) = uv$ is known as the independence copula.

4) If $C(u, v) = \min(u, v)$ then (*) reduces to

$$F_{X,Y}(x, y) = \min\left[F_X(x), F_Y(y)\right],$$

implying that X and Y are <u>completely dependent</u>.

5) <u>Definition of Copula</u>: $C : [0,1] \times [0,1] \longrightarrow [0,1]$ is a copula if it satisfies

i) $C(u, 0) = 0$

ii) $C(0, v) = 0$

iii) $C(u, 1) = u \quad \forall u$

iv) $C(1, v) = v \quad \forall v$

v) $\dfrac{\partial C(u, v)}{\partial u} \geqslant 0 \quad \forall u$

vi) $\dfrac{\partial C(u, v)}{\partial v} \geqslant 0 \quad \forall v$

Bivariate normal copula

$$\boxed{C(u,v) = \Phi_2\left(\Phi^{-1}(u), \Phi^{-1}(v)\right)}$$

Joint CDF of a bivariate normal RV

CDF of $N(0,1)$

biv normal distn is not a good model for financial data.

Bivariate t copula

$$C(u,v) = T_2\left(t_\nu^{-1}(u),\ t_\nu^{-1}(v)\right)$$

t_ν = CDF of a univariate Student's t distribution with ν degrees of freedom

T_2 = Joint CDF of a bivariate Student's t distribution with ν degrees of freedom.

Q3

$$C(u,v) = uv\,e^{-\theta \log u \log v}$$

i) $$C(u,0) = u \cdot 0 \cdot e^{-\theta \log u \cdot \log 0}$$
$$= 0$$

ii) $$C(0,v) = 0 \cdot v \cdot e^{-\theta \log 0 \cdot \log v}$$
$$= 0$$

iii) $$C(u,1) = u \cdot 1 \cdot e^{-\theta \log u \cdot \log 1}$$
$$= u$$

iv) $$C(1,v) = 1 \cdot v \cdot e^{-\theta \log 1 \cdot \log v}$$
$$= v$$

v) $$\frac{\partial C}{\partial u} = v \cdot e^{-\theta \log u \log v}$$
$$+ \cancel{u} v\, e^{-\theta \log u \cdot \log v} \cdot \left(-\frac{\theta}{\cancel{u}} \log v\right)$$

$$= (v - \theta v \log v)\, e^{-\theta \log u \log v}$$
$$= v\,(1 - \theta \log v)\, e^{-\theta \log u \log v}$$
$$\geq 0$$

vi) $$\frac{\partial C}{\partial v} = u\underbrace{\left(1 - \theta \log u\right)}_{\geq 0} \cdot e^{-\theta \log u \log v}$$

$$\geq 0$$

So, C is a copula.

LECTURE

25 NOVEMBER

9:00-10:00AM

MATH3/4/68181

- Unit Evaluation Questionaires will open on Monday 28 Nov
- Already prepared 200 different Qs.
- Will email the Q to you as soon as you complete UEQ.
- Deadine : 12:00 noon, 23 Dec Friday
- Email your answer to me as 1 single file.

Models for Stock

i) based on taking volatility as a RV.

X = Stock returns (observable)
V = Volatility (not observable)

The PDF of X

$$f_X(x) = \int_0^\infty \underbrace{f_{X|V}(x|v)}_{\text{Cond PDF of } X \text{ given } V} \underbrace{g(v)}_{\text{PDF of } V} dv$$

ii) X_t = Stock at time t

$$X_t - X_0 = \sum_{i=1}^{t} Z_i$$

$$E(X_t - X_0)^n = E\left(\sum_{i=1}^{t} Z_i\right)^n$$

iii) X_t = Stock at time t

$$\frac{X_t}{X_0} = \prod_{i=1}^{t} Z_i$$

$$E\left[\left(\frac{X_t}{X_0}\right)^n\right] = E\left[\prod_{i=1}^{t} Z_i^n\right]$$

eg 1

Suppose Z_i are indep RVs.

$$E\left[\left(\frac{X_t}{X_0}\right)^n\right] = \prod_{i=1}^{t} E(Z_i^n)$$

In particular

$$E\left[\left(\frac{X_t}{X_0}\right)\right] = \prod_{i=1}^{t} E(Z_i)$$

$$\mathrm{Var}\left[\frac{X_t}{X_0}\right] = \prod_{i=1}^{t} E(Z_i^2) - \prod_{i=1}^{t}\left(E(Z_i)\right)^2$$

eg 2

Suppose Z_i are IID.

$$E\left[\left(\frac{X_t}{X_0}\right)^n\right] = \prod_{i=1}^{t} E(Z^n) = \left(E(Z^n)\right)^t$$

In particular,

$$E\left[\left(\frac{X_t}{X_0}\right)\right] = \left(E(Z)\right)^t,$$

$$Var\left[\frac{X_t}{X_0}\right] = \left(E(Z^2)\right)^t - \left(E(Z)\right)^{2t}$$

eg 3

Suppose Z_i are indep $LN(\mu_i, \sigma_i^2)$

Lognormal

$$\frac{X_t}{X_0} = \prod_{i=1}^{t} Z_i$$

$$\Rightarrow \log\left(\frac{X_t}{X_0}\right) = \sum_{i=1}^{t} \log Z_i$$

$$\overset{\text{Math20802}}{=} \sum_{i=1}^{t} N(\mu_i, \sigma_i^2)$$

$$\overset{\text{Math20802}}{=} N\left(\sum_{i=1}^{t} \mu_i, \sum_{i=1}^{t} \sigma_i^2\right)$$

$$\Rightarrow \frac{X_t}{X_0} \sim LN\left(\sum_{i=1}^{t} \mu_i, \sum_{i=1}^{t} \sigma_i^2\right)$$

$$\Rightarrow E\left(\frac{X_t}{X_0}\right) = e^{\sum_{i=1}^{t} \mu_i + \frac{1}{2}\sum_{i=1}^{t} \sigma_i^2}$$

$$\mathrm{Var}\left(\frac{X_t}{X_0}\right) = \left[e^{\sum_{i=1}^{t} \sigma_i^2} - 1\right] \cdot e^{2\sum_{i=1}^{t} \mu_i + \sum_{i=1}^{t} \sigma_i^2}$$

If $X \sim LN(\mu, \sigma 2)$ then

$$E(X) = e^{\mu + \frac{\sigma^2}{2}}$$

$$Var(X) = \left[e^{\sigma^2} - 1\right] e^{2\mu + \sigma^2}$$

Income Modelling

Z = Reported income (Observable RV)

X = True income (Not observable RV)

~~Does~~ Is the distribution of Z consistent with ~~the~~ the distribution of X?

i) Over reporting of income

$$Z = \frac{X}{Y}, \quad Y \text{ is a RV in } (0,1)$$

ii) Under reporting of income

$$Z = XY, \quad Y \text{ is a RV is } (0,1)$$

i) Over reporting

Suppose Y has the PDF

$$f_Y(y) = c\, y^{c-1}, \quad 0 < y < 1$$

(Power function PDF).

Then X is Pareto distributed if and only if Z is also Pareto distributed.

Theorem I

ii) Under reporting Suppose Y has the PDF $f_Y(y) = c y^{c-1}$, $0 < y < 1$. Then X is Pareto distributed if and only if Z is also Pareto distributed.

Theorem 2

Theorems 1 and 2 imply that the distribution of Z is consistent with the distribution of X. Hence, Z can be modeled by a Pareto distribution without loss of generality.

Home work: prove Theorems 1 and 2.

EXAMPLE CLASS

28 NOVEMBER

12:00-13:00PM

MATH3/4/68181

Q1

(a) $f(x) = \dfrac{dF(x)}{dx}$

$$= e^{-(1+\xi x)^{-\frac{1}{\xi}}} \cdot (-1)\left(-\frac{1}{\xi}\right)(1+\xi x)^{-\frac{1}{\xi}-1} \cdot \xi$$

$$= (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}}$$

(b) $E(X^n) = \int x^n (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$

$$= \int \left(\frac{1+\xi x - 1}{\xi}\right)^n (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

Bin exp

$$= \xi^{-n} \int \sum_{k=0}^{n} \binom{n}{k} (1+\xi x)^k (-1)^{n-k} (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int (1+\xi x)^{k-\frac{1}{\xi}-1} \cdot e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

Set $y = (1+\xi x)^{-\frac{1}{\xi}}$

$1+\xi x = y^{-\xi}$

$x = \dfrac{y^{-\xi}-1}{\xi} \Rightarrow \dfrac{dx}{dy} = -y^{-\xi-1}$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k}$$

$$\cdot \int (y^{-\xi})^{k - \frac{1}{\xi} - 1} e^{-y} \left(-y^{-\xi-1}\right) dy$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^{\infty} y^{-\xi k} e^{-y} \, dy$$

$\boxed{\xi > 0}$: $\infty > 1 + \xi x > 0$

$\boxed{\xi < 0}$: $\infty > 1 + \xi x > 0$

$\boxed{\xi = 0}$: $1 + \xi x = 1 > 0$

$$y = (1 + \xi x)^{-\frac{1}{\xi}}$$

$\boxed{+\infty > 1 + \xi x > 0} \Rightarrow \boxed{\infty > y > 0}$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Gamma(1 - \xi k)$$

$$\boxed{\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} \, dt}$$

Q2

$$f(x) = (-1)\cdot\left(-\frac{1}{\xi}\right)(1+\xi x)^{-\frac{1}{\xi}-1}\cdot\xi$$

$$= (1+\xi x)^{-\frac{1}{\xi}-1} \to e^{-x} \quad \xi \to 0$$

$$E(X^n) = \int x^n \cdot (1+\xi x)^{-\frac{1}{\xi}-1}\,dx$$

$$= \int \left(\frac{1+\xi x-1}{\xi}\right)^n (1+\xi x)^{-\frac{1}{\xi}-1}\,dx$$

Bin Exp

$$= \xi^{-n}\int \sum_{k=0}^{n}\binom{n}{k}(1+\xi x)^k(-1)^{n-k}(1+\xi x)^{-\frac{1}{\xi}-1}\,dx$$

$$= \xi^{-n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\int (1+\xi x)^{k-\frac{1}{\xi}-1}\,dx$$

$$= \xi^{-n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\underbrace{\left[\frac{(1+\xi x)^{k-\frac{1}{\xi}}}{\left(k-\frac{1}{\xi}\right)\xi}\right]}_{=\Delta}$$

$$\Delta = \frac{1}{k\xi-1}\cdot(0-1) = \frac{1}{1-k\xi} \quad \text{if } \xi \geq 0$$

$$\Delta = \frac{1}{k\xi-1}(0-1) = \frac{1}{1-k\xi} \quad \text{if } \xi < 0$$

LECTURE

29 NOVEMBER

9:00-10:00AM

MATH3/4/68181

Bonus Q

- level > final exam
- independently
- 12:00 Fri 23 Dec
- email

Income Modeling

X = True income (not an observable RV)

Z = Reported income (observable RV)

- Over-reported income

$$Z = \frac{X}{Y}, \quad Y \text{ is a RV in } (0,1)$$

- Under-reported income

$$Z = XY, \quad Y \text{ is a RV in } (0,1)$$

Is the model for Z consistent with the model for X?

Theorem 1 Suppose Y is a RV with PDF $f_Y(y) = c\,y^{c-1}$, $0<y<1$. Then $Z = \frac{X}{Y}$ is Pareto distributed if and only if X is also Pareto distributed.

Theorem 2 Suppose Y is a RV with PDF $f_Y(y) = c\,y^{c-1}$, $0<y<1$. Then $Z = XY$ is Pareto distributed if and only if X is also Pareto distributed.

Homework: Prove Theorem 2.

Proof of Theorem 11:

i) Suppose Z is Pareto distributed with CDF

$$F_Z(z) = 1 - \left(\frac{K}{z}\right)^a, \quad z > K.$$

We want to show that X is also Pareto distributed.

$$\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(ZY \leq x) \\
&= P\left(Z \leq \frac{x}{Y}\right) \\
&= \int_0^1 P\left(Z \leq \frac{x}{y}\right) f_Y(y)\, dy \\
&= \int_0^1 F_Z\left(\frac{x}{y}\right) f_Y(y)\, dy \\
&= \int_0^1 \left[1 - \left(\frac{Ky}{x}\right)^a\right] c\, y^{c-1}\, dy \\
&= c \int_0^1 y^{c-1}\, dy - \frac{cK^a}{x^a} \int_0^1 y^{a+c-1}\, dy \\
&= c \left[\frac{y^c}{c}\right]_0^1 - \frac{cK^a}{x^a} \left[\frac{y^{a+c}}{a+c}\right]_0^1
\end{aligned}$$

Total Prob Rule

$$= 1 - \frac{c\,k^a}{x^a} \cdot \frac{1}{a+c}$$

$$= 1 - \frac{\frac{c}{a+c} \cdot k^a}{x^a}$$

$$= 1 - \frac{\left[\left(\frac{c}{a+c}\right)^{\frac{1}{a}} k\right]^a}{x^a}$$

$$= 1 - \frac{L^a}{x^a}, \quad \text{where } L = \left(\frac{c}{a+c}\right)^{\frac{1}{a}} k$$

$\Rightarrow$ X is Pareto distributed.

ii) Assume that X is Pareto distributed with CDF

$$F_X(x) = 1 - \left(\frac{M}{x}\right)^b, \quad x > M$$

We want to show that Z is also Pareto RV.

$$F_Z(z) = P(Z \leq z)$$

$$= P\left(\frac{X}{Y} \leq z\right)$$

$$= P(X \leq z \cdot Y)$$

Total Prob Rule

$$= \int_0^1 P(X \leq z \cdot y)\, f_Y(y)\, dy$$

$$= \int_0^1 F_X(z \cdot y)\, f_Y(y)\, dy$$

$$= \int_0^1 \left[1 - \left(\frac{M}{z \cdot y}\right)^b\right] c \cdot y^{c-1}\, dy$$

$$= c \cdot \int_0^1 y^{c-1}\, dy - \frac{c M^b}{z^b} \int_0^1 y^{c-b-1}\, dy$$

$$= c \left[\frac{y^c}{c}\right]_0^1 - \frac{c M^b}{z^b} \left[\frac{y^{c-b}}{c-b}\right]_0^1$$

$$= 1 - \frac{c M^b}{z^b} \cdot \frac{1}{c-b}$$

$$= 1 - \frac{\frac{c}{c-b} M^b}{Z^b}$$

$$= 1 - \frac{\left[\left(\frac{c}{c-b}\right)^{\frac{1}{b}} M\right]^b}{Z^b}$$

$$= 1 - \frac{N^b}{Z^b}, \quad \text{where } N = \left(\frac{c}{c-b}\right)^{\frac{1}{b}} M$$

$\Rightarrow$ Z is also Pareto distributed.

The proof of Theorem 1 is complete.

3rd years

- Fri 2 Dec – GARCH models (last lecture topic)
- next 2 weeks (weeks 11 & 12) will be revision for the final exam.

4/6 years

- Thurs 1 Dec – biv. extreme value models
- Thurs 7 Dec – " " " "
- Thurs 14 Dec – revision class

EXAMPLE CLASS

29 NOVEMBER

10:00-11:00AM

MATH3/4/68181

Q1

a) $$f(x) = \frac{dF(x)}{dx}$$

$$= e^{-(1+\xi x)^{-\frac{1}{\xi}}} (-1)\left(-\frac{1}{\xi}\right) \cdot (1+\xi x)^{-\frac{1}{\xi}-1} \cdot \xi$$

$$= (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}}$$

b) $$E(X^n) = \int x^n \cdot (1+\xi x)^{-\frac{1}{\xi}-1} \cdot e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

$$= \int \left(\frac{1+\xi x - 1}{\xi}\right)^n \cdot (1+\xi x)^{-\frac{1}{\xi}-1} \cdot e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

$$= \xi^{-n} \int ((1+\xi x) - 1)^n (1+\xi x)^{-\frac{1}{\xi}-1} \cdot e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

binomial exp

$$\overset{\downarrow}{=} \xi^{-n} \int \sum_{k=0}^{n} \binom{n}{k} (1+\xi x)^{k} (-1)^{n-k}$$

$$\cdot (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \cdot \int (1+\xi x)^{k-\frac{1}{\xi}-1}$$

$$\cdot e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

$$\boxed{\begin{aligned} &\text{Set} \quad y = (1+\xi x)^{-\frac{1}{\xi}} \\ &\Rightarrow y^{-\xi} = 1 + \xi x \\ &\Rightarrow x = \frac{y^{-\xi}-1}{\xi} \\ &\Rightarrow \frac{dx}{dy} = -y^{-\xi-1} \end{aligned}}$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_{\infty}^{0} (y^{-\xi})^{k-\frac{1}{\xi}-1}$$

$$\cdot e^{-y} \cdot (-y^{-\xi-1}) dy$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^\infty y^{-\xi k} e^{-y} \, dy$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \Gamma(1-\xi k)$$

To find out how these limits arise please consider the cases

$\xi = 0$

$\xi < 0$

$\xi > 0$

separately.

Q2

a) $$f(x) = \frac{dF(x)}{dx}$$

$$= (-1)\cdot\left(-\frac{1}{\xi}\right)\cdot(1+\xi x)^{-\frac{1}{\xi}-1}\cdot\xi$$

$$= (1+\xi x)^{-\frac{1}{\xi}-1}$$

b) $$E(X^n) = \int x^n \cdot (1+\xi x)^{-\frac{1}{\xi}-1}\,dx$$

$$= \int \left(\frac{1+\xi x-1}{\xi}\right)^n (1+\xi x)^{-\frac{1}{\xi}-1}\,dx$$

$$= \xi^{-n}\int (1+\xi x-1)^n (1+\xi x)^{-\frac{1}{\xi}-1}\,dx$$

bin exp

$$= \xi^{-n}\int \sum_{k=0}^{n}\binom{n}{k}(1+\xi x)^k(-1)^{n-k}\cdot(1+\xi x)^{-\frac{1}{\xi}-1}\,dx$$

$$= \xi^{-n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\underbrace{\int (1+\xi x)^{k-\frac{1}{\xi}-1}\,dx}_{=\Delta}$$

$$= \xi^{-n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n-k}}{1-k\xi}$$

$$\Delta = \left[\frac{(1+\xi x)^{k-\frac{1}{\xi}}}{\left(k-\frac{1}{\xi}\right)\cdot\xi} \right]_0^{\infty} \quad \text{if } \xi \geq 0$$

$$= 0 - \frac{1}{k\xi - 1} = \frac{1}{1-k\xi}$$

$$\Delta = \left[\frac{(1+\xi x)^{k-\frac{1}{\xi}}}{\left(k-\frac{1}{\xi}\right)\xi} \right]_0^{-\frac{1}{\xi}} \quad \text{if } \xi < 0$$

$$= 0 - \frac{1}{k\xi - 1} = \frac{1}{1-k\xi}$$

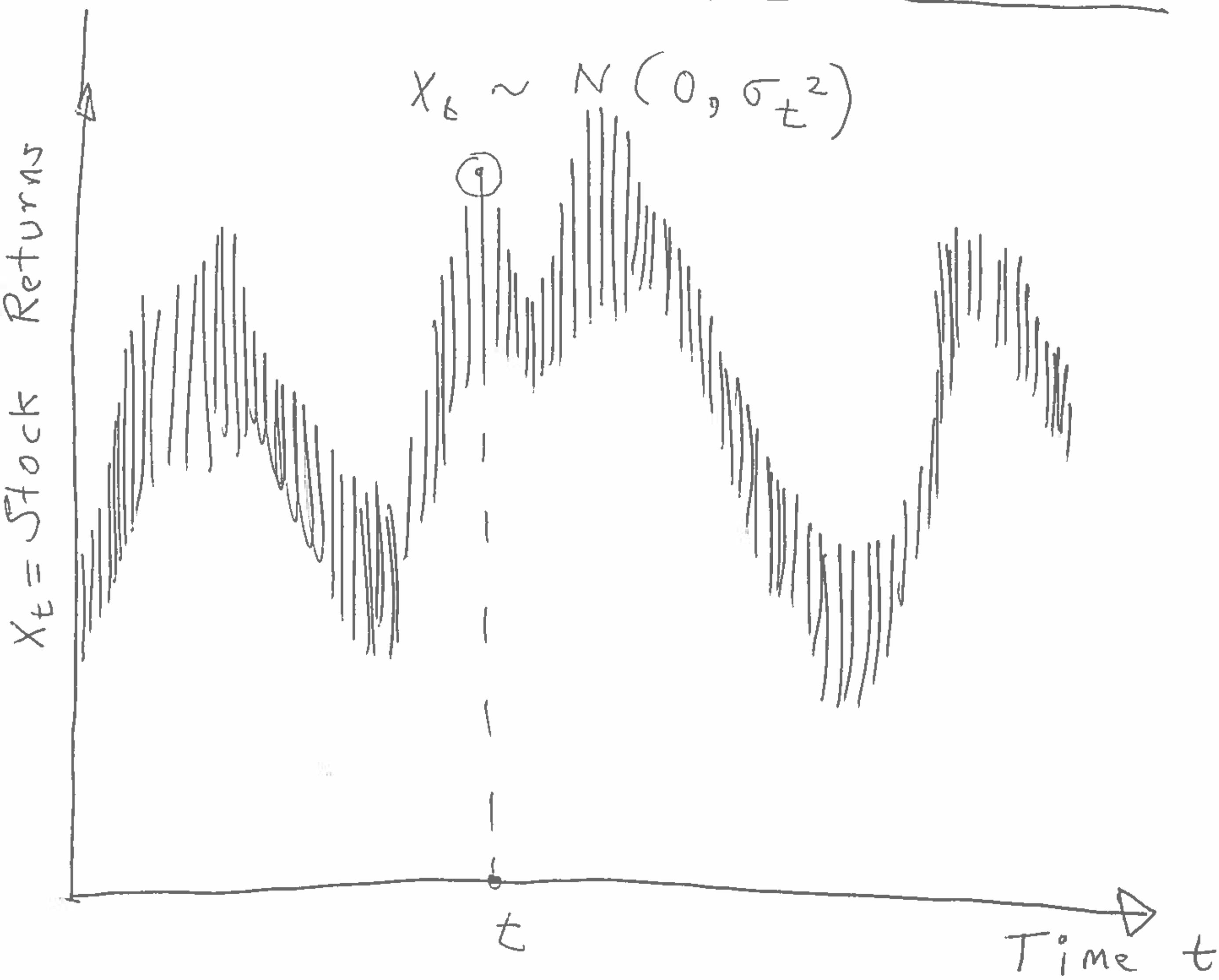
LECTURE

2 DECEMBER

9:00-10:00AM

MATH3/4/68181

GARCH Type Models



$$X_t \sim N(0, \sigma_t^2)$$

$$\Rightarrow \boxed{X_t = \sigma_t Z_t}$$

where $Z_t \sim N(0, 1)$

$$E(X_t) = E(\sigma_t Z_t) = \sigma_t E(Z_t) = 0$$

$$\begin{aligned} Var(X_t) &= Var(\sigma_t Z_t) \\ &= \sigma_t^2 \, Var(Z_t) = \sigma_t^2 \end{aligned}$$

σ_t = volatility process

Z_t = innovation process

- Z_t can follow any distribution, not just $N(0, 1)$.
- σ_t is usually taken a function past σ_s, $s < t$ (past volatilities) and past Z_s, $s < t$ (past innovations)

- ARCH(q) model

$$X_t = \sigma_t Z_t$$

where

$$\underbrace{\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2}$$

volatility at time t depends on the past q ~~innovations.~~ stock returns

- GARCH(p, q) model

$$X_t = \sigma_t Z_t$$

where

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2 \underbrace{+ \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2}$$

volatility at time t depends on the previous q stock returns as well as the previous p volatilities

- **NGARCH model**

$$X_t = \sigma_t Z_t$$

where

$$\sigma_t^2 = \omega + \alpha\left(X_{t-1} - \theta\sigma_{t-1}\right)^2 + \beta\sigma_{t-1}^2$$

volatility at time t is a function of the stock return on the previous day and volatility on the previous day.

- <u>QGARCH</u> <u>model</u>

$$X_t = \sigma_t Z_t$$

where

$$\sigma_t^2 = K + \alpha X_{t-1}^2 + \beta \sigma_{t-1}^2 + \phi X_{t-1}$$

volatility at time depends on the stock return on the previous day as well as the volatility on the previous day.

For a negative stock return on the previous day, the volatility on day t will be smaller.

For a positive stock return on day t-1, the volatility on day t will be larger.

- GJR- GARCH model

$$X_t = \sigma_t Z_t$$

where

$$\sigma_t^2 = K + \delta \sigma_{t-1}^2 + \alpha X_{t-1}^2 + \phi X_{t-1}^2 I_{t-1}$$

and

$$I_{t-1} = \begin{cases} 0 & \text{if } X_{t-1} \geqslant 0 \\ 1 & \text{if } X_{t-1} < 0 \end{cases}$$

If $X_{t-1} < 0$ then volatility on day t will be larger

If $X_{t-1} \geqslant 0$ then volatility on day t will be smaller

Ex 1

Consider the ARCH(1) model

$$X_t = \sigma_t Z_t$$

where $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$

and $Z_t \sim N(0,1)$.

Find the MLEs of α_0 and α_1.

$$Z_t = \frac{X_t}{\sigma_t} = \frac{X_t}{\sqrt{\alpha_0 + \alpha_1 X_{t-1}^2}}$$

$$\sim N(0,1)$$

So,

$$L(\alpha_0, \alpha_1) = \prod_{t=1}^{n} \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{Z_t^2}{2}} \right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\sum_{t=1}^{n} Z_t^2}$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\sum_{t=1}^{n} \frac{X_t^2}{\alpha_0 + \alpha_1 X_{t-1}^2}}$$

The log-likelihood is

$$\log L = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\frac{X_t^2}{\alpha_0 + \alpha_1 X_{t-1}^2}$$

The partial derivatives are

$$\frac{\partial \log L}{\partial \alpha_0} = \frac{1}{2}\sum_{t=1}^{n}\frac{X_t^2}{\left(\alpha_0 + \alpha_1 X_{t-1}^2\right)^2} = 0 \quad \text{— (1)}$$

$$\frac{\partial \log L}{\partial \alpha_1} = \frac{1}{2}\sum_{t=1}^{n}\frac{X_t^2 X_{t-1}^2}{\left(\alpha_0 + \alpha_1 X_{t-1}^2\right)^2} = 0 \quad \text{— (2)}$$

The MLEs of α_0 and α_1 are the simultaneous solutions of (1) and (2).

In R, fGARCH can compute the MLEs of GARCH type models.

Ex 2

Consider the GARCH (1, 1) model

$$X_t = \sigma_t Z_t$$

where

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

and $Z_t \sim N(0, 1)$.

Find the MLEs of α_0, α_1 and β_1.

$$Z_t = \frac{X_t}{\sigma_t} = \frac{X_t}{\sqrt{\alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2}}$$

$$\sim N(0, 1)$$

So,

$$L(\alpha_0, \alpha_1, \beta_1) = \prod_{t=1}^{n} \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{Z_t^2}{2}} \right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} \sum_{t=1}^{n} \frac{X_t^2}{(\alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2)}}$$

EXAMPLE CLASS

5 DECEMBER

12:00-13:00PM

MATH3/4/68181

Q1 ARCH(q) model:

$$e_t = \sigma_t Z_t$$

where $\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_q e_{t-q}^2$

$$Z_t \sim N(0, 1)$$

$$\begin{aligned}
E(e_t) &= E(\sigma_t Z_t) \\
&= E\{E[(\sigma_t Z_t \mid \sigma_t)]\} \\
&= E\{\sigma_t \underline{E(Z_t)}\} \\
&= E\{\sigma_t \cdot 0\} = 0
\end{aligned}$$

$$\begin{aligned}
E(e_t^2) &= E(\sigma_t^2 Z_t^2) \\
&= E\{E[\sigma_t^2 Z_t^2 \mid \sigma_t]\} \\
&= E\{\sigma_t^2 E[Z_t^2]\} \\
&= E\{\sigma_t^2 \cdot 1\} \\
&= E\{\sigma_t^2\} \\
&= E\{\alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_q e_{t-q}^2\} \\
&= \alpha_0 + \alpha_1 E(e_{t-1}^2) + \cdots + \alpha_q E(e_{t-q}^2
\end{aligned}$$

$$\Rightarrow E(e_t^2) = \alpha_0 + \alpha_1 E(e_{t-1}^2) + \cdots + \alpha_q E(e_{t-q}^2)$$

Assume <u>stationarity</u> and let
$E(e_t^2) = \sigma^2$.

$$\Rightarrow \sigma^2 = \alpha_0 + \alpha_1 \sigma^2 + \cdots + \alpha_q \sigma^2$$

$$\Rightarrow \sigma^2 = \frac{\alpha_0}{1 - \alpha_1 - \cdots - \alpha_q}.$$

Q2 GARCH (p, q) model

$$e_t = \sigma_t Z_t$$

where

$$\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_q e_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2$$

$$E(e_t) = E\{E(\sigma_t Z_t \mid \sigma_t)\}$$
$$= E\{\sigma_t E(Z_t)\}$$
$$= 0$$

$$E(\sigma_t^2) = E\{E(\sigma_t^2 Z_t^2 / \sigma_t)\}$$
$$= E\{\sigma_t^2 E(Z_t^2)\}$$
$$= E\{\sigma_t^2\}$$
$$= E\{\alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_q e_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2\}$$
$$= \alpha_0 + \alpha_1 E(e_{t-1}^2) + \cdots + \alpha_q E(e_{t-q}^2) + \beta_1 E(\sigma_{t-1}^2) + \cdots + \beta_p E(\sigma_{t-p}^2)$$

$$\sigma^2 = \alpha_0 + \alpha_1 \sigma^2 + \cdots + \alpha_q \sigma^2 + \beta_1 \sigma^2 + \cdots + \beta_p \sigma^2$$ assuming stationarity

$$\Rightarrow \quad \sigma^2 = \frac{\alpha_0}{1 - \alpha_1 - \dots - \alpha_q - \beta_1 - \dots - \beta_p}$$

Q3

NGARCH model

$$e_t = \sigma_t Z_t$$

where

$$\sigma_t^2 = w + (e_{t-1} - \theta\sigma_{t-1})^2 + \beta\sigma_{t-1}^2$$

$$Z_t \sim N(0,1)$$

$$E[e_t] = E\{E(\sigma_t Z_t \mid \sigma_t)\}$$
$$= E\{\sigma_t E(Z_t)\} = 0$$

$$E[e_t^2] = E\{E(\sigma_t^2 Z_t^2 \mid \sigma_t)\}$$
$$= E\{\sigma_t^2 E(Z_t^2)\}$$
$$= E\{\sigma_t^2\}$$
$$= E\{w + \alpha(e_{t-1} - \theta\sigma_{t-1})^2 + \beta\sigma_{t-1}^2\}$$
$$= E\{w + \alpha e_{t-1}^2 - 2\alpha\theta \boxed{e_{t-1}\sigma_{t-1}}$$
$$+ \alpha\theta^2\sigma_{t-1}^2 + \beta\sigma_{t-1}^2\}$$
$$= w + \alpha E(e_{t-1}^2) - 2\alpha\theta E(e_{t-1}\sigma_{t-1})$$
$$+ \cancel{(\alpha\theta^2 + \beta)} E(\sigma_{t-1}^2)$$

$$E(e_{t-1}\sigma_{t-1}) = E\{E(e_{t-1}\sigma_{t-1} \mid \sigma_{t-1})\}$$
$$= E\{\sigma_{t-1} E(e_{t-1})\}$$
$$= 0$$

$$E(e_t^2) = \omega + \alpha E(e_{t-1}^2) + (\alpha\theta^2 + \beta) E(\sigma_{t-1}^2)$$

$$\Rightarrow \sigma^2 = \omega + \alpha\sigma^2 + (\alpha\theta^2 + \beta)\sigma^2,$$

assuming stationarity

$$\Rightarrow \sigma^2 = \frac{\omega}{1 - \alpha - \alpha\theta^2 - \beta}.$$

LECTURE

6 DECEMBER

9:00-10:00AM

MATH3/4/68181

Exam

- 5 questions for yr 3 students, answer any 4.
- 2 hrs

- 8 questions for yrs 4 & 6 students, answer 2 of the first 3 questions and 4 of the remaining
- 3 hrs

- Details of material covered by the exam will be emailed to you later this week.

- Syllabus for Year 3 has been completed
- Years 4 & 6 have more to cover

Exam 2014/15

Q8 $X_1, X_2, \ldots, X_\alpha$ IID with CDF

$$F(x) = 1 - \left(\frac{K}{x}\right)^a , \quad x > K$$

Let $Y = \min(X_1, X_2, \ldots, X_\alpha)$

(a)
$$\begin{aligned}
F_Y(y) &= P(Y < y) \\
&= P(\min X_i < y) \\
&= 1 - P(\min X_i > y) \\
&= 1 - P(X_1 > y, X_2 > y, \ldots, X_\alpha > y) \\
\text{indep} \quad &= 1 - (P(X_1 > y))^\alpha \\
&= 1 - (1 - P(X_1 \leq y))^\alpha \\
&= 1 - \left(\cancel{1} - \left(\cancel{1} - \left(\frac{K}{y}\right)^a\right)\right)^\alpha \\
&= 1 - \left(\frac{K}{y}\right)^{a\alpha}
\end{aligned}$$

(b)
$$f_Y(y) = \frac{a\alpha K^{a\alpha}}{y^{a\alpha+1}}$$

(c) $$E(Y^n) = \int_k^\infty y^n \cdot \frac{a\alpha k^{a\alpha}}{y^{a\alpha+1}}\, dy$$

$$= a\alpha k^{a\alpha} \int_k^\infty y^{n-a\alpha-1}\, dy$$

$$= a\alpha k^{a\alpha} \left[\frac{y^{n-a\alpha}}{n-a\alpha} \right]_k^\infty$$

$$= a\alpha k^{a\alpha} \left[0 - \frac{k^{n-a\alpha}}{n-a\alpha} \right] \quad \text{if } n < a\alpha$$

$$= \frac{a\alpha k^n}{a\alpha - n} \quad \text{if } n < a\alpha$$

$$E(Y) = \frac{a\alpha k}{a\alpha - 1} \quad \text{if } 1 < a\alpha$$

$$Var(Y) = \frac{a\alpha k^2}{a\alpha - 2} - \left(\frac{a\alpha k}{a\alpha - 1}\right)^2 \quad \text{if } 2 < a\alpha$$

(d)

$$\mathrm{VaR}_p(Y) = F_Y^{-1}(p)$$

$$F_Y(y) = 1 - \left(\frac{K}{y}\right)^{a\alpha} = p$$

$$\Rightarrow \left(\frac{K}{y}\right)^{a\alpha} = 1 - p$$

$$\Rightarrow \frac{K}{y} = (1-p)^{\frac{1}{a\alpha}}$$

$$\Rightarrow y = K(1-p)^{-\frac{1}{a\alpha}}$$

$$\Rightarrow \mathrm{VaR}_p(Y) = K(1-p)^{-\frac{1}{a\alpha}}$$

(e)

$$ES_p(Y) = \frac{1}{p}\int_0^p F_Y^{-1}(t)\,dt$$

$$= \frac{K}{p}\int_0^p (1-t)^{-\frac{1}{a\alpha}}\,dt$$

$$= \frac{K}{p}\left[\frac{(1-t)^{1-\frac{1}{a\alpha}}}{(-1)\left(1-\frac{1}{a\alpha}\right)}\right]_0^p$$

$$= \frac{Ka\alpha}{p(1-a\alpha)}\left[(1-p)^{1-\frac{1}{a\alpha}} - 1\right]$$

(f) $$L(a, k) = \prod_{i=1}^{n} \left\{ \frac{a\alpha k^{a\alpha}}{y_i^{a\alpha+1}} I\{y_i \geq k\} \right\}$$

$$I\{A\} = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false} \end{cases}$$

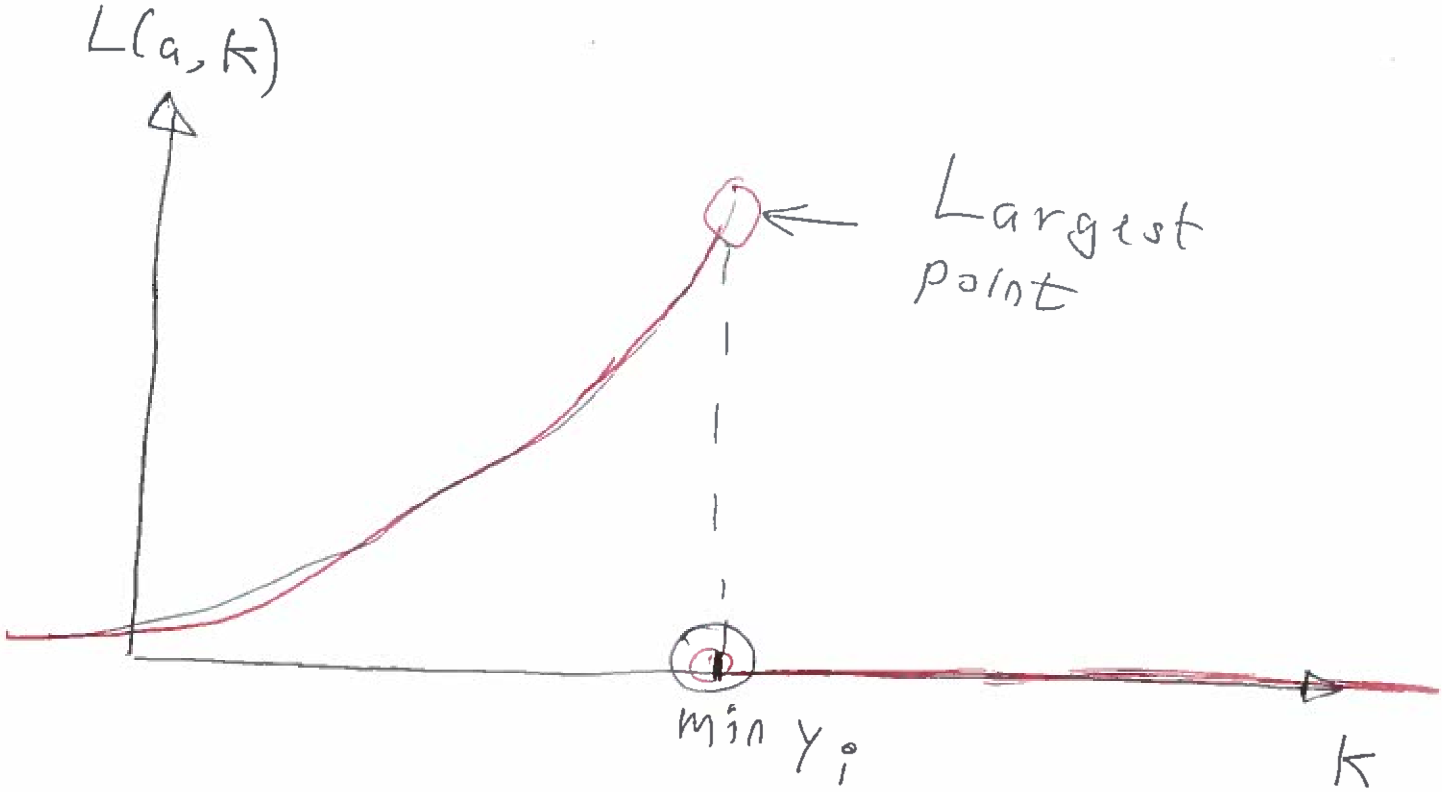
$$= \frac{(a\alpha)^n k^{na\alpha}}{\left(\prod_{i=1}^{n} y_i\right)^{a\alpha+1}} \left(\prod_{i=1}^{n} I\{y_i \geq k\} \right)$$

$$= \frac{(a\alpha)^n k^{na\alpha}}{\left(\prod_{i=1}^{n} y_i\right)^{a\alpha+1}} I\{\min y_i \geq k\}$$

$$\log L = n \log(a\alpha) + na\alpha \log k - (a\alpha+1) \sum_{i=1}^{n} \log y_i + \log I\{\min y_i \geq k\}$$

Use the <u>standard</u> <u>approach</u> to find the MLE of a

Use the <u>indicator</u> <u>function</u> <u>approach</u> to find the MLE of k.



$$\Rightarrow \quad \hat{k} = \min y_i$$

i) Write down the L using indicator functions

ii) graph L vs the parameter of interest

iii) read the largest value of the graph

iv) take the corresponding parameter value as the MLE.

" Indicator Function Approach "

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} + n\alpha \log K$$

$$- \alpha \sum_{i=1}^{n} \log y_i = 0$$

$$\Rightarrow \quad \frac{n}{a} = -n\alpha \log K + \alpha \sum_{i=1}^{n} \log y_i$$

$$\Rightarrow \quad \hat{a} = n\left[-n\alpha \log \hat{K} + \alpha \sum_{i=1}^{n} \log y_i\right]^{-1}$$

$$= n\left[-n\alpha \log(\min y_i) + \alpha \sum_{i=1}^{n} \log y_i\right]^{-1}.$$

EXAMPLE CLASS

6 DECEMBER

10:00-11:00AM

MATH3/4/68181

Q1 ARCH (q) model

$$e_t = \sigma_t Z_t \quad , \quad Z_t \sim N(0,1)$$

where $\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \dots + \alpha_q e_{t-q}^2$

$$E(e_t) = E(\sigma_t Z_t)$$

$$= E[E(\sigma_t Z_t \mid \sigma_t)]$$

$$\boxed{E(X) = E[E(X \mid Y)]}$$

$$= E[\sigma_t E(Z_t)]$$

$$= E[\sigma_t \cdot 0] = 0$$

$$E(e_t^2) = E(\sigma_t^2 Z_t^2)$$

$$= E[E(\sigma_t^2 Z_t^2 \mid \sigma_t)]$$

$$= E[\sigma_t^2 E(Z_t^2)]$$

$$= E[\sigma_t^2 \cdot 1]$$

$$= E[\sigma_t^2]$$

$$= E[(\alpha_0 + \alpha_1 e_{t-1}^2 + \dots + \alpha_q e_{t-q}^2)]$$

$$= \alpha_0 + \alpha_1 E(e_{t-1}^2) + \dots + \alpha_q E(e_{t-q}^2)$$

Assume $\{e_t\}$ is stationary for t sufficiently large.

Let $E(e_t^2) = Var(e_t) = \sigma^2$ for t sufficiently large.

$$\Rightarrow \sigma^2 = \alpha_0 + \alpha_1 \sigma^2 + \cdots + \alpha_q \sigma^2$$

$$\Rightarrow \boxed{\sigma^2 = \frac{\alpha_0}{1 - \alpha_1 - \cdots - \alpha_q}}.$$

Q2

GARCH(p, q) model

$$e_t = \sigma_t Z_t \quad , \quad Z_t \sim N(0,1)$$

where

$$\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_q e_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2$$

$$E(e_t) = E(\sigma_t Z_t)$$
$$= E[E(\sigma_t Z_t | \sigma_t)]$$
$$= E[\sigma_t \underline{E(Z_t)}] = 0.$$

$$E(e_t^2) = E[E(\underline{\sigma_t^2} Z_t^2 | \sigma_t)]$$
$$= E[\sigma_t^2 . \underline{E(Z_t^2)}]$$
$$= E[\sigma_t^2]$$
$$= E[\alpha_0 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_q e_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2]$$
$$= \alpha_0 + \alpha_1 E[e_{t-1}^2] + \cdots + \alpha_q E[e_{t-q}^2] + \beta_1 E[\sigma_{t-1}^2] + \cdots + \beta_p E[\sigma_{t-p}^2]$$

Assume stationarity for all t large.

Let $E[e_t^2] = Var(e_t) = \sigma^2$ for all t large.

$$\sigma^2 = \alpha_0 + \alpha_1 \sigma^2 + \cdots + \alpha_q \sigma^2 + \beta_1 \sigma^2 + \cdots + \beta_p \sigma^2$$

$$\sigma^2 = \frac{\alpha_0}{1 - \alpha_1 - \cdots - \alpha_q - \beta_1 - \cdots - \beta_p}$$

Q3

NGARCH model

$$e_t = \sigma_t Z_t \quad , \quad Z_t \sim N(0,1)$$

where $\sigma_t^2 = \omega + \alpha(e_{t-1} - \theta\sigma_{t-1})^2 + \beta\sigma_{t-1}^2$

$$E(e_t) = E\left[E[\sigma_t Z_t \mid \sigma_t]\right]$$
$$= E[\sigma_t E[Z_t]] = 0$$

$$E(e_t^2) = E\left[E(\sigma_t^2 Z_t^2 \mid \sigma_t)\right]$$
$$= E[\sigma_t^2 E(Z_t^2)] = E[\sigma_t^2]$$
$$= E[\omega + \alpha(e_{t-1} - \theta\sigma_{t-1})^2 + \beta\sigma_{t-1}^2]$$
$$= E[\omega + \alpha e_{t-1}^2 - 2\alpha\theta e_{t-1}\sigma_{t-1} + \alpha\theta^2\sigma_{t-1}^2 + \beta\sigma_{t-1}^2]$$
$$= \omega + \alpha E[e_{t-1}^2] - \cancel{2\alpha\theta E[e_{t-1}\sigma_{t-1}]} = 0$$
$$+ (\alpha\theta^2 + \beta) E[\sigma_{t-1}^2]$$

$$E[e_{t-1}\sigma_{t-1}] = E[E(\sigma_{t-1} e_{t-1} \mid \sigma_{t-1})]$$
$$= E[\sigma_{t-1} \underbrace{E(e_{t-1})}_{=0}] = 0$$

$$\Rightarrow E(e_t^2) = \omega + \alpha E[e_{t-1}^2] + (\alpha\theta^2 + \beta) E[\sigma_{t-1}^2]$$

Assume stationarity as before.

$$\sigma^2 = \omega + \alpha\sigma^2 + (\alpha\theta^2 + \beta)\sigma^2$$

$$\Rightarrow \sigma^2 = \frac{\omega}{1 - \alpha - (\alpha\theta^2 + \beta)}.$$

Q5

$$E[e_t] = 0 \quad \checkmark$$

$$E[e_t^2] = E[\sigma_t^2]$$

$$= k + \delta E[\sigma_{t-1}^2] + \alpha E[e_{t-1}^2]$$

$$+ \phi E[e_{t-1}^2 I_{t-1}]$$

$$E[e_{t-1}^2 I_{t-1}]$$

$$= E[e_{t-1}^2 \cdot 0] P(e_{t-1} \geq 0)$$

$$+ E[e_{t-1}^2] P(e_{t-1} < 0)$$

$$= E[e_{t-1}^2] \cdot P(e_{t-1} < 0)$$

LECTURE

9 DECEMBER

9:00-10:00AM

MATH3/4/68181

Exam 2014/15

X = Stock Returns

$X \mid \theta \sim \text{Uni}[-\theta, \theta]$

(a) Suppose θ has PDF $\frac{\lambda}{\theta^2} e^{-\frac{\lambda}{\theta}}, \theta > 0$

$$F_X(x) = \int \underbrace{F_{X|\theta}(x|\theta)}_{\text{cond CDF of } X|\theta} \underbrace{g(\theta)}_{\text{PDF of } \theta} d\theta$$

$$= \int_0^\infty \frac{x - (-\theta)}{2\theta} \cdot \frac{\lambda}{\theta^2} \cdot e^{-\frac{\lambda}{\theta}} d\theta$$

$$= \frac{\lambda x}{2} \int_0^\infty \frac{1}{\theta^3} e^{-\frac{\lambda}{\theta}} d\theta$$

$$+ \frac{\lambda}{2} \int_0^\infty \frac{1}{\theta^2} e^{-\frac{\lambda}{\theta}} d\theta$$

Set $y = \frac{\lambda}{\theta} \Rightarrow \theta = \frac{\lambda}{y}$

$\Rightarrow \frac{d\theta}{dy} = -\frac{\lambda}{y^2}$

$$= \frac{\lambda x}{2} \int_\infty^0 \frac{y^3}{\lambda^3} e^{-y} \left(-\frac{\lambda}{y^2}\right) dy$$

$$+ \frac{\lambda}{2} \int_\infty^0 \frac{y^2}{\lambda^2} e^{-y} \left(-\frac{\lambda}{y^2}\right) dy$$

$$= \frac{x}{2\lambda} \int_0^\infty y\, e^{-y}\, dy$$

$$+ \frac{1}{2} \int_0^\infty e^{-y}\, dy$$

$$= \frac{x}{2\lambda} \cdot \Gamma(2) + \frac{1}{2} \Gamma(1) = \frac{x+\lambda}{2\lambda}.$$

(b) $$f_X(x) = \frac{d}{dx} F_X(x) = \frac{1}{2\lambda}$$

(c) $$E(X) = \int_{-\lambda}^{\lambda} x \cdot \frac{1}{2\lambda}\, dx$$

$$= \frac{1}{2\lambda} \cdot \left[\frac{x^2}{2}\right]_{-\lambda}^{\lambda}$$

$$= \frac{1}{2\lambda} \left[\frac{\lambda^2}{2} - \frac{(-\lambda)^2}{2}\right] = 0$$

(d) $$E(X^2) = \int_{-\lambda}^{\lambda} x^2 \cdot \frac{1}{2\lambda}\, dx$$

$$= \frac{1}{2\lambda} \left[\frac{x^3}{3}\right]_{-\lambda}^{\lambda}$$

$$= \frac{1}{2\lambda} \left[\frac{\lambda^3}{3} - \frac{(-\lambda)^3}{3}\right]$$

$$= \frac{\lambda^2}{3}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{\lambda^2}{3}$$

Rules for manipulating products of indicator functions

1) $$\prod_{i=1}^{n} I\{X_i < A\}$$

$$= I\{\max X_i < A\}$$

2) $$\prod_{i=1}^{n} I\{X_i > B\}$$

$$= I\{\min X_i > B\}$$

3) $$I\{A > x\} \cdot I\{A > y\}$$

$$= I\{A > \max(x, y)\}$$

(e) Indicator function approach

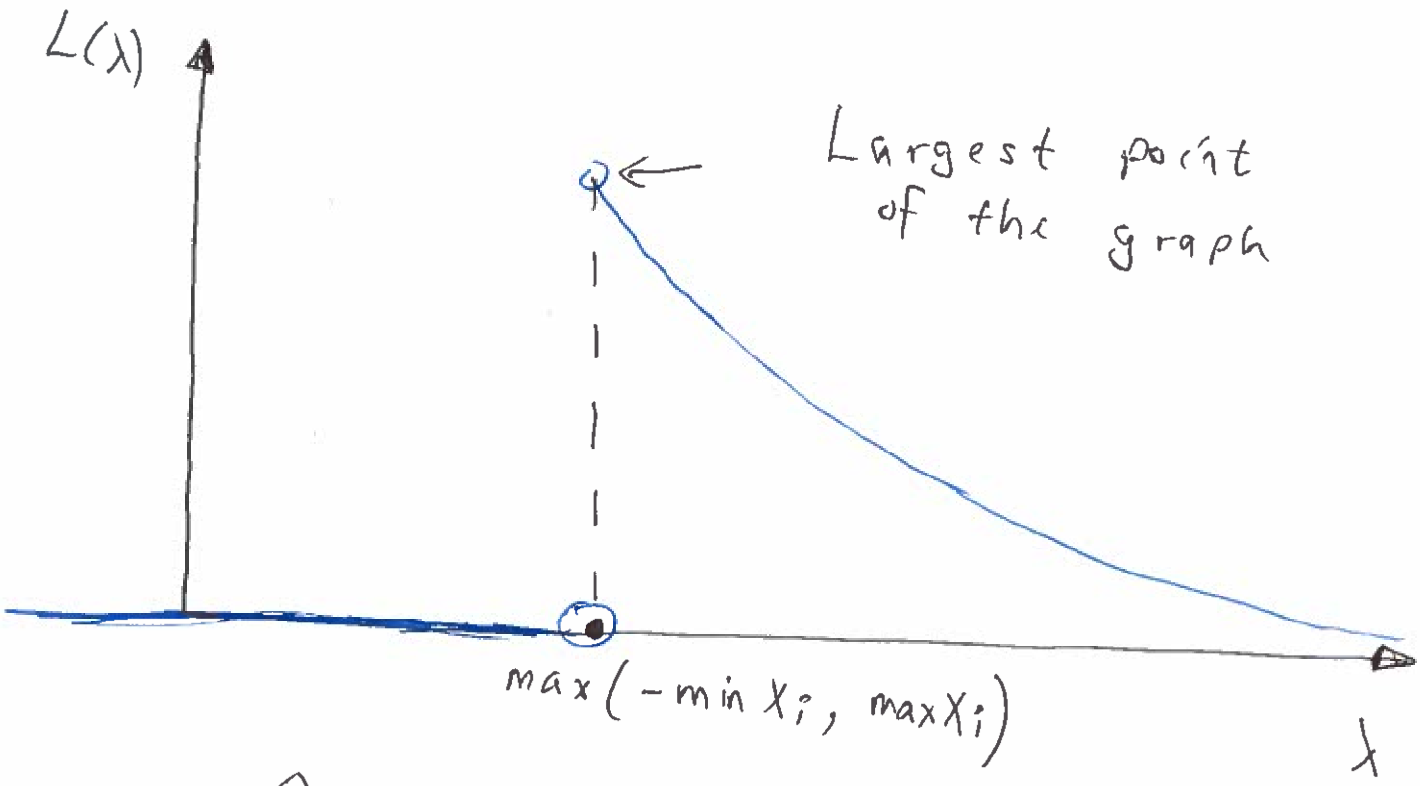
$$L(\lambda) = \prod_{i=1}^{n} \left[\frac{1}{2\lambda} \cdot I\{-\lambda < x_i < \lambda\} \right]$$

$$= \frac{1}{(2\lambda)^n} \left[\prod_{i=1}^{n} I\{-\lambda < x_i < \lambda\} \right]$$

$$= \frac{1}{(2\lambda)^n} I\{\min X_i > -\lambda\} \cdot I\{\max X_i < \lambda\}$$

$$= \frac{1}{(2\lambda)^n} I\{\lambda > -\min X_i\} \cdot I\{\lambda > \max X_i\}$$

$$= \boxed{\frac{1}{(2\lambda)^n}} I\{\lambda > \max(-\min X_i, \max X_i)\}$$



$$\hat{\lambda} = \max(-\min X_i, \max X_i)$$

Exam 2014/15

b(i)

$$f(x) = \frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}}, \quad -\infty < x < \infty$$

$\boxed{x > 0}$

$$F(x) = \int_{-\infty}^{x} \frac{1}{2\lambda} e^{-\frac{|y|}{\lambda}} dy$$

$$= \left(\int_{-\infty}^{+\infty} - \int_{x}^{\infty}\right) \frac{1}{2\lambda} e^{-\frac{|y|}{\lambda}} dy$$

$$= 1 - \int_{x}^{\infty} \frac{1}{2\lambda} e^{-\frac{|y|}{\lambda}} dy$$

$$= 1 - \int_{x}^{\infty} \frac{1}{2\lambda} e^{-\frac{y}{\lambda}} dy$$

$$= 1 - \frac{1}{2\lambda} \left[\frac{e^{-\frac{y}{\lambda}}}{\left(-\frac{1}{\lambda}\right)} \right]_{x}^{\infty}$$

$$= 1 - \frac{1}{2\lambda} \left[0 - \left(-\lambda e^{-\frac{x}{\lambda}}\right) \right]$$

$$= 1 - \frac{1}{2} e^{-\frac{x}{\lambda}}, \quad x > 0$$

$\boxed{x \leq 0}$

$$F(x) = \int_{-\infty}^{x} \frac{1}{2\lambda} e^{-\frac{|y|}{\lambda}} \, dy$$

$$= \frac{1}{2\lambda} \int_{-\infty}^{x} e^{\frac{y}{\lambda}} \, dy$$

$$= \frac{1}{2\lambda} \left[\frac{e^{\frac{y}{\lambda}}}{\left(\frac{1}{\lambda}\right)} \right]_{-\infty}^{x}$$

$$= \frac{1}{2\lambda} \left[\lambda e^{\frac{x}{\lambda}} - 0 \right]$$

$$= \frac{1}{2} e^{\frac{x}{\lambda}} , \quad x \leq 0 .$$

<u>b (ii)</u>

$$1 - \frac{1}{2} e^{-\frac{x}{\lambda}} = p$$

$$\Rightarrow \frac{1}{2} e^{-\frac{x}{\lambda}} = 1 - p$$

$$\Rightarrow e^{-\frac{x}{\lambda}} = 2(1-p)$$

$$\Rightarrow x = -\lambda \log \left[2(1-p) \right], \quad p \geq \frac{1}{2}$$

$$\frac{1}{2} e^{\frac{x}{\lambda}} = p$$

$$\Rightarrow e^{\frac{x}{\lambda}} = 2p$$

$$\Rightarrow x = \lambda \log \left[2p \right], \quad p \leq \frac{1}{2}$$

$$\mathrm{VaR}_p(X) = \begin{cases} -\lambda \log[2(1-p)] & p > \frac{1}{2} \\ \lambda \log[2p] & p \le \frac{1}{2} \end{cases}$$

b (iii)

$$ES_p(X) = \frac{1}{p} \int_0^p \mathrm{VaR}_t(X)\, dt$$

$$= \begin{cases} \frac{1}{p}\int_0^{\frac{1}{2}} \lambda \log(2t)\,dt - \frac{1}{p}\int_{\frac{1}{2}}^{p} \lambda \log[2(1-t)]\,dt & p > \frac{1}{2} \\ \frac{\lambda}{p}\int_0^p \log(2t)\,dt & p \le \frac{1}{2} \end{cases}$$

Integration by parts.

(c) (i)

$$L(\lambda) = \prod_{i=1}^{n} \frac{1}{2\lambda} e^{-\frac{|x_i|}{\lambda}}$$

$$= \frac{1}{(2\lambda)^n} e^{-\frac{1}{\lambda}\sum_{i=1}^{n}|x_i|}$$

(ii) $$\log L(\lambda) = -n\log(2\lambda) - \frac{1}{\lambda}\sum_{i=1}^{n}|x_i|$$

$$\frac{d\log L}{d\lambda} = -\frac{n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n}|x_i| = 0$$

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n}|x_i|$$

$$\frac{d^2\log L}{d\lambda^2} = \frac{n}{\lambda^2} - \frac{2}{\lambda^3}\sum_{i=1}^{n}|x_i|$$

$$= \frac{n}{\lambda^2}\left[1 - \frac{2}{n\lambda}\sum_{i=1}^{n}|x_i|\right]$$

$$\overset{\lambda=\hat{\lambda}}{=} \frac{n}{\hat{\lambda}^2}[1-2] < 0$$

$\Rightarrow$ $\hat{\lambda}$ is an MLE.

EXAMPLE CLASS

12 DECEMBER

12:00-13:00PM

MATH3/4/68181

REVISION

$$\bar{F}(x, y) = P(X > x,\ Y > y)$$

" Joint survival function "

$$F(x, y) = P(X < x,\ Y < y)$$

" Joint CDF "

$f(x, y)$ " Joint PDF "

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} \bar{F}(x, y)$$

Ex

Suppose $X_1, X_2, \ldots, X_p$ are losses on p investments with joint survival function

$$\bar{F}(x_1, x_2, \ldots, x_p) = e^{-x_1 - x_2 - \cdots - x_p}$$

Derive the PDF and CDF of the total portfolio loss $S = X_1 + X_2 + \cdots + X_p$

The joint PDF of $(X_1, X_2, \ldots, X_p)$ is

$$f(x_1, x_2, \ldots, x_p) = (-1)^p \frac{\partial^p}{\partial x_1 \partial x_2 \cdots \partial x_p} e^{-x_1 - x_2 - \cdots - x_p}$$

$$= (-1)^p \frac{\partial^{p-1}}{\partial x_1 \partial x_2 \cdots \partial x_{p-1}} (-1) e^{-x_1 - x_2 - \cdots - x_p}$$

$$= (-1)^p \frac{\partial^{p-2}}{\partial x_1 \partial x_2 \cdots \partial x_{p-2}} (-1)^2 e^{-x_1 - x_2 - \cdots - x_p}$$

$\vdots$

$$= (-1)^p \cdot (-1)^p e^{-x_1 - x_2 - \cdots - x_p}$$

$$= e^{-x_1 - x_2 - \cdots - x_p}$$

$$\bar{F}(x_1, x_2, \ldots, x_p)$$

$$= P(X_1 > x_1, X_2 > x_2, \ldots, X_p > x_p)$$

Joint survival function of $(X_1, X_2, \ldots, X_p)$

$$F(x_1, x_2, \ldots, x_p) = P(X_1 < x_1, X_2 < x_2, \ldots, X_p < x_p)$$

Joint CDF of $(X_1, X_2, \ldots, X_p)$

$$f(x_1, x_2, \ldots, x_p) = \frac{\partial^p}{\partial x_1 \partial x_2 \cdots \partial x_p} F(x_1, x_2, \ldots, x_p)$$

Joint PDF

$$= (-1)^p \frac{\partial^p}{\partial x_1 \partial x_2 \cdots \partial x_p} \bar{F}(x_1, x_2, \ldots, x_p)$$

$$S' = X_1 + X_2 + \cdots + X_p$$

$$\Rightarrow \quad f_{S'}(s) = e^{-s}$$

$$F_{S'}(s) = \int_0^s e^{-t}\,dt$$

$$= \left[-e^{-t}\right]_0^s$$

$$= -e^{-s} - (-1)$$

$$= 1 - e^{-s}$$

(b) $$F(x) = 1 - (1 - x^b)^a$$

$$F(x) = 1 \Rightarrow 1 - (1 - x^b)^a = 1$$

$$\Rightarrow (1 - x^b)^a = 0$$

$$\Rightarrow x = 1 = w(F)$$

$$\lim_{t \to 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)}$$

$$= \lim_{t \to 0} \frac{\cancel{1} - \{\cancel{1} - (1 - (1 - tx)^b)^a\}}{\cancel{1} - \{\cancel{1} - (1 - (1 - t)^b)^a\}}$$

$$= \lim_{t \to 0} \left[\frac{1 - (1 - tx)^b}{1 - (1 - t)^b} \right]^a$$

$$= \lim_{t \to 0} \left[\frac{\cancel{1} - (\cancel{1} - \cancel{b}tx)}{\cancel{1} - (\cancel{1} - \cancel{b}t)} \right]^a \qquad \boxed{(1 - z)^\alpha \approx 1 - \alpha z}$$

$$= x^a$$

$\Rightarrow$ F belongs to the Weibull max domain.

LECTURE

13 DECEMBER

9:00-10:00AM

MATH3/4/68181

REVISION

Exam. 2014/15

$$F(x) = 1 - (1 + x^c)^{-k}$$

$$F(x) = 1$$

$$\Rightarrow 1 - (1 + x^c)^{-k} = 1$$

$$\Rightarrow (1 + x^c)^{-k} = 0$$

$$\Rightarrow 1 + x^c = \infty$$

$$\Rightarrow x = \infty = w(F)$$

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t \to \infty} \frac{\cancel{1} - [\cancel{1} - (1 + (tx)^c)^{-k}]}{\cancel{1} - [\cancel{1} - (1 + t^c)^{-k}]}$$

$$= \lim_{t \to \infty} \left(\frac{1 + (tx)^c}{1 + t^c} \right)^{-k}$$

$$= \lim_{t \to \infty} \left(\frac{\underbrace{\frac{1}{t^c}}_{0} + x^c}{\underbrace{\frac{1}{t^c}}_{0} + 1} \right)^{-k}$$

$$= (x^c)^{-k} = x^{-ck}$$

F belongs to the Frechet max domain

Exam 2015/16 B1.

(a) $X_i \sim Exp(\lambda)$ IID

$$M_{X_i}(t) = E[e^{tX_i}]$$

$$= \int_0^\infty e^{tx} \cdot \underbrace{\lambda e^{-\lambda x}}\, dx$$

$$= \lambda \int_0^\infty e^{-\cancel{\lambda(1-t)}\, x(\lambda-t)} dx$$

$$= \lambda \left[\frac{e^{-x(\lambda-t)}}{-(\lambda-t)} \right]_0^\infty$$

$$= \lambda \left[0 - \frac{1}{-(\lambda-t)} \right]$$

$$= \frac{\lambda}{\lambda - t} \quad \text{if } \lambda > t$$

(b)

$$M_{T|N=n}(t) = E\left[e^{tT} \mid N=n\right]$$

$$= E\left[e^{t(X_1+X_2+\cdots+X_N)} \mid N=n\right]$$

$$= E\left[e^{t(X_1+X_2+\cdots+X_n)}\right]$$

$$= E\left[e^{tX_1} e^{tX_2} \cdots e^{tX_n}\right]$$

indep

$$= E\left[e^{tX_1}\right] E\left[e^{tX_2}\right] \cdots E\left[e^{tX_n}\right]$$

$$= \frac{\lambda}{\lambda-t} \cdot \frac{\lambda}{\lambda-t} \cdots \frac{\lambda}{\lambda-t}$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^n$$

(c) $\Rightarrow$ $T \mid N=n \sim \text{Gamma}(\lambda, n)$

(d) $$M_T(t) = \sum_{n=\cancel{0}}^{\infty} M_{T|N=n}(t)\, P(N=n)$$

Total Prob Rule

$$= \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda-t}\right)^n \theta(1-\theta)^{n-1}$$

$$= \frac{\lambda\theta}{\lambda - t} \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda - t}\right)^{n-1} (1-\theta)^{n-1}$$

$[m = n-1]$

$$= \frac{\lambda\theta}{\lambda - t} \sum_{m=0}^{\infty} \left[\frac{\lambda(1-\theta)}{\lambda - t}\right]^{m}$$

$$= \frac{\lambda\theta}{\lambda - t} \cdot \frac{1}{1 - \frac{\lambda(1-\theta)}{\lambda - t}}$$

$$\boxed{\sum_{m=0}^{\infty} x^m = \frac{1}{1-x}}$$

$$= \frac{\lambda\theta}{\lambda - t - \lambda(1-\theta)}$$

$$\boxed{M_T(t) = \frac{\lambda\theta}{\lambda\theta - t}}$$

$$\Rightarrow \quad T \sim Exp(\lambda\theta)$$

(e)

$$E(T) = M'_T(0)$$

$$E(T^2) = M''_T(0)$$

$$M'_T(t) = \frac{\lambda\theta}{(\lambda\theta - t)^2} \Rightarrow M'_T(0) = \frac{1}{\lambda\theta}$$

$$M''_T(t) = \frac{2\lambda\theta}{(\lambda\theta - t)^3} \Rightarrow M''_T(0) = \frac{2}{(\lambda\theta)^2}$$

$$E(T) = \frac{1}{\lambda\theta}$$

$$Var(T) = E(T^2) - (E(T))^2 = \frac{2}{(\lambda\theta)^2} - \frac{1}{(\lambda\theta)^2}$$

$$= \frac{1}{(\lambda\theta)^2}$$

(f) $T \sim Exp(\lambda\theta)$

$$F_T(t) = 1 - e^{-\lambda\theta t} = p$$

$$\Rightarrow e^{-\lambda\theta t} = 1-p$$

$$\Rightarrow -\lambda\theta t = \log(1-p)$$

$$\Rightarrow t = -\frac{1}{\lambda\theta}\log(1-p)$$

$$\Rightarrow VaR_p(T) = -\frac{1}{\lambda\theta}\log(1-p)$$

(g)

$$ES_p(T) = \frac{1}{p}\int_0^p VaR_u(T)\,du$$

$$= -\frac{1}{\lambda\theta p}\int_0^p 1\cdot\log(1-u)\,du$$

$$= -\frac{1}{\lambda\theta p}\left\{\Big[u\cdot\log(1-u)\Big]_0^p - \int_0^p u\cdot\left(-\frac{1}{1-u}\right)du\right\}$$

$$= -\frac{1}{\lambda\theta p}\left\{p\cdot\log(1-p) - 0 + \int_0^p \frac{(u-1)+1}{1-u}\,du\right\}$$

$$= -\frac{1}{\lambda\theta p}\left\{p\cdot\log(1-p) + \int_0^p\left(-1+\frac{1}{1-u}\right)du\right\}$$

$$= -\frac{1}{\lambda\theta p}\left\{p\cdot\log(1-p) + \Big[-u - \log(1-u)\Big]_0^p\right\}$$

$$= -\frac{1}{\lambda\theta p}\left\{p\cdot\log(1-p) - p - \log(1-p) - 0\right\}$$

Exam 2015/16 B4 (v)

$$\widehat{VaR}_0(X) = \hat{k}$$

$$\widehat{ES}_0(X) = \hat{k}$$

$$\hat{k} = \min X_i \qquad \text{Let } Z = \min X_i$$

$$F_Z(z) = P(Z < z) = 1 - P(Z \geq z)$$

$$= 1 - P(\min X_i \geq z)$$

$$= 1 - \left(P(X \geq z)\right)^n$$

$$= 1 - \left(\frac{k}{z}\right)^{an}$$

$$f_Z(z) = \frac{an\, k^{an}}{z^{an+1}}$$

$$E[Z] = an\, k^{an} \int_k^{\infty} z^{-an}\, dz$$

$$= \frac{an\, k^{an}}{1 - an} \cdot k^{1-an}$$

$$= \frac{an\, k}{1 - an} \neq k$$

$\Rightarrow$ $\hat{k}$ is biased.

Exam 2015/16 B3 (d)

$$f(x) = 0.5\, e^{-|x|}, \quad -\infty < x < +\infty$$

$$\omega(F) = +\infty$$

$$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$\overset{LH}{=} \lim_{t \to \infty} \frac{\cancel{-} f(t + x\gamma(t)) \cdot (1 + x\gamma'(t))}{\cancel{-} f(t)}$$

$$= \lim_{t \to \infty} \frac{\cancel{0.5}\, e^{-|t + x\gamma(t)|}\,(1 + x\gamma'(t))}{\cancel{0.5}\, e^{-|t|}}$$

$$= \lim_{t \to \infty} \frac{e^{-t - x\gamma(t)}\,(1 + x\gamma'(t))}{e^{-t}}$$

$$= \lim_{t \to \infty} e^{-x\gamma(t)} \cdot (1 + x\gamma'(t))$$

$$= e^{-x} \quad \text{if} \quad \gamma(t) \equiv 1$$

$\Rightarrow$ F belongs to the Gumbel max domain.

EXAMPLE CLASS

13 DECEMBER

10:00-11:00AM

MATH3/4/68181

~~09~~

Suppose a portfolio has k investments. Assume the losses on the k investments $(X_1, X_2, \cdots, X_k)$ has

$$\bar{F}(x_1, x_2, \cdots, x_k) = e^{-x_1-x_2-\cdots-x_k}$$

Find the PDF ant CDF of the total portfolio loss $S = X_1 + X_2 + \cdots + X_k$.

$$f(x_1, x_2, \cdots, x_k) = (-1)^k \frac{\partial^k e^{-x_1-x_2-\cdots-x_k}}{\partial x_1 \partial x_2 \cdots \partial x_k}$$

$$= (-1)^k \frac{\partial^{k-1}}{\partial x_1 \partial x_2 \cdots \partial x_{k-1}} \left(- e^{-x_1-x_2-\cdots-x_k}\right)$$

$$= (-1)^k \frac{\partial^{k-2}}{\partial x_1 \partial x_2 \cdots \partial x_{k-2}} \left((-1)^2 e^{-x_1-x_2-\cdots-x_k}\right)$$

$$\vdots$$

$$= (-1)^k \cdot (-1)^k e^{-x_1-x_2-\cdots-x_k}$$

$$= e^{-x_1-x_2-\cdots-x_k}$$

$$= e^{-(x_1+x_2+\cdots+x_k)}$$

$$f_S(s) = e^{-s}$$

Suppose $(X_1, X_2, \ldots, X_k)$ is a random vector.

$$\bar{F}(x_1, x_2, \ldots, x_k)$$

$$= P(X_1 > x_1, X_2 > x_2, \ldots, X_k > x_k)$$

"Joint survival function of $(X_1, X_2, \ldots, X_k)$"

$$F(x_1, x_2, \ldots, x_k)$$

$$= P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_k \leq x_k)$$

"Joint CDF of $(X_1, X_2, \ldots, X_k)$"

$f(x_1, x_2, \ldots, x_k)$ "Joint PDF of $(X_1, X_2, \ldots, X_k)$"

$$f(x_1, x_2, \ldots, x_k) = \left(\frac{\partial^k \bar{F}}{\partial x_1 \partial x_2 \cdots \partial x_k}\right) \cdot (-1)^k$$

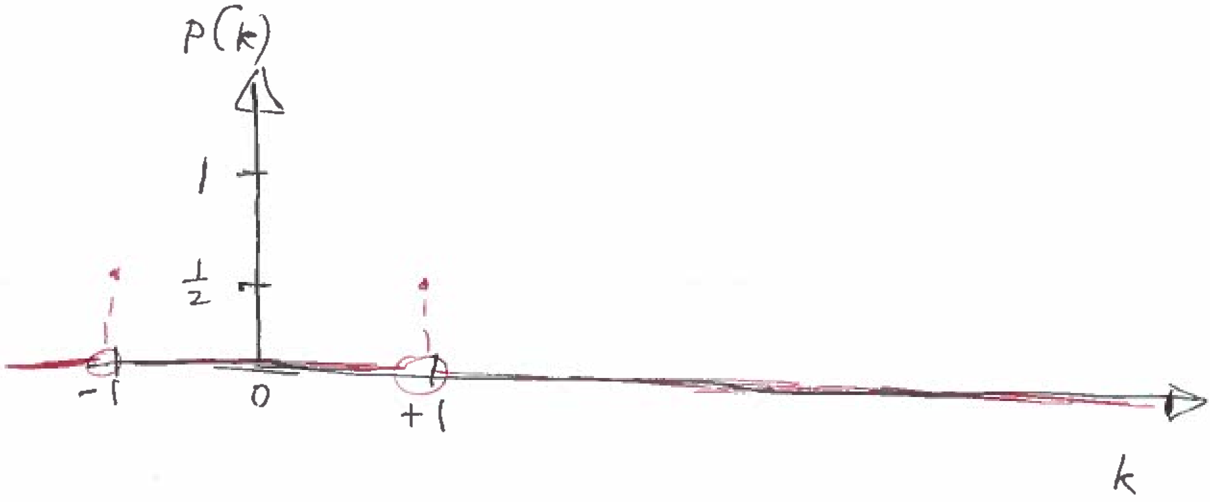
$$= \left(\frac{\partial^k F}{\partial x_1 \partial x_2 \cdots \partial x_k}\right)$$

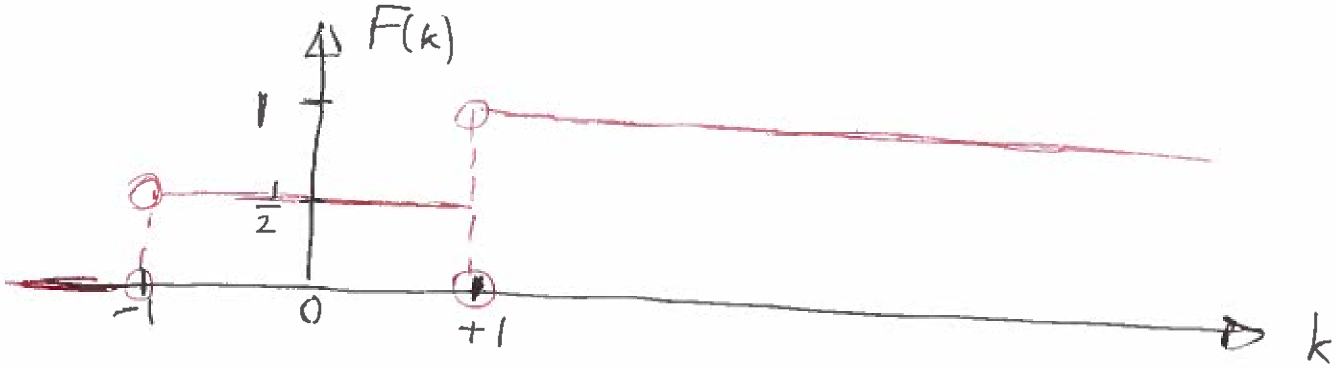
$$F_S(s) = \int_0^s e^{-u}\,du$$

$$= \left[-e^{-u}\right]_0^s$$

$$= -e^{-s} - (-1) = 1 - e^{-s}$$

Exam 2015/16 B3 (b)





$$F(k) = 1 \Rightarrow k = +1 = w(F)$$

$$\lim_{k \to w(F)} \frac{P(X=k)}{1-F(k-1)} = \frac{P(X=1)}{1-F(1-1)}$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 \neq 0$$

ETT does not hold.

Exam 2015/16 B3 (a)

$$f(x) = C\, x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 < x < 1$$

$$\omega(F) = 1$$

$$\lim_{t \to 0} \frac{1 - F(\omega(F) - tx)}{1 - F(\omega(F) - t)}$$

$$= \lim_{t \to 0} \frac{1 - F(1 - tx)}{1 - F(1-t)}$$

$$\overset{LH}{=} \lim_{t \to 0} \frac{-f(1-tx)\cdot(-x)}{-f(1-t)\cdot(-1)}$$

$$= \lim_{t \to 0} \frac{\cancel{C}\cdot(1-tx)^{\alpha-1}(\cancel{1}-(\cancel{1}-tx))^{\beta-1}\cdot x}{\cancel{C}\cdot(1-t)^{\alpha-1}(\cancel{1}-(\cancel{1}-t))^{\beta-1}}$$

$$= \lim_{t \to 0} \left(\frac{1-tx}{1-t}\right)^{\alpha-1} \cdot \frac{(tx)^{\beta-1}}{t^{\beta-1}} \cdot x$$

$$= x^{\beta}$$

$\Rightarrow$ F belongs to the Weibull max domain.

Exam 2015/16 B3(c)

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2}, \quad -\infty < x < +\infty$$

$$\omega(F) = +\infty$$

$$\lim_{t \to +\infty} \frac{1-F(tx)}{1-F(t)}$$

$$\overset{LH}{=} \lim_{t \to +\infty} \frac{-f(tx) \cdot x}{-f(t)}$$

$$= \lim_{t \to +\infty} \frac{\cancel{\frac{1}{\pi}} \cdot \frac{1}{1+(tx)^2} \cdot x}{\cancel{\frac{1}{\pi}} \cdot \frac{1}{1+t^2}}$$

$$= \lim_{t \to \infty} \frac{1+t^2}{1+(tx)^2} \cdot x$$

$$= \lim_{t \to \infty} \frac{\frac{1}{t^2} + 1}{\frac{1}{t^2} + x^2} \cdot x = \frac{1}{x}$$

0

0

$\Rightarrow$ F belongs to the Frechet max domain.

LECTURE

16 DECEMBER

9:00-10:00AM

MATH3/4/68181

REVISION

Exam 2012/13 Q6

$X \sim \mathrm{Uni}[a, b]$

$$F(x) = \frac{x-a}{b-a} = p$$

$$\Rightarrow x = a + p \cdot (b-a)$$

$$= VaR_p(X)$$

$$ES_p(X) = \frac{1}{p} \int_0^p VaR_t(X)\, dt$$

$$= \frac{1}{p} \int_0^p \left[a + t \cdot (b-a)\right] dt$$

$$= \frac{1}{p} \cdot \left[a \cdot t + \frac{t^2}{2} \cdot (b-a)\right]_0^p$$

$$= \frac{1}{p} \cdot \left[a \cdot p + \frac{p^2}{2} \cdot (b-a)\right]$$

$$= a + \frac{p}{2} \cdot (b-a)$$

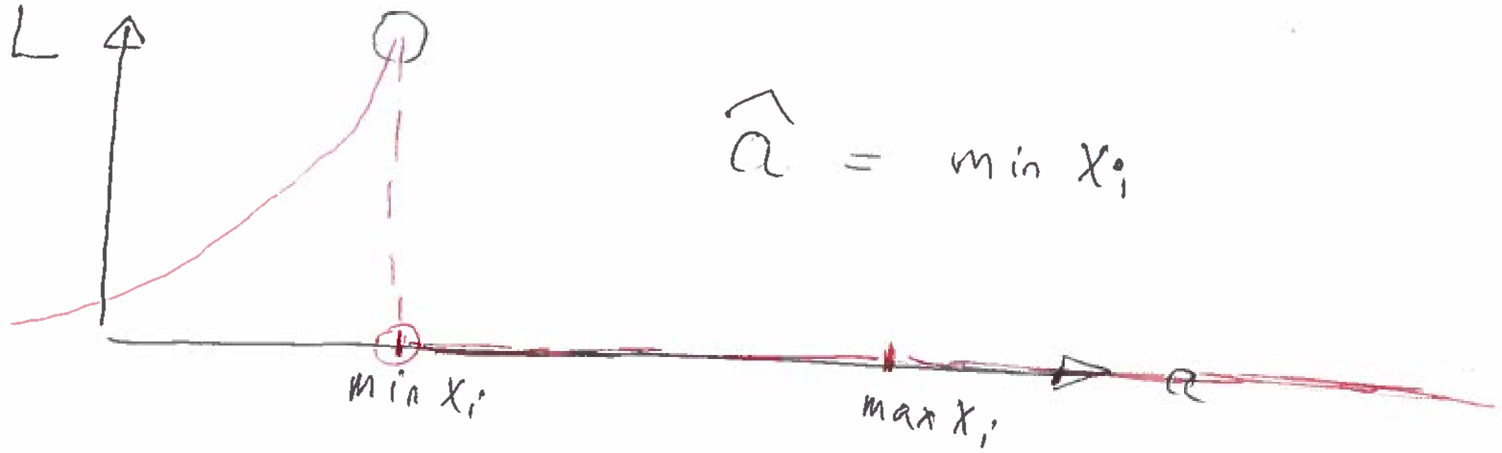
(i) <u>Indicator function Approach</u>

$$L(a, b) = \prod_{i=1}^{n} \left[\frac{1}{b-a} I\{a \leq X_i \leq b\} \right]$$

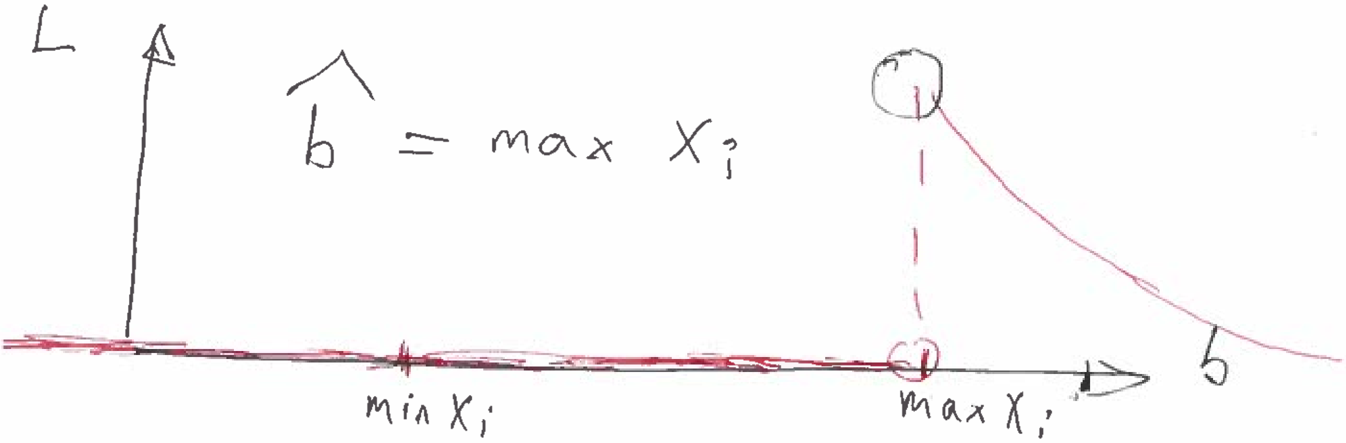
$$= \frac{1}{(b-a)^n} \cdot \left[\prod_{i=1}^{n} I\{a \leq X_i \leq b\} \right]$$

$$= \frac{1}{(b-a)^n} \cdot \prod_{i=1}^{n} I\{X_i \geq a\} \cdot I\{X_i \leq b\}$$

$$= \frac{1}{(b-a)^n} I\{\min X_i \geq a\} I\{\max X_i \leq b\}$$



$$\hat{a} = \min X_i$$



$$\hat{b} = \max X_i$$

(iv)

$$\mathrm{VaR}_p(X) = a + p(b-a)$$

$$\Rightarrow \widehat{\mathrm{VaR}_p}(X) = \hat{a} + p(\hat{b} - \hat{a})$$

$$= \min X_i + p(\max X_i - \min X_i)$$

$$\mathrm{ES}_p(X) = a + \frac{p}{2}(b-a)$$

$$\Rightarrow \widehat{\mathrm{ES}_p}(X) = \hat{a} + \frac{p}{2}(\hat{b} - \hat{a})$$

$$= \min X_i + \frac{p}{2}(\max X_i - \min X_i)$$

(v) $E[\widehat{\mathrm{VaR}_p}(X)]$

$$= E[\min X_i] + p\{E[\max X_i] - E[\min X_i]\}$$

$$= (1-p) \cdot \underbrace{E[\min X_i]} + p \cdot \underbrace{E[\max X_i]}$$

show this $\neq$ $\mathrm{VaR}_p(X) \Rightarrow \widehat{\mathrm{VaR}_p}(X)$ is biased.

Let $U = \min X_i$

$$F_U(u) = P[\min X_i < u]$$
$$= 1 - P[\min X_i \geq u]$$
$$= 1 - P[X_1 \geq u, \ldots, X_n \geq u]$$
$$= 1 - (P[X > u])^n$$
$$= 1 - (1 - P[X \leq u])^n$$
$$= 1 - \left(1 - \frac{u-a}{b-a}\right)^n$$
$$= 1 - \left(\frac{b-u}{b-a}\right)^n$$

$$f_U(u) = \frac{n(b-u)^{n-1}}{(b-a)^n}$$

$$E[U] = n\int_a^b u \cdot \frac{(b-u)^{n-1}}{(b-a)^n}\,du$$
$$= n\int_a^b [b-(b-u)]\frac{(b-u)^{n-1}}{(b-a)^n}\,du$$
$$= nb\int_a^b \frac{(b-u)^{n-1}}{(b-a)^n}\,du - n\int_a^b \frac{(b-u)^n}{(b-a)^n}\,du$$
$$= \frac{nb}{(b-a)^n}\left[\frac{(b-u)^n}{(-n)}\right]_a^b - \frac{n}{(b-a)^n}\left[\frac{(b-u)^{n+1}}{(-(n+1))}\right]_a^b$$

$$= b - \frac{n \cdot (b-a)}{n+1}$$

$$= \frac{b + na}{n+1} = E[\min X_i]$$

Similarly, find $E[\max X_i]$.

Questions that are examinable for Math 38181

Exam 2012/13 — Q1, Q2, Q3, Q4, Q5

Exam 2013/14 — Q2, Q3, Q4, Q5, Q6

Exam 2014/15 — Q2, Q3, Q4, Q5, Q6

Exam 2015/16 — B1, B2, B3, B4, B5

Questions that are examinable for Math 4168181

Exam 2012/13 — Q1, Q2, Q3, Q4, Q5, Q6, Q7

Exam 2013/14 — Q2, Q3, Q4, Q5, Q6, Q7, Q8

Exam 2014/15 — Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8

Exam 2015/16 — A1 to A3, B1 to B5