LECTURE

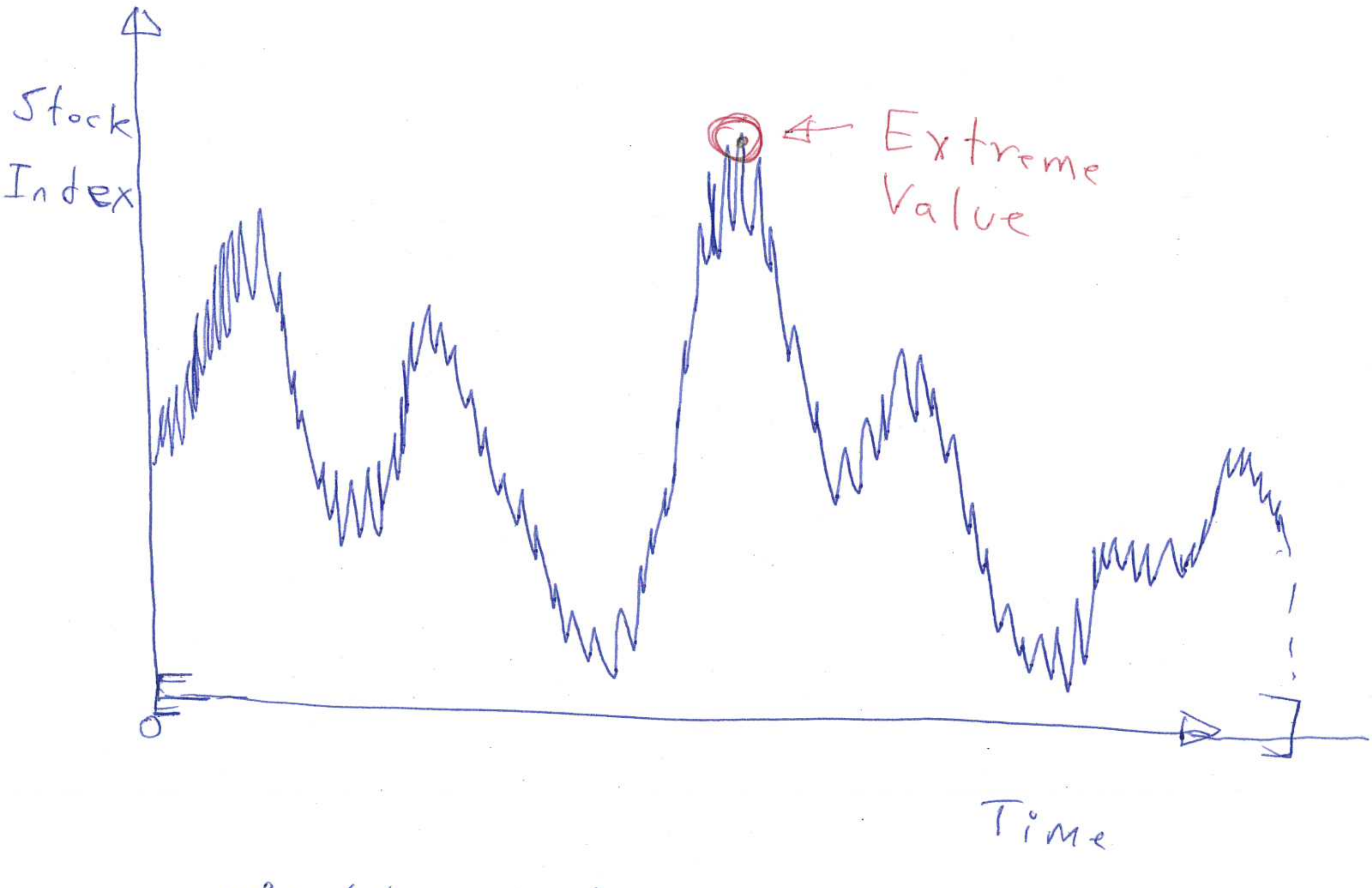
29 SEPTEMBER

9:00-10:00AM

MATH3/4/68181

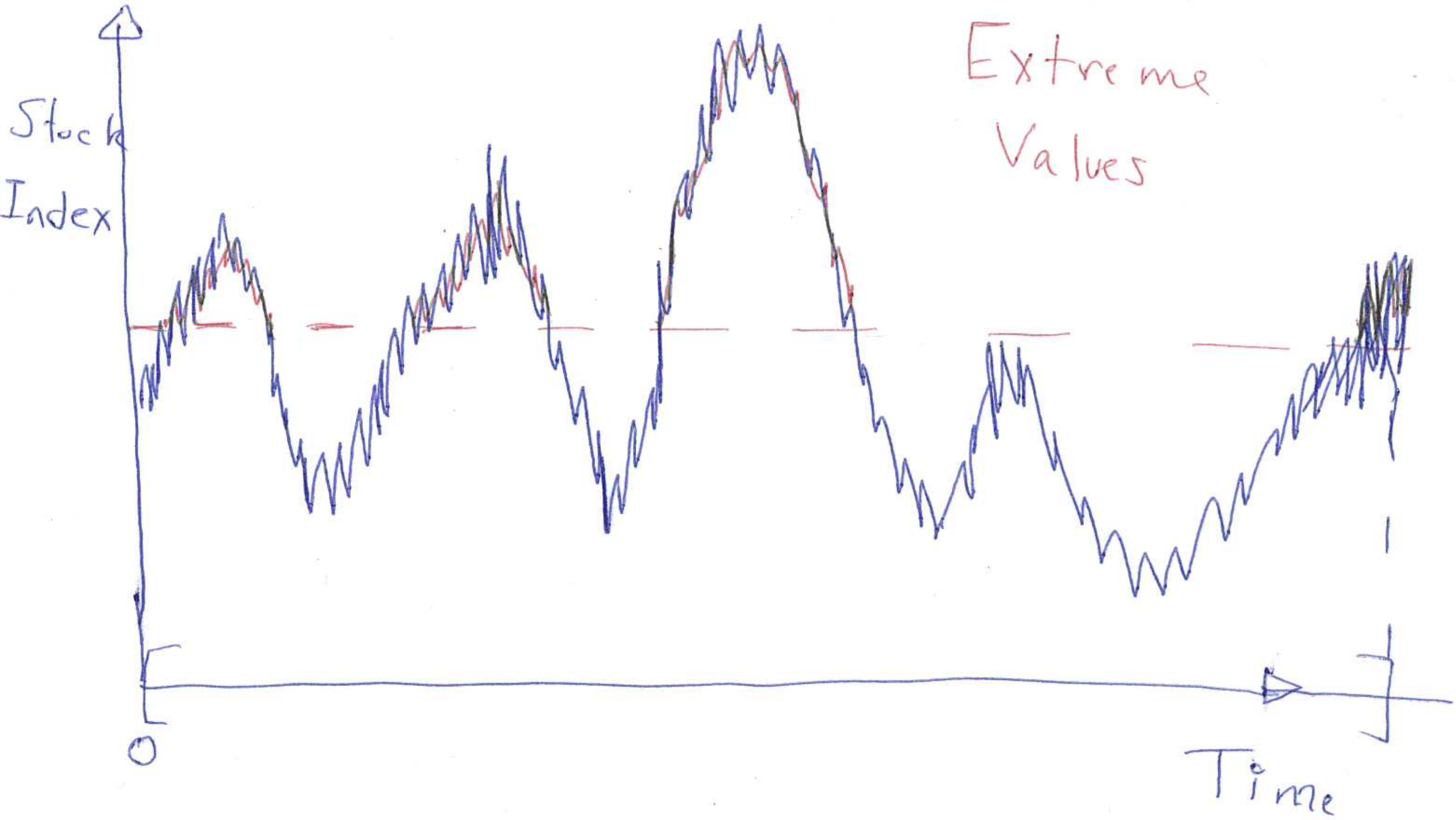
Extreme Values

1) What is an extreme value?

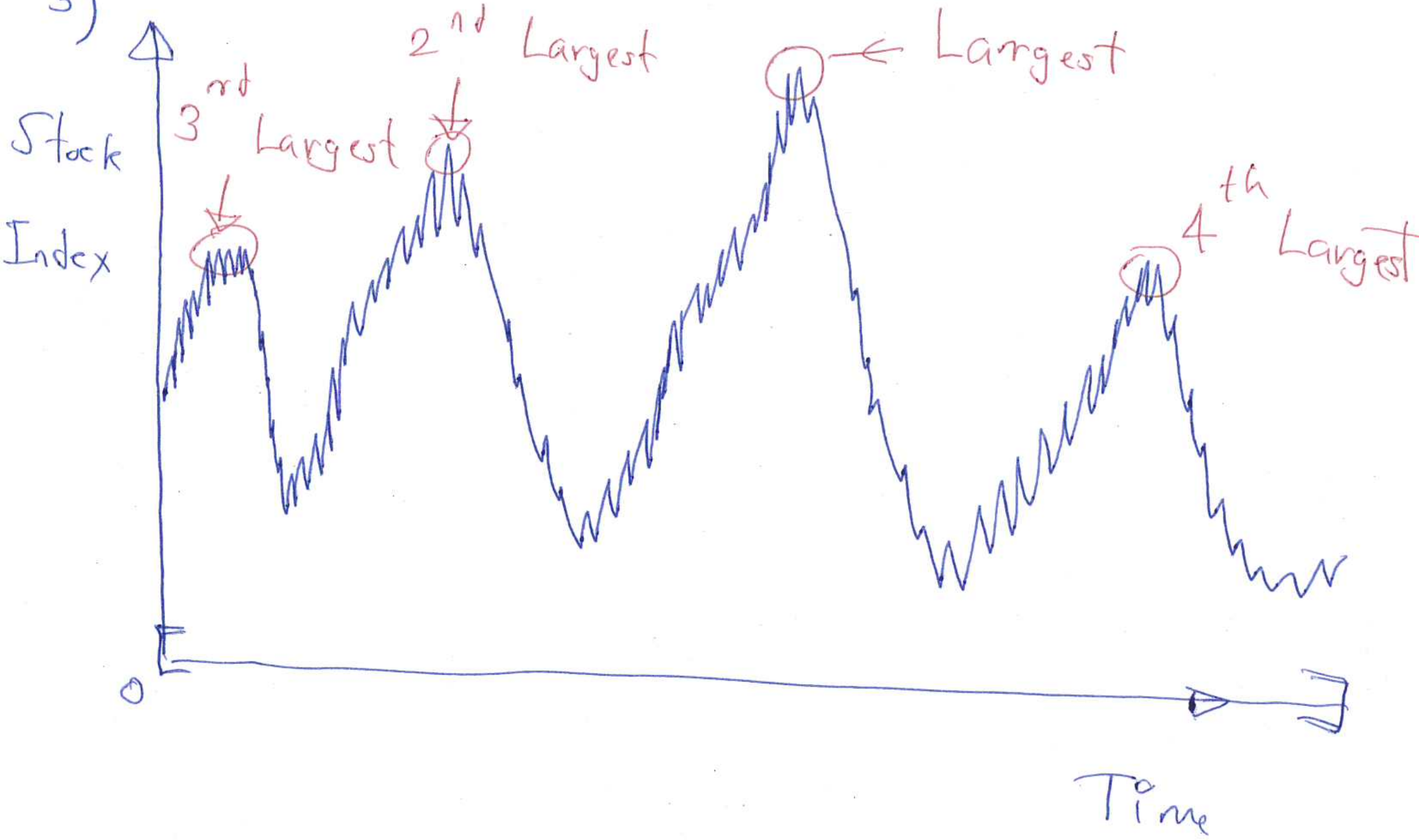


$$\min(X_1, \ldots, X_n) = -\max(-X_1, \ldots, -X_n)$$

2)



3)


Stock Index
3rd Largest
2nd Largest
Largest
4th Largest
0
Time

Suppose $X_1, X_2, \ldots, X_n$ are IID with CDF $F(\cdot)$. The extreme value is

$$M_n = \max(X_1, X_2, \ldots, X_n).$$

What distribution does M_n have?

$$\begin{aligned}
&P(M_n \leq x) \\
&= P(\max(X_1, \ldots, X_n) \leq x) \\
&= P(X_1 \leq x, \ldots, X_n \leq x) \\
&\overset{\text{indep}}{=} P(X_1 \leq x) \cdots P(X_n \leq x) \\
&= F(x) \cdots F(x) = F^n(x)
\end{aligned}$$

$$\lim_{n \to \infty} P(M_n \leq x) = \lim_{n \to \infty} F^n(x)$$

$$= \begin{cases} 0 & \text{if } 0 < F(x) < 1 \\ 1 & \text{if } F(x) = 1 \end{cases}$$

"degenerate" limit

LECTURE

1 OCTOBER

12:00-13:00PM

MATH4/68181

Suppose $X_1, X_2, \ldots, X_n$ are IID with CDF $F(\cdot)$. Let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

$$\bar{X} \xrightarrow[n \to \infty]{} \mu$$

population mean

SLLN

$$\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \xrightarrow[n \to \infty]{} N(0, 1)$$

CLT

We want to look at the limit of

$$\frac{M_n - b_n}{a_n}, \quad a_n > 0, \ b_n \in \mathbb{R}$$

as $n \longrightarrow \infty$.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x \right)$$

$$= \Pr(M_n \leq a_n x + b_n)$$

$$= P(X_1 \leq a_n x + b_n, \ldots, X_n \leq a_n x + b_n)$$

$$\overset{\text{indep}}{=} P(X_1 \leq a_n x + b_n) \cdots P(X_n \leq a_n x + b_n)$$

$$= F(a_n x + b_n) \cdots F(a_n x + b_n)$$

$$= F^n (a_n x + b_n).$$

Does this have a $\boxed{\text{non-degenerate}}$ limit as $n \to \infty$?

Extremal Types Theorem (ETT)

If there exists $a_n > 0$ & $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) \longrightarrow G(x)$$

as $n \to \infty$ for a non-degenerate G then G must be of the $\boxed{\text{same}}$ $\boxed{\text{type}}$ as one of:

I : $\Lambda(x) = e^{-e^{-x}}$, $-\infty < x < \infty$

[Gumbel]

II : $\Phi_\alpha(x) = \begin{cases} 0 & x < 0 \\ e^{-x^{-\alpha}} & x > 0 \end{cases}$

[Frechet]

III : $\Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha} & x < 0 \\ 1 & x > 0 \end{cases}$

[Weibull]

G_1 & G_2 are said to be of the same type if $G_1(x) = G_2(ax+b)$ for all x, $a>0$, $b \in \mathbb{R}$.

eg.

(i) $e^{-e^{-2x+3}}$ & $e^{-e^{-x}}$

are of the same type.

(ii) $e^{-x^{-1}}$ & $e^{-3x^{-1}}$

are of the same type

(iii) e^{x} & e^{-x-2}

are not of the same type.

Given F, which of the 3 limits ~~would~~ (if any) would be attained?

I: $\exists\ \gamma(t) > 0$ such that

$$\lim_{t \uparrow \omega(F)} \frac{1-F(t+x\gamma(t))}{1-F(t)} \longrightarrow e^{-x}$$

II: $\omega(F) = \infty$ & $\lim_{t \uparrow \infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha}$

III: $\omega(F) < \infty$ & $\lim_{t \downarrow 0} \frac{1-F(\omega(F)-tx)}{1-F(\omega(F)-t)} = x^{\alpha}$

$\omega(F) = \sup\{x : F(x) < 1\}$
"upper end point of F"

Given a F, $\omega(F)$ must satisfy $F(\omega(F)) = 1$. All you need to do is to solve for $\boxed{F(x) = 1}$.

<u>eg</u>

(i) $F(x) = 1 - e^{-x}$, $x > 0$

$1 - e^{-x} = 1$

$\Rightarrow$ $e^{-x} = 0$

$\Rightarrow$ $x = +\infty$. $\Rightarrow$ $w(F) = +\infty$.

(ii) $F(x) = x$, $0 < x < 1$

$F(x) = 1$

$\Rightarrow$ $x = 1$ $\Rightarrow$ $w(F) = 1$.

Conditions in terms of PDF f

I: $\dfrac{b(tx)}{b(t)} \to 1$ as $t \to \infty$,

where $b(t) = t\, f\left(F^{-1}\left(1 - \frac{1}{t}\right)\right)$

II: $\omega(F) = \infty$ & $\lim\limits_{t \to \infty} F^{-1}\left(1 - \frac{1}{t}\right) b(t) = \alpha$

III: $\omega(F) < \infty$ & $\lim\limits_{t \to \infty} \left\{ \omega(F) - F^{-1}\left(1 - \frac{1}{t}\right) \right\} b(t) = \alpha$

eg

$$F(x) = 1 - e^{-x}, \quad x > 0$$

$$f(x) = e^{-x}, \quad x > 0$$

$$F^{-1}(x) = -\log(1-x)$$

$$b(t) = t \cdot f\left(F^{-1}\left(1-\frac{1}{t}\right)\right)$$

$$= t \cdot f\left(-\log\left(1-\left(1-\frac{1}{t}\right)\right)\right)$$

$$= t \cdot f(\log t)$$

$$= t \cdot e^{-\log t} = 1$$

$$\frac{b(tx)}{b(t)} = \frac{1}{1} = 1 \Rightarrow \text{(I) is satisfied}$$

$\Rightarrow \exists \; a_n > 0 \;\&\; b_n \in \mathbb{R}$ such that

$$P\left(\frac{M_n - b_n}{a_n} \leq x\right) \to e^{-e^{-x}}, \quad n \to \infty.$$

eg

$$F(x) = x \quad , \quad 0 < x < 1$$
$$f(x) = 1$$
$$F^{-1}(x) = x$$

$$b(t) = t \cdot f\left(1 - \frac{1}{t}\right) = t \cdot 1 = t$$

$$\frac{b(tx)}{b(t)} = \frac{tx}{t} = x \neq 1$$

$\Rightarrow$ (I) is not satified

(II) is also not satified because $\omega(F) = 1$.

$$\left\{\omega(F) - F^{-1}\left(1 - \frac{1}{t}\right)\right\} b(t)$$
$$= \left\{1 - \left(1 - \frac{1}{t}\right)\right\} \cdot t = \frac{1}{t} \cdot t = 1$$

$\Rightarrow$ (III) is satified with $\alpha = 1$.

$\exists$ $a_n > 0$ & $b_n \in \mathbb{R}$ such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \to e^{-(-x)^1} = e^{x}$$

LECTURE

2 OCTOBER

9:00-10:00AM

MATH3/4/68181

We want to look at the limit of

$$\frac{M_n - b_n}{a_n}, \quad a_n > 0, \quad b_n \in \mathbb{R}$$

as $n \to \infty$.

$$P\left(\frac{M_n - b_n}{a_n} < x \right)$$

$$= P\left(M_n \leq a_n x + b_n \right)$$

$$= P(X_1 \leq a_n x + b_n, \dots, X_n \leq a_n x + b_n)$$

$$\overset{\text{indep}}{=} P(X_1 \leq a_n x + b_n) \cdots P(X_n \leq a_n x + b_n)$$

$$= F(a_n x + b_n) \cdots F(a_n x + b_n)$$

$$= F^n (a_n x + b_n)$$

Does this have a non-degenerate limit as $n \to \infty$?

Extremal Types Theorem (ETT)

If there exists $a_n > 0$ & $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) \longrightarrow G(x), \quad n \to \infty$$

for a non-degenerate G then G must be of the same type as

$$\text{I}: \ \Lambda(x) = e^{-e^{-x}}, \quad -\infty < x < \infty$$

[Gumbel]

$$\text{II}: \ \Phi_\alpha(x) = \begin{cases} e^{-x^{-\alpha}}, & x > 0 \\ 0, & x < 0 \end{cases}$$

[Fréchet]

$$\text{III}: \ \Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha}, & x < 0 \\ 1, & x > 0 \end{cases}$$

[Weibull]

How to determine which of the three limits is attained for a given F?

G_1 & G_2 are of the <u>same type</u> if $G_1(x) = G_2(ax+b)$ for all x, $a>0$, $b \in \mathbb{R}$.

<u>eg</u> (i) $\boxed{e^{-e^{-x}}}$ & $\boxed{e^{-e^{-2x-3}}}$

are of the same type.

(ii) $\boxed{e^{-x^{-1}}}$ & $\boxed{e^{-(3x+2)^{-1}}}$

are of the same type.

(iii) $\boxed{e^{-x^{-1}}}$ & $\boxed{e^{-(-6x+1)^{-1}}}$

are not of the same type.

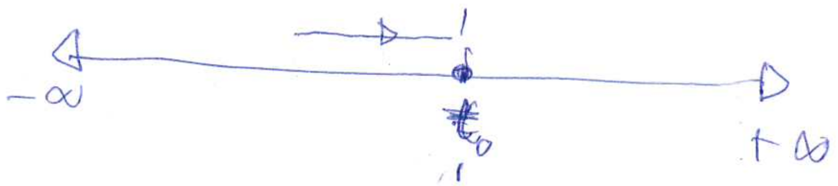
Contitions for the 3 limits

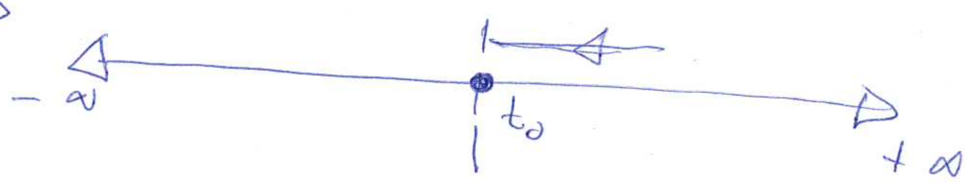
I: $\exists\ \gamma(t) > 0$ such that

$$\lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = e^{-x}$$

II: $w(F) = \infty$ & $\displaystyle\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}$

III: $w(F) < \infty$ & $\displaystyle\lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^{\alpha}$

$t \uparrow t_0 \iff$ 

$t \downarrow t_0 \iff$ 

$$w(F) = \sup\{x : F(x) < 1\}$$

" upper end point of F "

$w(F)$ is found simply as the root of $\boxed{F(x) = 1.}$

<u>eg</u> (i) $F(x) = 1 - e^{-x}$

$$\Rightarrow F(x) = 1 \iff 1 - e^{-x} = 1$$

$$\iff e^{-x} = 0 \iff x = +\infty$$

$$\iff w(F) = +\infty.$$

(ii) $F(x) = 1 - \frac{1}{x}, \quad x \geq 1$

$$\Rightarrow F(x) = 1 \iff 1 - \frac{1}{x} = 1 \iff \frac{1}{x} = 0$$

$$\iff x = +\infty$$

$$\iff w(F) = +\infty.$$

eg

1) $F(x) = 1 - e^{-x}, \quad x > 0$

$F^{-1}(x) = -\log(1-x)$

$\omega(F) = +\infty$

$$\lim_{t \uparrow +\infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \lim_{t \uparrow +\infty} \frac{\cancel{1} - (\cancel{1} - e^{-t - x\gamma(t)})}{\cancel{1} - (\cancel{1} - e^{-t})}$$

$$= \lim_{t \uparrow +\infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}} = \lim_{t \uparrow +\infty} e^{-x\gamma(t)}$$

$$= e^{-x} \quad \text{if } \boxed{\gamma(t) = 1}$$

$\Rightarrow$ Condition (I) is satisfied

$\Rightarrow \exists\ a_n > 0$ & $b_n \in \mathbb{R}$ such that

$$\boxed{P\left(\frac{M_n - b_n}{a_n} \leq x\right) \to e^{-e^{-x}}, \quad n \to \infty}$$

$a_n = \gamma\left(F^{-1}\left(1 - \frac{1}{n}\right)\right) = 1$

$b_n = F^{-1}\left(1 - \frac{1}{n}\right) = -\log\left(1 - \left(1 - \frac{1}{n}\right)\right) = \log n$

$$\Rightarrow P\left(\frac{M_n - \log n}{1} \leq x\right) \to e^{-e^{-x}}, \quad n \to \infty$$

eg

2) $F(x) = 1 - \frac{1}{x}, \quad x \geq 1 \qquad F^{-1}(x) = \frac{1}{1-x}$

$\omega(F) = +\infty.$

$$\lim_{t \uparrow +\infty} \frac{1 - F(\cancel{t} + x\gamma(t))}{1 - F(t)} = \lim_{t \uparrow +\infty} \frac{\cancel{1} - (\cancel{1} - \frac{1}{t + x\gamma(t)})}{\cancel{1} - (\cancel{1} - \frac{1}{t})}$$

$$= \lim_{t \uparrow +\infty} \frac{\frac{1}{t + x\gamma(t)}}{\frac{1}{t}} = \lim_{t \uparrow +\infty} \frac{t}{t + x\gamma(t)}$$

$$= \lim_{t \uparrow +\infty} \frac{1}{1 + x\frac{\gamma(t)}{t}} \neq e^{-x}$$

$\Rightarrow$ Condition (I) is not satisfied

$$\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t \uparrow \infty} \frac{\cancel{1} - (\cancel{1} - \frac{1}{tx})}{\cancel{1} - (\cancel{1} - \frac{1}{t})}$$

$$= \lim_{t \uparrow \infty} \frac{\frac{1}{tx}}{\frac{1}{t}} = \frac{1}{x} = x^{-1}$$

$\Rightarrow$ Condition (II) is satisfied with $\boxed{\alpha = 1}$

$\Rightarrow \exists\ a_n > 0$ & $b_n \in \mathbb{R}$ such that

$$\boxed{P\left(\frac{M_n - b_n}{a_n} < x\right) \to \begin{cases} e^{-x^{-1}} & , x > 0 \\ 0 & , x < 0 \end{cases} \quad n \to \infty}$$

$a_n = F^{-1}(1 - \frac{1}{n}) = \frac{1}{1 - (1 - \frac{1}{n})} = n, \quad b_n = 0$

$$\Rightarrow \boxed{P\left(\frac{M_n}{n} < x\right) \to \begin{cases} e^{-x^{-1}} & , x > 0 \\ 0 & , x < 0 \end{cases}}$$

How to determine a_n & b_n if one of the limits is attained?

I : $a_n = \gamma\left(F^{-1}\left(1-\frac{1}{n}\right)\right)$, $b_n = F^{-1}\left(1-\frac{1}{n}\right)$

II : $a_n = F^{-1}\left(1-\frac{1}{n}\right)$, $b_n = 0$

III : $a_n = w(F) - F^{-1}\left(1-\frac{1}{n}\right)$, $b_n = w(F)$.

LECTURE

6 OCTOBER

9:00-10:00AM

MATH3/4/68181

More Examples

1) $F(x) = [1-e^{-x}]^{\alpha}$, $x > 0$

$F^{-1}(x) = -\log(1-x^{\frac{1}{\alpha}})$

$\omega(F)$ is the value satisfying

$$F(x) = 1$$

$$\Leftrightarrow [1-e^{-x}]^{\alpha} = 1$$

$$\Leftrightarrow 1-e^{-x} = 1 \Leftrightarrow e^{-x} = 0 \Leftrightarrow x = +\infty.$$

$$\Leftrightarrow \omega(F) = +\infty.$$

$$\lim_{t \uparrow \infty} \frac{1-[1-e^{-t-x\gamma(t)}]^{\alpha}}{1-[1-e^{-t}]^{\alpha}}$$

Aside: $[1-y]^{\alpha} \approx 1-\alpha y$ for small y

$$= \lim_{t \uparrow \infty} \frac{\cancel{1}-[\cancel{1}-\alpha e^{-t-x\gamma(t)}]}{\cancel{1}-[\cancel{1}-\alpha e^{-t}]} = \lim_{t \uparrow \infty} \frac{\cancel{\alpha} e^{-t-x\gamma(t)}}{\cancel{\alpha} e^{-t}}$$

$$= e^{-x\gamma(t)} = e^{-x} \text{ if } \boxed{\gamma(t) = 1}.$$

$\exists$ $a_n > 0$ & $b_n \in \mathbb{R}$ such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \to e^{-e^{-x}}, \quad n \to \infty$$

$a_n = \gamma(F^{-1}(1-\frac{1}{n})) = 1$

$b_n = F^{-1}(1-\frac{1}{n}) = -\log[1-(1-\frac{1}{n})^{\frac{1}{\alpha}}]$

2) $\phi(x) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty$

$$F(x) = \Phi(x) \Rightarrow F^{-1}(x) = \Phi^{-1}(x)$$

$w(F)$ is the value satisfying

$$F(x) = 1$$

$$\Leftrightarrow \Phi(x) = 1 \Leftrightarrow x = +\infty$$

$$\Leftrightarrow w(F) = +\infty.$$

$$\lim_{t \uparrow \infty} \frac{1 - \Phi(t + x\gamma(t))}{1 - \Phi(t)} \underset{\text{L'Hopital's Rule}}{=} \lim_{t \uparrow \infty} \frac{-\phi(t + x\gamma(t)) \cdot (1 + x\gamma'(t))}{-\phi(t)}$$

$$= \lim_{t \uparrow \infty} \frac{\cancel{\frac{1}{\sqrt{2\pi}}} e^{-\frac{(t + x\gamma(t))^2}{2}} (1 + x\gamma'(t))}{\cancel{\frac{1}{\sqrt{2\pi}}} e^{-\frac{t^2}{2}}}$$

$$= \lim_{t \uparrow \infty} e^{-tx\gamma(t) - \frac{x^2(\gamma(t))^2}{2}} \underbrace{(1 + x\gamma'(t))}_{\to 1}$$

$$= e^{-x} \quad \text{if} \quad \gamma(t) = \frac{1}{t}, \quad \gamma'(t) = -\frac{1}{t^2}$$

$\exists$ $a_n > 0$, $b_n \in \mathbb{R}$ such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \to e^{-e^{-x}}, \quad n \to \infty$$

$$a_n = \gamma\left(\Phi^{-1}\left(1 - \frac{1}{n}\right)\right) = \left[\Phi^{-1}\left(1 - \frac{1}{n}\right)\right]^{-1}$$

$$b_n = \Phi^{-1}\left(1 - \frac{1}{n}\right).$$

3) $F(x) = 1 - \frac{1}{\log x}$, $x > e$

$w(F)$ is the value satisfying

$$1 - \frac{1}{\log x} = 1 \Leftrightarrow - \frac{1}{\log x} = 0 \Leftrightarrow x = +\infty$$

$$\Leftrightarrow w(F) = +\infty.$$

I : $$\lim_{t \uparrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \lim_{t \uparrow \infty} \frac{1 - \left(1 - \frac{1}{\log(t + x\gamma(t))}\right)}{1 - \left(1 - \frac{1}{\log t}\right)}$$

$$= \lim_{t \uparrow \infty} \frac{\log t}{\log(t + x\gamma(t))} \overset{\text{L'H Rule}}{=} \lim_{t \uparrow \infty} \frac{\frac{1}{t}}{\frac{1 + x\gamma'(t)}{t + x\gamma(t)}}$$

$$= \lim_{t \uparrow \infty} \frac{t + x\gamma(t)}{t} \cdot \frac{1}{1 + x\gamma'(t)} = \lim_{t \uparrow \infty} \left(1 + x\frac{\gamma(t)}{t}\right) \frac{1}{1 + x\gamma'(t)}$$

$\neq e^{-x} \Rightarrow$ (I) is not satisfied

II : $$\lim_{t \uparrow \infty} \frac{\cancel{1} - \left(\cancel{1} - \frac{1}{\log(tx)}\right)}{\cancel{1} - \left(\cancel{1} - \frac{1}{\log t}\right)} = \lim_{t \uparrow \infty} \frac{\log t}{\log(tx)}$$

$$= \lim_{t \uparrow \infty} \frac{\log t}{\log t + \log x} = 1 \neq x^{\alpha} \Rightarrow$$ (II) is not satisfied

III : $\cancel{\lim_{t \uparrow \infty}}$ $w(F) = +\infty \not< \infty \Rightarrow$ (III) is not satisfied

$\Rightarrow$ None of the 3 conditions is satisfied

EXAMPLE CLASS

6 OCTOBER

10:00-11:00AM

MATH3/4/68181

1)

$$\Lambda(x) = e^{-e^{-x}}$$

$$\Lambda'(x) = e^{-e^{-x}} (-1)\, e^{-x} (-1)$$

$$= e^{-e^{-x}} e^{-x}$$

$$\Phi_\alpha(x) = \begin{cases} e^{-x^{-\alpha}} & , x > 0 \\ 0 & , x < 0 \end{cases}$$

$$\Phi_\alpha'(x) = \begin{cases} e^{-x^{-\alpha}} (-1)(-\alpha)\, x^{-\alpha-1} & , x > 0 \\ 0 & , x < 0 \end{cases}$$

$$= \begin{cases} \alpha x^{-\alpha-1} e^{-x^{-\alpha}} & , x > 0 \\ 0 & , x < 0 \end{cases}$$

$$\Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha} & x < 0 \\ 1 & x > 0 \end{cases}$$

$$\Psi_\alpha'(x) = \begin{cases} e^{-(-x)^\alpha} (-1)\, \alpha (-x)^{\alpha-1} (-1) & , x < 0 \\ 0 & , x > 0 \end{cases}$$

$$= \begin{cases} \alpha (-x)^{\alpha-1} e^{-(-x)^\alpha} & , x < 0 \\ 0 & , x > 0 \end{cases}$$

2. $$\int_{-\infty}^{\infty} x \cdot e^{-x} e^{-e^{-x}} \, dx$$

$$= \int_{\infty}^{0} (-\log y) \cdot y \cdot e^{-y} \left(-\frac{dy}{y}\right)$$

$$\boxed{\begin{aligned} y &= e^{-x} \\ x &= -\log y \\ \frac{dx}{dy} &= -\frac{1}{y} \end{aligned}}$$

$$= -\int_{0}^{\infty} \log y \cdot e^{-y} \, dy$$

$$\boxed{\begin{aligned} \frac{d}{da} y^a &= y^a \log y \\ \frac{d}{da} y^a \Big|_{a=0} &= y^a \log y \Big|_{a=0} \\ &= y^0 \cdot \log y \\ &= \log y \end{aligned}}$$

$$= -\int_{0}^{\infty} \left(\frac{d}{da} y^a \Big|_{a=0} \right) e^{-y} \, dy$$

$$= -\frac{d}{da} \left(\int_{0}^{\infty} y^a e^{-y} \, dy \right) \Big|_{a=0}$$

$$= -\frac{d}{da} \Gamma(a+1) \Big|_{a=0} = -\Gamma'(a+1) \Big|_{a=0} = -\Gamma'(1).$$

4) $$\Lambda^n(x) = \Lambda(\alpha_n x + \beta_n)$$

$$\Leftrightarrow \left[e^{-e^{-x}}\right]^n = e^{-e^{-\alpha_n x - \beta_n}}$$

$$\Leftrightarrow e^{-ne^{-x}} = e^{-e^{-\alpha_n x - \beta_n}}$$

$$\Leftrightarrow -ne^{-x} = -e^{-\alpha_n x - \beta_n}$$

$$\Leftrightarrow ne^{-x} = e^{-\alpha_n x - \beta_n}$$

$$\Leftrightarrow \log n - x = -\alpha_n x - \beta_n$$

$$\Leftrightarrow -1 = -\alpha_n \quad \& \quad \log n = -\beta_n$$

$$\Leftrightarrow \alpha_n = 1 \quad \& \quad \beta_n = -\log n$$

10) $F(x) = 1 - \left(\frac{k}{x}\right)^a \quad , \quad x \geq k \qquad F^{-1}(x) = k(1-x)^{-\frac{1}{a}}$

$w(F)$ is the value satisfying

$$F(x) = 1 \Leftrightarrow 1 - \left(\frac{k}{x}\right)^a = 1 \Leftrightarrow \left(\frac{k}{x}\right)^a = 0$$

$$\Leftrightarrow x = +\infty \Leftrightarrow w(F) = +\infty .$$

(I): $$\lim_{t \uparrow \infty} \frac{1 - \left[1 - \left(\frac{k}{t + x\gamma(t)}\right)^a\right]}{1 - \left[1 - \left(\frac{k}{t}\right)^a\right]} = \lim_{t \uparrow \infty} \left(\frac{t}{t + x\gamma(t)}\right)^a$$

$$= \lim_{t \uparrow \infty} \left(\frac{1}{1 + x \cdot \frac{\gamma(t)}{t}}\right)^a \neq e^{-x} \Rightarrow \text{(I) is not satisfied}$$

(II): $$\lim_{t \uparrow \infty} \frac{1 - \left[1 - \left(\frac{k}{tx}\right)^a\right]}{1 - \left[1 - \left(\frac{k}{t}\right)^a\right]} = \lim_{t \uparrow \infty} \left(\frac{t}{tx}\right)^a = x^{-a}$$

$\Rightarrow$ (II) is satisfied

$\exists$ $a_n > 0$ & $b_n \in \mathbb{R}$ such that:

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \to \begin{cases} e^{-(x)^{-a}} & x > 0 \\ 0 & x < 0 \end{cases}$$

$$a_n = k\left(1 - \left(1 - \frac{1}{n}\right)\right)^{-\frac{1}{a}} = k n^{\frac{1}{a}} \quad , \quad b_n = 0$$

L'Hopital's Rule

$$\boxed{\lim_{x \uparrow \infty} \frac{g_1(x)}{g_2(x)} = \lim_{x \uparrow \infty} \frac{g_1'(x)}{g_2'(x)}}$$

LECTURE

8 OCTOBER

12:00-13:00PM

MATH4/68181

11). $F(x) = k \cdot \int_0^{G(x)} t^{a-1}(1-t)^{b-1} e^{-ct}\, dt$

i) assume G has the Gumbel limit and then show that F also has the Gumbel limit

ii) assume G has the Frechet limit and then show that F also has the Frechet limit

iii) assume G has the Weibull limit and then show that F also has the Weibull limit.

11)

i) Assume G has Gumbel Limit. This means $\exists \gamma(t) > 0$ such that

$$\boxed{\lim_{t \uparrow w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x}}$$

We want to show that this holds with G replaced by F.

$$\lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} \overset{LH}{=} \lim_{t \uparrow w(F)} \frac{-f(t + x\gamma(t))(1 + x\gamma'(t))}{-f(t)}$$

$$= \lim_{t \uparrow w(F)} \frac{K g(t + x\gamma(t)) (G(t + x\gamma(t)))^{a-1} [1 - G(t + x\gamma(t))]^{b-1} e^{-cG(t + x\gamma(t))} (1 + x\gamma'(t))}{K g(t) (G(t))^{a-1} \cdot [1 - G(t)]^{b-1} e^{-cG(t)}}$$

$$\overset{\text{assumption}}{=} \lim_{t \uparrow w(G)} \frac{g(t + x\gamma(t))}{g(t)} \Bigg[\frac{\overset{\to 1}{G(t + x\gamma(t))}}{\underset{\to 1}{G(t)}}\Bigg]^{a-1} \Bigg[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\Bigg]^{b-1}_{\to e^{-x}} \frac{e^{-c\overset{\to 1}{G(t + x\gamma(t))}}}{e^{-c\underset{\to 1}{G(t)}}} (1 + x\gamma'(t))$$

$$= \lim_{t \uparrow w(G)} e^{-x(b-1)} \left[\frac{g(t + x\gamma(t))}{g(t)} (1 + x\gamma'(t))\right]$$

$$\overset{LH}{=} \lim_{t \uparrow w(G)} e^{-x(b-1)} \Bigg[\underbrace{\frac{1 - G(t + x\gamma(t))}{1 - G(t)}}_{\to e^{-x}}\Bigg] = e^{-bx}$$

same type as e^{-x}

So, F also has the Gumbel limit.

ii) Assume that G has the Frechet limit. That is

$$\lim_{t \uparrow \infty} \frac{1-G(tx)}{1-G(t)} = x^{-\alpha}, \quad \alpha > 0$$

We want to show that this holds with G replaced by F.

$$\lim_{t \uparrow \infty} \frac{1-F(tx)}{1-F(t)} \overset{LH}{=} \lim_{t \uparrow \infty} \frac{-x f(tx)}{-f(t)}$$

$$= \lim_{t \uparrow \infty} \frac{x \cdot \cancel{k}\, g(tx) \left(G(tx)\right)^{a-1} \left[1-G(tx)\right]^{b-1} e^{-c\,G(tx)}}{\cancel{k}\, g(t) \left(G(t)\right)^{a-1} \left[1-G(t)\right]^{b-1} e^{-c\,G(t)}}$$

$$= \lim_{t \uparrow \infty} \frac{x \cdot g(tx)}{g(t)} \left[\frac{G(tx)}{G(t)}\right]^{a-1} \left[\frac{1-G(tx)}{1-G(t)}\right]^{b-1} \frac{e^{-c\,G(tx)}}{e^{-c\,G(t)}}$$

$1 \quad 1 \quad x^{-\alpha} \quad 1 \quad 1$

$$= \lim_{t \uparrow \infty} x^{-(b-1)\alpha} \left[\frac{x\,g(tx)}{g(t)}\right] \overset{LH}{=} \lim_{t \uparrow \infty} x^{-(b-1)\alpha} \left[\boxed{\frac{1-G(tx)}{1-G(t)}}\right] \rightarrow x^{-\alpha}$$

$$= x^{-(b-1)\alpha} \cdot x^{-\alpha} = x^{-b\alpha}$$

same type as $x^{-\alpha}$,

So, F also has the Frechet limit.

(iii) Assume that G has the Weibull limit. That is

$$\lim_{t \downarrow 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^{\alpha}, \quad \alpha > 0.$$

We want to show that this holds with G replaced by F. Assume $w(F) = w(G)$.

$$\lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} \overset{LH}{=} \lim_{t \downarrow 0} \frac{-f(w(F) - tx) \cdot (-x)}{-f(w(F) - t) \cdot (-1)}$$

$$= \lim_{t \downarrow 0} \frac{x \cdot \cancel{K} \cdot g(w(F) - tx)\,\big(G(w(F) - tx)\big)^{a-1}\,[1 - G(w(F) - tx)]^{b-1}\, e^{-cG(w(F) - tx)}}{\cancel{K} \cdot g(w(F) - t)\,\big(G(w(F) - t)\big)^{a-1}\,[1 - G(w(F) - t)]^{b-1}\, e^{-cG(w(F) - t)}}$$

$$\overset{\text{assumption}}{=} \lim_{t \downarrow 0} x \cdot \frac{g(w(G) - tx)}{g(w(G) - t)} \left[\frac{\overbrace{G(w(G) - tx)}^{\to 1}}{\underbrace{G(w(G) - t)}_{\to 1}}\right]^{a-1} \left[\underbrace{\frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)}}_{\to x^{\alpha}}\right]^{b-1} \frac{e^{-c\overbrace{G(w(G) - tx)}^{\to 1}}}{e^{-c\underbrace{G(w(G) - t)}_{\to 1}}}$$

$$= \lim_{t \downarrow 0} \left[x \cdot \frac{g(w(G) - tx)}{g(w(G) - t)}\right] \cdot (x^{\alpha})^{b-1}$$

$$\overset{LH}{=} \lim_{t \downarrow 0} \left[\overbrace{\frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)}}^{\to x^{\alpha}}\right] \cdot x^{\alpha(b-1)} = x^{\alpha b}, \text{ same type as } x^{\alpha}$$

So, F also has the Weibull limit.

LECTURE

9 OCTOBER

9:00-10:00AM

MATH3/4/68181

It is not convenient to have 3 distns to model sample maxima.

Can the 3 distn be combined into one? Yes.

The <u>GEV</u> (Generalised Extreme Value) distn combines the 3 into 1 form.

Its cdf is

$$G(x) = e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}},$$

$-\infty < \mu < \infty$ "location parameter"

$\sigma > 0$ "scale parameter"

$-\infty < \xi < \infty$ "shape parameter"

Domain of x: $1+\xi\frac{x-\mu}{\sigma} > 0$.

GEV contains the 3 limits as particular cases. Why?

$\boxed{\xi = 0}$: $G(x) = e^{-e^{-\frac{x-\mu}{\sigma}}}$, same type as Gumbel

$$\boxed{\left(1+\frac{y}{n}\right)^n \to e^y \quad \text{as } n \to \infty}$$

$\boxed{\xi > 0}$: $G(x) = e^{-\left(\frac{\sigma + \xi x - \xi \mu}{\sigma}\right)^{-\frac{1}{\xi}}}$

$$= e^{-\left(\frac{\xi}{\sigma} x + \frac{\sigma - \xi\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

same type as $e^{-x^{-\frac{1}{\xi}}}$

the Fréchet

$\boxed{\xi < 0}$: $G(x) = e^{-\left(\frac{\sigma + \xi x - \xi \mu}{\sigma}\right)^{-\frac{1}{\xi}}}$

$$= e^{-\left(-\left(\frac{-\xi}{\sigma} x + \frac{\xi\mu - \sigma}{\sigma}\right)\right)^{-\frac{1}{\xi}}}$$

same type as $e^{-(-x)^{-\frac{1}{\xi}}}$

the Weibull case

GEV contains the Gumbel, Fréchet & Weibull limits as particular cases.

Suppose $X_1, X_2, \ldots, X_n$ are iid obsns on sample maxima. We need to estimate μ, σ, & ξ. We use MLE.

$$g(x) = \frac{1}{\sigma}\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \cdot e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

The likelihood function is

$$L(\mu, \sigma, \xi) = \prod_{i=1}^{n}\left[\frac{1}{\sigma}\left(1+\xi\frac{X_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} e^{-\left(1+\xi\frac{X_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}\right]$$

$$\log L = -n\log\sigma - \left(\frac{1}{\xi}+1\right)\sum_{i=1}^{n}\log\left(1+\xi\frac{X_i-\mu}{\sigma}\right) - \sum_{i=1}^{n}\left(1+\xi\frac{X_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}}$$

$\boxed{\text{MLE}}$ $\boxed{\text{equations}}$ $\boxed{\text{for}}$ $\boxed{\text{GEV}}$

$$\frac{\partial \log L}{\partial \mu} = \frac{1+\xi}{\sigma} \sum_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-1}$$

$$- \frac{1}{\sigma} \sum_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi} - 1} = 0 \quad \text{— (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1+\xi}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu) \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-1}$$

$$- \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu) \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi} - 1} = 0$$

— (2)

$$\frac{\partial \log L}{\partial \xi} = \frac{1}{\xi^2} \sum_{i=1}^{n} \log \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)$$

$$- \frac{1}{\sigma}\left(\frac{1}{\xi} + 1\right) \sum_{i=1}^{n} (x_i - \mu) \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-1}$$

$$- \sum_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi}} \left[\frac{1}{\xi^2} \log \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)\right.$$

$$\left. - \frac{1}{\xi} \frac{x_i - \mu}{\sigma} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-1}\right] = 0$$

— (3)

The MLEs of μ, σ & ξ are the solns of $\frac{\partial \log L}{\partial \mu} = 0$, $\frac{\partial \log L}{\partial \sigma} = 0$ & $\frac{\partial \log L}{\partial \xi} = 0$.

<u>fgev</u> in R solves these equations for μ, σ & ξ

Generalized Pareto Distribution

$$P(X > x+u \mid X > u)$$

$$= P(\underbrace{X-u}_{} > x \mid \underbrace{X-u}_{} > 0)$$

exceeded amount

$$\longrightarrow \left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}} \quad \text{as} \quad u \to \infty$$

Pickands (1975)

$$\Rightarrow \frac{1-F(x+u)}{1-F(u)} \approx \left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}} \quad \text{if}$$

u is large

$$\Leftrightarrow \quad 1-F(x+u) \approx [1-F(u)]\left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$\Leftrightarrow \quad F(x+u) \approx 1-[1-F(u)]\left(1+\xi\frac{x}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$\Leftrightarrow \quad \boxed{F(y) \approx 1-[1-F(u)]\left(1+\xi\frac{(y-u)}{\sigma}\right)^{-\frac{1}{\xi}}}$$

GP model

$-\infty < \xi < \infty$ "shape parameter"

$\sigma > 0$ "scale parameter"

Suppose $x_1, x_2, \ldots, x_n$ are data on exceedances. We want to estimate ξ & σ. We use MLE. The density of F is

$$f(y) = \frac{1}{\sigma}\left[1-F(u)\right]\left(1+\xi\frac{(y-u)}{\sigma}\right)^{-\frac{1}{\xi}-1}.$$

The likelihood function is

$$L(\sigma,\xi) = \prod_{i=1}^{n}\left\{\frac{1}{\sigma}\left[1-F(u)\right]\left[1+\xi\frac{x_i-u}{\sigma}\right]^{-\frac{1}{\xi}-1}\right\}$$

$$= \frac{1}{\sigma^n}\left[1-F(u)\right]^n\left\{\prod_{i=1}^{n}\left[1+\xi\frac{x_i-u}{\sigma}\right]\right\}^{-\frac{1}{\xi}-1}$$

The log-likelihood is

$$\log L = -n\log\sigma + n\log\left[1-F(u)\right] - \left(\frac{1}{\xi}+1\right)\sum_{i=1}^{n}\log\left[1+\xi\frac{x_i-u}{\sigma}\right]$$

MLE Equations for GP

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1+\xi}{\sigma^2} \sum_{i=1}^{n} (x_i - u)\left[1+\frac{\xi}{\sigma}(x_i-u)\right]^{-1}$$

$$= 0$$

$$\frac{\partial \log L}{\partial \xi} = \frac{1}{\xi^2} \sum_{i=1}^{n} \log\left[1+\frac{\xi}{\sigma}(x_i-u)\right]$$

$$-\frac{1}{\sigma}\left(\frac{1}{\xi}+1\right)\sum_{i=1}^{n} (x_i-u)\left[1+\frac{\xi}{\sigma}(x_i-u)\right]^{-1}$$

$$= 0$$

The MLEs are the solutions of

$$\frac{\partial \log L}{\partial \sigma} = 0 \quad \& \quad \frac{\partial \log L}{\partial \xi} = 0.$$

fpot in R solves these equations for σ & ξ.

Suppose we consider r largest obsns.

It can be shown that their joint PDF is

$$f(x_1, x_2, \dots, x_r) = \sigma^{-r} e^{-\left(1+\xi \frac{x_r - \mu}{\sigma}\right)^{-1/\xi}} \cdot e^{-\left(\frac{1}{\xi}+1\right) \sum_{i=1}^{r} \log\left(1+\xi \frac{x_i-\mu}{\sigma}\right)}$$

$$x_1 > x_2 > \dots > x_r$$

$-\infty < \mu < \infty$ "location parameter"

$\sigma > 0$ "scale parameter"

$-\infty < \xi < \infty$ "shape parameter"

$\boxed{\text{MLE}}$ $\boxed{\text{Equations}}$ $\boxed{\text{for}}$ $\boxed{R}$ $\boxed{\text{Largest}}$

$$\frac{\partial \log L}{\partial \mu} = -\frac{1}{\sigma}\sum_{i=1}^{n}\left(1+\xi\frac{X_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1}$$

$$+\frac{\xi+1}{\sigma}\sum_{i=1}^{n}\sum_{j=1}^{r}\left(1+\xi\frac{X_{i,j}-\mu}{\sigma}\right)^{-1} = 0$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{nr}{\sigma} - \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_{i,r}-\mu)\left(1+\xi\frac{X_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1}$$

$$+\frac{\xi+1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{r}(X_{i,j}-\mu)\left(1+\xi\frac{X_{i,j}-\mu}{\sigma}\right)^{-1}$$

$$= 0$$

$$\frac{\partial \log L}{\partial \xi} = -\sum_{i=1}^{n}\left(1+\xi\frac{X_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\left[\frac{1}{\xi^2}\log\left(1+\xi\frac{X_{i,r}-\mu}{\sigma}\right)\right.$$

$$\left.-\frac{1}{\sigma\xi}(X_{i,r}-\mu)\left(1+\xi\frac{X_{i,r}-\mu}{\sigma}\right)^{-1}\right]$$

$$+\frac{1}{\xi^2}\sum_{i=1}^{n}\sum_{j=1}^{r}\log\left(1+\xi\frac{X_{i,j}-\mu}{\sigma}\right)$$

$$-\frac{1}{\sigma}\left(\frac{1}{\xi}+1\right)\sum_{i=1}^{n}\sum_{j=1}^{r}(X_{i,j}-\mu)\left(1+\xi\frac{X_{i,j}-\mu}{\sigma}\right)^{-1}$$

$$= 0$$

LECTURE

13 OCTOBER

9:00-10:00AM

MATH3/4/68181

Friday 16 Oct 9:00am

Stopford Lect Th 6

ETT does not always hold, i.e. contitions (I) - (III) fail to hold for certain F.

It can be a big job checking conditions (I) - (III) for a given F.

Is there a short cut to find out none of contitions (I) - (III) are satisfied?

Suppose F is the CDF of a discrete RV X. Then ETT will not hold (i.e. none of conditions (I) - (III) will be satisfied] if

$$\boxed{\lim_{k \uparrow w(F)} \frac{P(X=k)}{1-F(k-1)} \neq 0.}$$

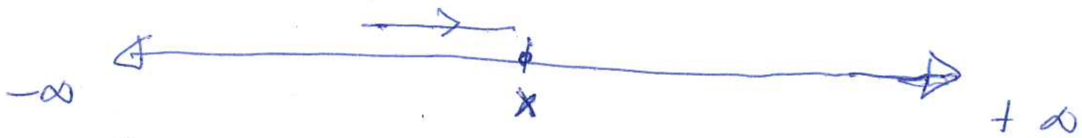
OR

$$\boxed{\lim_{k \uparrow w(F)} \frac{P(X=k)}{\sum_{j=k}^{\infty} P(X=j)} \neq 0}$$

Suppose F is the CDF of a continuous RV X. Let $f(x) = dF(x)/dx$.

Then ETT will not hold (i.e. none of the conditions (I) - (III) will be satisfied) if

$$\lim_{t \uparrow \omega(F)} \frac{f(x)}{1 - F(x^-)} \neq 0$$



eg 1

$$X \sim \text{Geom}(p)$$

$$P(X = k) = p(1-p)^{k-1}, \quad k \geq 1$$

$$F(k) = P(X \leq k) = 1 - (1-p)^k$$

show this

$w(F)$ is the solution of

$$F(\cancel{w(F)}\,X) = 1 \iff 1-(1-p)^X = 1$$

$$\iff (1-p)^X = 0$$

$$\iff X = +\infty$$

$$\Rightarrow w(F) = +\infty$$

$$\lim_{k \uparrow \infty} \frac{p(1-p)^{k-1}}{X - [X - (1-p)^{k-1}]} = p \neq 0$$

$\Rightarrow$ None of conditions (I) - (III) will be satisfied.

$\Rightarrow$ EVT will not hold.

eg 2

$X \sim$ Discrete Uniform on $\{1, 2, \ldots, n\}$.

$$P(X = k) = \frac{1}{n}$$

$$F(k) = \frac{k}{n}$$

$w(F)$ is the solution of

$$F(x) = 1 \iff \frac{x}{n} = 1 \iff x = n$$
$$\Rightarrow w(F) = n.$$

$$\lim_{k \uparrow n} \frac{\frac{1}{n}}{1 - \frac{k-1}{n}} = \frac{\frac{1}{n}}{1 - \frac{n-1}{n}} = 1 \neq 0$$

$\Rightarrow$ None of conditions (I) - (III) will be satisfied

$\Rightarrow$ ETT will not hold.

eg 3

$X \sim \text{Bernoulli}(p)$

$$X = \begin{cases} 1 & \text{with prob} \quad p \\ 0 & \quad '' \qquad '' \quad 1-p \end{cases}$$

$$F(k) = \begin{cases} 0 & \text{if} \quad k < 0 \\ 1-p & \text{if} \quad k = 0 \\ 1 & \text{if} \quad k \geq 1 \end{cases}$$

$w(F)$ is the solution of

$$F(x) = 1 \iff x = 1$$

$$\lim_{k \uparrow 1} \frac{P(X=k)}{1-F(k-1)} = \frac{P(X=1)}{1-F(0)} = \frac{p}{1-(1-p)} = 1 \neq 0$$

$\Rightarrow$ None of conditions (I)-(III) will be satisfied

$\Rightarrow$ ETT will not hold.

eg 4

$$P(X=k) = \frac{k^{-s}}{\zeta(s)}, \quad k \geq 1$$

$\zeta(\cdot)$ is Riemann zeta function.

It can be shown that $\omega(F) = +\infty$.

$$\lim_{k \uparrow \infty} \frac{\cancel{\frac{k^{-s}}{\zeta(s)}}}{\sum_{j=k}^{\infty} \cancel{\frac{j^{-s}}{\zeta(s)}}} = \lim_{k \uparrow \infty} \frac{k^{-s}}{\sum_{j=k}^{\infty} j^{-s}}$$

$$= \lim_{k \uparrow \infty} \frac{k^{-s}}{\int_k^{\infty} x^{-s}\, dx} = \lim_{k \uparrow \infty} \frac{k^{-s}}{\left[\frac{x^{1-s}}{1-s}\right]_k^{\infty}}$$

$$= \lim_{k \uparrow \infty} \frac{k^{-s}}{0 - \frac{k^{1-s}}{1-s}} \quad \text{if } 1-s < 0$$

$$= \lim_{k \uparrow \infty} -\frac{1-s}{k} = 0 \Rightarrow$$ One of the conditions (I)-(III) will hold.

Homework: Check which condition is satisfied.

EXAMPLE CLASS

13 OCTOBER

10:00-11:00AM

MATH3/4/68181

Q2

$$P(k) = \begin{cases} 1 & k = k_0 \\ 0 & k \neq k_0 \end{cases}$$

$$F(k) = \begin{cases} 0 & \text{if } k < k_0 \\ 1 & \text{if } k \geq k_0 \end{cases}$$

$\omega(F)$ is the solution of

$$F(x) = 1 \iff \cancel{X} = \cancel{k_0} \Rightarrow \omega(F) = k_0$$

$$\lim_{k \uparrow k_0} \frac{P(X=k)}{1 - F(k-1)} = \frac{P(k_0)}{1 - F(k_0 - 1)} = \frac{1}{1 - 0} = 1 \neq 0$$

$\Rightarrow$ ETT fails to hold.

Q5

$$F(k) = 1 - \log_2 \frac{k+2}{k+1}$$

$w(F)$ is the solution of

$$F(x) = 1$$

$$\Leftrightarrow \quad 1 - \log_2 \frac{k+2}{k+1} = 1 \quad \Leftrightarrow \quad \log_2 \left(\frac{k+2}{k+1}\right) = 0$$

$$\Leftrightarrow \quad \frac{k+2}{k+1} = 1 \quad \not\Leftrightarrow \quad k = +\infty \quad \Rightarrow \quad w(F) = +\infty.$$

$$\lim_{k \uparrow \infty} \frac{-\log_2\left[1 - (k+1)^{-2}\right]}{\cancel{1} - \left[\cancel{1} - \log_2 \frac{k+1}{k}\right]} = \lim_{k \uparrow \infty} \frac{\log_2\left[1 - (k+1)^{-2}\right]}{\log_2 \frac{k+1}{k}}$$

$$\overset{LH}{\underset{\downarrow}{=}} \lim_{k \uparrow \infty} \frac{\underbrace{\frac{1}{1-(k+1)^{-2}}}_{1} \cdot \cancel{\frac{1}{\log 2}} \cdot (-1)(-2)(k+1)^{-3}(-1)}{\underbrace{\frac{k}{k+1}}_{1} \cdot \cancel{\frac{1}{\log 2}} \cdot (-1)\, k^{-2}}$$

$$\boxed{\frac{\partial \log_a z}{\partial z} = \frac{1}{z \log a}}$$

$$= \lim_{k \uparrow \infty} + \frac{2k^2}{(k+1)^3} = 0 \Rightarrow \text{ETT will hold.}$$

Q6

$$p(k) = \frac{p^k}{1+\theta}\left[\theta(1-2p) + (1-p)(1+\theta k)\right]$$

$$F(k) = \sum_{j=0}^{k} p(j)$$

$$= \sum_{j=0}^{k} \frac{p^j}{1+\theta}\left[\theta(1-2p) + (1-p)(1+\theta j)\right]$$

$$\vdots$$

$$= 1 - \frac{1+\theta+\theta k}{1+\theta}\, p^k$$

$w(F)$ is the value satisfying

$$F(x) = 1$$

$$\Leftrightarrow \quad 1 - \frac{1+\theta+\theta x}{1+\theta}\, p^x = 1 \quad \Leftrightarrow \quad x = +\infty$$

$$\lim_{k \uparrow \infty} \frac{\frac{p^k}{1+\theta}\left[\theta(1-2p) + (1-p)(1+\theta k)\right]}{1 - \left[1 - \frac{1+\theta+\theta(k-1)}{1+\theta}\, p^{k-1}\right]} = 1 - p \neq 0$$

$\Rightarrow$ ETT fails to hold.

LECTURE

15 OCTOBER

12:00-13:00PM

MATH4/68181

Financial Ratios

eg

1) $\text{Current ratio} = \dfrac{\text{Current assets } (X)}{\text{Current liabilities } (Y)}$

2) $\text{Sales margin} = \dfrac{\overset{(X)}{\text{Sales}} - \overset{(Y)}{\text{Costs}}}{\text{Sales}(X)}$

3) $\text{Changes in Capital} = \dfrac{\overset{(X)}{\text{Closing capital}} - \overset{(Y)}{\text{Opening Capital}}}{\text{Opening Capital}(Y)}$

We want to be able to predict values of these ratios into the future.

We need to model X & Y.

The most popular model for income is the Pareto distribution.
(Italian economist)

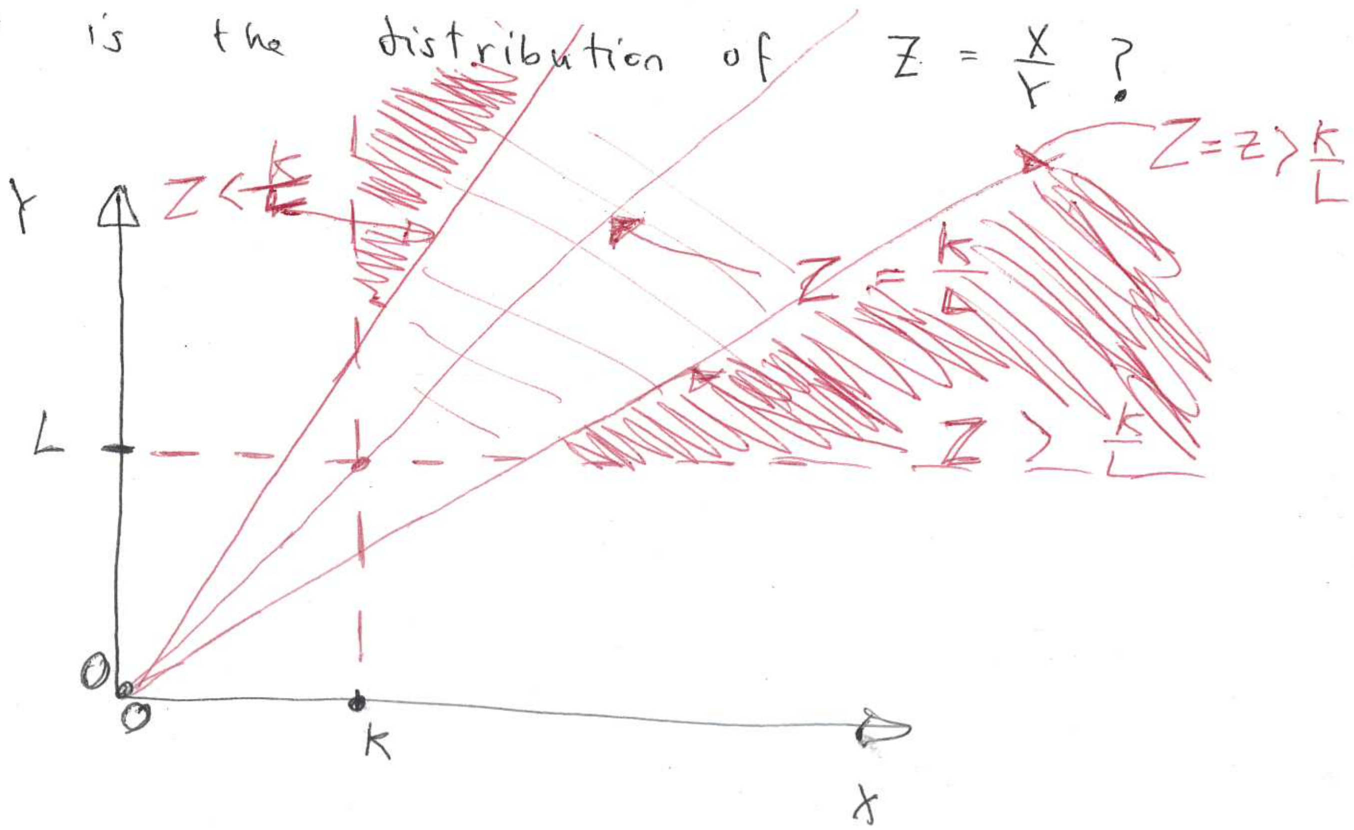
There are several Pareto distribution. The simplest is known as the Pareto type I distribution.

Suppose X & Y are indep RVs following the Pareto type I distribution with the CDFs

$$F_X(x) = 1 - \left(\frac{K}{x}\right)^a, \quad x \geq K.$$

$$F_Y(y) = 1 - \left(\frac{L}{y}\right)^b, \quad y \geq L.$$

What is the distribution of $Z = \frac{X}{Y}$?



$$\boxed{z < \frac{k}{L}}$$

$$P(Z < z) = \int_k^\infty \int_{x/z}^\infty \frac{ak^a}{x^{a+1}} \cdot \frac{bL^b}{y^{b+1}}\, dy\, dx$$

$$= abk^aL^b \int_k^\infty \frac{1}{x^{a+1}} \left[\frac{y^{-b}}{(-b)}\right]_{\frac{x}{z}}^\infty dx$$

$$= a\cancel{b}k^aL^b \int_k^\infty \frac{1}{x^{a+1}} \cdot \frac{1}{\cancel{b}} \cdot \left(\frac{x}{z}\right)^{-b} dx$$

$$= ak^aL^bz^b \int_k^\infty x^{-a-b-1}\, dx$$

$$= ak^aL^bz^b \left[\frac{x^{-a-b}}{-a-b}\right]_k^\infty$$

$$= ak^aL^bz^b \cdot \frac{k^{-a-b}}{a+b} = \frac{ak^{-b}L^bz^b}{a+b}.$$

$$\boxed{z > \frac{k}{L}}$$

$$P(Z < z) = 1 - P(Z > z)$$

$$= 1 - \int_{L}^{\infty} \int_{yz}^{\infty} \frac{bL^b}{y^{b+1}} \frac{ak^a}{x^{a+1}} \, dx \, dy$$

$$= 1 - abk^a L^b \int_{L}^{\infty} \frac{1}{y^{b+1}} \left[\frac{x^{-a}}{-a} \right]_{yz}^{\infty} dy$$

$$= 1 - \cancel{a}bk^a L^b \int_{L}^{\infty} \frac{1}{y^{b+1}} \frac{(yz)^{-a}}{\cancel{a}} dy$$

$$= 1 - bk^a L^b z^{-a} \int_{L}^{\infty} y^{-a-b-1} dy$$

$$= 1 - bk^a L^b z^{-a} \left[\frac{y^{-a-b}}{-a-b} \right]_{L}^{\infty}$$

$$= 1 - bk^a L^b z^{-a} \frac{L^{-a-b}}{a+b}$$

$$= 1 - \frac{b}{a+b} \cdot k^a L^{-a} z^{-a}$$

Z = Current Ratio

$$F_Z(z) = \begin{cases} \dfrac{a K^{-b} L^{b} z^{b}}{a+b}, & z < \dfrac{K}{L} \\[2ex] 1 - \dfrac{b}{a+b} K^{a} L^{-a} z^{-a}, & z \geq \dfrac{K}{L} \end{cases}$$

$$F_Z(z) = 0.001$$

$E(Z)$ expected current ratio

$Var(Z)$ variability of " "

$$f_Z(z) = \begin{cases} \dfrac{ab K^{-b} L^{b} z^{b-1}}{a+b}, & z < \dfrac{K}{L} \\[2ex] \dfrac{ab K^{a} L^{-a} z^{-a}}{a+b}, & z \geq \dfrac{K}{L} \end{cases}$$

$$F_Z^{-1}(p) = \begin{cases} \left(\frac{(a+b)}{a}\frac{K^b}{L^b}p\right)^{\frac{1}{b}} & p < \frac{a}{a+b} \\ \left(\frac{a+b}{b}\frac{L^a}{K^a}(1-p)\right)^{-\frac{1}{a}} & p \geq \frac{a}{a+b} \end{cases}$$

LECTURE

16 OCTOBER

9:00-10:00AM

MATH3/4/68181

Portfolio

A portfolio is a collection of investments.

Suppose there are m investments.

X_1 = loss on investment 1

X_2 = " " " 2

X_3 = " " " 3

$\vdots$

X_m = " " " m

- $X_1, X_2, \ldots, X_m$ are RVs
- m may also be a RV
- $X_1, X_2, \ldots, X_m$ could be IID RVs

  OR dependent RVs

Cases

i) $X_1, X_2, \ldots, X_m$ IID & m fixed

ii) $X_1, X_2, \ldots, X_m$ INID & m fixed
(independent & not identically distributed)

iii) $X_1, X_2, \ldots, X_m$ dep RVs & m fixed

iv) $X_1, X_2, \ldots, X_m$ IID RVs & m is a RV

v) $X_1, X_2, \ldots, X_m$ INID RVs & m is a RV

vi) $X_1, X_2, \ldots, X_m$ dep RVs & m is a RV.

Variables of interest for a portfolio

$T = X_1 + X_2 + \dots + X_m$ "total portfolio loss"

$T = w_1 X_1 + w_2 X_2 + \dots + w_m X_m$ "weighted porfolio loss"

weights

eg $w_1 = w_2 = \dots = w_m = \frac{1}{m}$ "average portfolio loss"

$\max(X_1, X_2, \dots, X_m)$ "maximum portfolio loss"

$\min(X_1, X_2, \dots, X_m)$ "minimum portfolio loss" i)

$$F_T(t) = P(T < t)$$

$$= P(X_1 + \dots + X_m < t)$$

$$= \int\int \cdots \int_{\{X_1 + \dots + X_m < t\}} f_{X_1}(x_1) \dots f_{X_m}(x_m) \, dx_1 \dots dx_m$$

$$f_T(t) = \int\int \cdots \int f_{X_1}(x_1) \, f_{X_2}(x_2) \dots f_{X_m}(t - x_1 - \dots - x_{m-1}) \, dx_1 \, dx_2 \dots dx_{m-1}$$

$$E(T) = E(X_1 + \dots + X_m) = E(X_1) + \dots + E(X_m)$$

$$Var(T) = Var(X_1 + \dots + X_m) \underset{\text{IID}}{=} Var(X_1) + \dots + Var(X_m)$$

eg $\boxed{m = 2}$ $\quad f_T(t) = \int f_{X_1}(x_1) \, f_{X_2}(t - x_1) \, dx_1$

iii) $F_T(t) = P(T < t)$

$$= P(X_1 + \dots + X_m < t)$$

$$= \int\int_{\{x_1 + \dots + x_m < t\}} \dots \int \boxed{f_{X_1, \dots, X_m}(x_1, \dots, x_m)} \, dx_1 \dots dx_m$$

Joint PDF of $(X_1, \dots, X_m)$

$$f_T(t) = \int\int \dots \int f_{X_1, \dots, X_m}(x_1, \dots, x_{m-1}, t - x_1 - \dots - x_{m-1}) \, dx_1 \dots dx_{m-1}$$

$$E(T) = E(X_1 + \dots + X_m) = E(X_1) + \dots + E(X_m)$$

$$\mathrm{Var}(T) = \mathrm{Var}(X_1 + \dots + X_m)$$

$$= \mathrm{Var}(X_1) + \dots + \mathrm{Var}(X_m) + \sum_{i \neq j} \boxed{\mathrm{Cov}}(X_i, X_j)$$

Covariance

iv) $F_T(t) = P(T < t)$

$= P(X_1 + \dots + X_M < t)$

Total Prob Thm

$= \sum_{m=1}^{\infty} P(X_1 + \dots + X_M < t \mid M = m) P(M=m)$

$= \sum_{m=1}^{\infty} \left[\int\int_{X_1 + \dots + X_m < t} \cdots \int f_{X_1}(x_1) \cdots f_{X_m}(x_m) \, dx_1 \cdots dx_m \right]$

$\cdot P(M=m)$

$f_T(t) = \sum_{m=1}^{\infty} \left[\int\int \cdots \int f_{X_1}(x_1) \cdots f_{X_{m-1}}(x_{m-1}) f_{X_m}(t - x_1 - \dots - x_{m-1}) \right.$

$\left. dx_1 \cdots dx_{m-1} \right]$

$E(X) = E[E(X|Y)]$

$\cdot P(M=m)$

$E(T) = E(X_1 + \dots + X_M)$

$= E[E(X_1 + \dots + X_M \mid M)]$

$= E[E(X_1) + \dots + E(X_M)]$

$= E[M E(X)] = E(M) \cdot E(X)$

$Var(T) = E(T^2) - (E(T))^2$

$= E(T^2) - (E(M) \cdot E(X))^2$

$$E(T^2) = E\left[E\left((X_1 + \cdots + X_m)^2 \mid m\right)\right]$$

$$= E\left[E\left(\sum_{i=1}^{m} X_i^2 + \sum_{i \neq j} X_i X_j \mid m\right)\right]$$

$$= E\left[m E(X^2) + m(m-1)\left(E(X)\right)^2 \right]$$

$$= E(m)\, E(X^2) + \left(E(m^2) - E(m)\right)\left(E(X)\right)^2.$$

$$\mathrm{Var}(T) = E(m) \cdot E(X^2) + \left(E(m^2) - E(m)\right)\left(E(X)\right)^2 - \left(E(m) \cdot E(X)\right)^2.$$

LECTURE

20 OCTOBER

9:00-10:00AM

MATH3/4/68181

http://www.maths.manchester.ac.uk/

~saraLees/extremes.html.

ii)

$$F_T(t) = P(T < t) = P(X_1 + \dots + X_m < t)$$

$$= \underset{\{x_1 + \dots + x_m < t\}}{\int\int \dots \int} f_{X_1}(x_1) \dots f_{X_m}(x_m)\, dx_1 \dots dx_m$$

$$f_T(t) = \int\int \dots \int f_{X_1}(x_1)\, f_{X_2}(x_2) \dots f_{X_m}(t - x_1 - \dots - x_{m-1})\, dx_1\, dx_2 \dots dx_{m-1}$$

$$E(T) = E(X_1) + \dots + E(X_m)$$

$$Var(T) = Var(X_1) + \dots + Var(X_m)$$

v) $F_T(t) = P(T < t) = P(X_1 + \cdots + X_M < t)$

Total Prob Thm

$$= \sum_{m=1}^{\infty} P(X_1 + \cdots + X_M < t \mid M = m)\, P(M = m)$$

$$= \sum_{m=1}^{\infty} \left[\underset{x_1 + \cdots + x_m < t}{\int\int \cdots \int} f_{X_1}(x_1) \cdots f_{X_m}(x_m)\, dx_1 \cdots dx_m \right] \cdot P(M = m)$$

$$f_T(t) = \sum_{m=1}^{\infty} \left[\int\int \cdots \int f_{X_1}(x_1) \cdots f_{X_{m-1}}(x_{m-1}) \cdot f_{X_m}(t - x_1 - \cdots - x_{m-1})\, dx_1 \cdots dx_{m-1} \right] \cdot P(M = m)$$

$$E(T) = \sum_{m=1}^{\infty} \left[E(X_1) + \cdots + E(X_m) \right] P(M = m)$$

<u>Proof</u>: $E(T) = E(X_1 + \cdots + X_M)$

$E(X) = E[E(X|Y)]$

$$= E\left[E(X_1 + \cdots + X_M \mid M) \right]$$

$$= E\left[E(X_1) + \cdots + E(X_M) \right]$$

$$= \sum_{m=1}^{\infty} \left[E(X_1) + \cdots + E(X_m) \right] P(M = m)$$

vi)

$$F_T(t) = P(T < t) = P(X_1 + \cdots + X_M < t)$$

Total Prob Thm

$$\xrightarrow{} \sum_{m=1}^{\infty} P(X_1 + \cdots + X_M < t \mid M = m) \quad P(M = m)$$

$$= \sum_{m=1}^{\infty} \left[\int\int_{\{X_1 + \cdots + X_m < t\}} \cdots \int \boxed{f_{X_1, \ldots, X_m}} (x_1, \ldots, x_m) \, dx_1 \cdots dx_m \right] P(M=m)$$

Join PDF of $(X_1, \ldots, X_m)$

$$f_T(t) = \sum_{m=1}^{\infty} \left[\int \int \cdots \int f_{X_1, \ldots, X_m}(x_1, \ldots, x_{m-1}, t - x_1 - \cdots - x_{m-1}) \, dx_1 \cdots dx_{m-1} \right] P(M=m)$$

$$E(T) = \sum_{m=1}^{\infty} \left[E(X_1) + \cdots + E(X_m) \right] P(M=m)$$

$$Var(T) = E(T^2) - (E(T))^2$$

$$= E((X_1 + \cdots + X_M)^2) - (E(T))^2$$

$E(X) = E[E(X|Y)]$

$$= E\left[E((X_1 + \cdots + X_M)^2 \mid M) \right] - (E(T))^2$$

$$= E\left[\sum_{i=1}^{M} E(X_i^2) + \sum_{i \neq j} E(X_i X_j) \Big| M \right] - (E(T))^2$$

$$= \sum_{m=1}^{\infty} \left[\sum_{i=1}^{m} E(X_i^2) + \sum_{i \neq j} E(X_i X_j) \right] P(M=m) - (E(T))^2$$

eg 1

X_i IID $N(\mu, \sigma^2)$ & m fixed

$$T = X_1 + \cdots + X_m \sim N(m\mu, m\sigma^2)$$

Aside: $M_{X_i}(t) = E[e^{tX_i}] = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$$M_T(t) = E\left[e^{t(X_1 + \cdots + X_m)}\right]$$

$$\overset{\text{indep}}{=} E[e^{tX_1}] \cdots E[e^{tX_m}]$$

$$= M_{X_1}(t) \cdots M_{X_m}(t)$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdots e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$= e^{m\mu t + \frac{m\sigma^2 t^2}{2}},$$ the MGF of $N(m\mu, m\sigma^2)$

$$F_T(t) = \Phi\left(\frac{t - m\mu}{\sqrt{m}\sigma}\right)$$

$$f_T(t) = \frac{1}{\sqrt{2\pi}\sqrt{m}\sigma} e^{-\frac{(t - m\mu)^2}{2m\sigma^2}}$$

$E(T) = m\mu$, $\quad Var(T) = m\sigma^2$

eg 2 $\quad X_i \sim N(\mu_i, \sigma_i^2)$ indep & m fixed

$$T = X_1 + \cdots + X_m \sim N\left(\sum_{i=1}^m \mu_i, \sum_{i=1}^m \sigma_i^2\right)$$

$$F_T(t) = \Phi\left(\frac{t - \sum_{i=1}^m \mu_i}{\sqrt{\left(\sum_{i=1}^m \sigma_i^2\right)}}\right)$$

$$f_T(t) = \frac{1}{\sqrt{2\pi}\sqrt{\sum_{i=1}^m \sigma_i^2}} e^{-\frac{\left(t - \sum_{i=1}^m \mu_i\right)^2}{2\left(\sum_{i=1}^m \sigma_i^2\right)}}$$

$$E(T) = \sum_{i=1}^m \mu_i, \quad Var(T) = \sum_{i=1}^m \sigma_i^2$$

EXAMPLE CLASS

20 OCTOBER

10:00-11:00AM

MATH3/4/68181

Q1)

$$L(\sigma) = \prod_{i=1}^{n} \left[\frac{1}{\sigma} e^{-\frac{x_i}{\sigma}} e^{-e^{-\frac{x_i}{\sigma}}} \right]$$

$$= \frac{1}{\sigma^n} e^{-\frac{1}{\sigma}\sum_{i=1}^{n} x_i} e^{-\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}}$$

$$\log L(\sigma) = -n \log \sigma - \frac{1}{\sigma}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}$$

$$\frac{d \log L}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}} = 0$$

$$\Leftrightarrow \sigma n = \left[+ \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}} \right]$$

$$\Leftrightarrow \sigma = + \bar{x} - \frac{1}{n}\sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}}$$

Q2

$$L(\sigma,\lambda) = \prod_{i=1}^{n} \left[\lambda \sigma^{\lambda} x_i^{-\lambda-1} e^{-\sigma^{\lambda} x_i^{-\lambda}} \right]$$

$$= \lambda^{n} \sigma^{n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{-\lambda-1} e^{-\sigma^{\lambda} \sum_{i=1}^{n} x_i^{-\lambda}}$$

$$\log L(\sigma,\lambda) = n\log\lambda + n\lambda \log\sigma - (\lambda+1)\sum_{i=1}^{n} \log x_i$$
$$- \sigma^{\lambda} \sum_{i=1}^{n} x_i^{-\lambda}$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} - \lambda \sigma^{\lambda-1} \sum_{i=1}^{n} x_i^{-\lambda} = 0 \quad -(1)$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n\log\sigma - \sum_{i=1}^{n} \log x_i - \sigma^{\lambda} \sum_{i=1}^{n} x_i^{-\lambda} \log\left(\frac{\sigma}{x_i}\right) = 0 \quad -(2)$$

$$\boxed{\frac{dx^a}{da} = x^a \log x}$$

$$(1) \Rightarrow \frac{n}{\sigma^{\lambda}} = \sum_{i=1}^{n} x_i^{-\lambda} \Rightarrow \sigma = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{-\lambda}\right)^{-\frac{1}{\lambda}}$$

$$(2) \Rightarrow \frac{n}{\lambda} - \frac{n}{\lambda} \log\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{-\lambda}\right) - \sum_{i=1}^{n} \log x_i$$
$$- \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{-\lambda}\right)^{-1} \sum_{i=1}^{n} x_i^{-\lambda} \left[-\frac{1}{\lambda} \log\left(\frac{1}{n}\sum_{j=1}^{n} x_j^{-\lambda}\right) - \log x_i \right] = 0 \quad -(3)$$

(3) only depends on λ.

Q3

$$L(\sigma,\lambda) = \lambda^n \sigma^{-n\lambda} \left(\prod_{i=1}^{n} x_i\right)^{\lambda-1} e^{-\sigma^{-\lambda}\sum_{i=1}^{n} x_i^{\lambda}}$$

$$\log L = n\log\lambda - n\lambda\log\sigma + (\lambda-1)\sum_{i=1}^{n}\log x_i$$
$$- \sigma^{-\lambda}\sum_{i=1}^{n} x_i^{\lambda}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n\lambda}{\sigma} - \lambda\sigma^{-\lambda-1}\sum_{i=1}^{n} x_i^{\lambda} = 0 \quad -(1)$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - n\log\sigma + \sum_{i=1}^{n}\log x_i$$
$$- \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{\lambda}\log\left(\frac{x_i}{\sigma}\right) = 0 \quad -(2)$$

LECTURE

22 OCTOBER

12:00-13:00PM

MATH4/68181

$$M = \max(X_1, X_2, \ldots, X_m)$$ "maximum portfolio loss"

i) $$F_M(x) = P(M \leq x)$$

$$= P(\max(X_1, \ldots, X_m) \leq x)$$

$$= P(X_1 \leq x, \ldots, X_m \leq x)$$

$$\xrightarrow{\text{IID}} = P(X_1 \leq x) \cdots P(X_m \leq x)$$

$$= F_{X_1}(x) \cdots F_{X_m}(x) = \prod_{j=1}^{m} F_{X_j}(x)$$

$$f_M(x) = \frac{d}{dx} \prod_{j=1}^{m} F_{X_j}(x)$$

product rule
$$= \sum_{k=1}^{m} f_{X_k}(x) \left[\prod_{\substack{j=1 \\ j \neq k}}^{m} F_{X_j}(x) \right].$$

ii) $$F_M(x) = \prod_{j=1}^{m} F_{X_j}(x)$$

$$f_M(x) = \sum_{k=1}^{m} f_{X_k}(x) \left[\prod_{\substack{j=1 \\ j \neq k}}^{m} F_{X_j}(x) \right]$$

(iii)

$$F_M(x) = P(M < x)$$
$$= P(\max(X_1, \ldots, X_m) < x)$$
$$= P(X_1 < x, \ldots, X_m < x)$$
$$= \boxed{F_{X_1, \ldots, X_m}}(x, \ldots, x)$$

Joint CDF of $(X_1, \ldots, X_m)$

$$f_M(x) = \frac{d}{dx} F_{X_1, \ldots, X_m}(x, \ldots, x)$$

~~iii)~~ iv)

$$P(M < x) = P(\max(X_1, \dots, X_M) < x)$$

$$= P(X_1 < x, \dots, X_M < x)$$

Total Prob Rule

$$= \sum_{m=1}^{\infty} P(X_1 < x, \dots, X_M < x \mid M = m)\, P(M = m)$$

$$= \sum_{m=1}^{\infty} P(X_1 < x, \dots, X_m < x)\; P(M = m)$$

indep

$$= \sum_{m=1}^{\infty} P(X_1 < x) \dots P(X_m < x)\; P(M = m)$$

$$= \sum_{m=1}^{\infty} F_{X_1}(x) \dots F_{X_m}(x) \cdot P(M = m)$$

$$= \sum_{m=1}^{\infty} \left[\prod_{j=1}^{m} F_{X_j}(x)\right] \cdot P(M = m)$$

$$f_M(x) = \sum_{m=1}^{\infty} \frac{d}{dx}\left[\prod_{j=1}^{m} F_{X_j}(x)\right] P(M = m)$$

Product Rule

$$= \sum_{m=1}^{\infty} \sum_{k=1}^{m} f_{X_k}(x) \left[\prod_{\substack{j=1 \\ j \neq k}}^{m} F_{X_j}(x)\right] P(M = m)$$

v) $$F_M(x) = \sum_{m=1}^{\infty} \left[\prod_{j=1}^{m} F_{X_j}(x) \right] P(M=m)$$

$$f_M(x) = \sum_{m=1}^{\infty} \sum_{k=1}^{m} f_{X_k}(x) \left[\prod_{\substack{j=1 \\ j \neq k}}^{m} F_{X_j}(x) \right] P(M=m)$$

(vi) $F_M(x) = P(\max(X_1, \ldots, X_M) < x)$

$$= P(X_1 < x, \ldots, X_M < x)$$

Total Prob Rule

$$= \sum_{m=1}^{\infty} P(X_1 < x, \ldots, X_M < x \mid M=m) P(M=m)$$

$$= \sum_{m=1}^{\infty} P(X_1 < x, \ldots, X_m < x) P(M=m)$$

$$= \sum_{m=1}^{\infty} \boxed{F_{X_1, \ldots, X_m}} (x, \ldots, x) \; P(M=m)$$

Joint CDF of $(X_1, \ldots, X_m)$

$$f_M(x) = \sum_{m=1}^{\infty} \left[\frac{d}{dx} F_{X_1, \ldots, X_m}(x, \ldots, x) \right] P(M=m)$$

$$\bar{F}(x, y) = P(X > x, Y > y)$$

"survival function"

$$\bar{F}(x, y) = 1 - P(X < x) - P(Y < y) + P(X < x, Y < y)$$

$$= 1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)$$

$F_{X,Y}(x, y) = P(X < x, Y < y)$ "Joint CDF"

$F_X(x) = P(X < x)$ "Marginal CDF of X"

$F_Y(y) = P(Y < y)$ " " " of Y "

$\bar{F}_{X,Y}(x, y) = P(X > x, Y > y)$ "Joint survival function"

$$\bar{F}_{X,Y}(x, y) = 1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y) \quad (*)$$

$$F_{X,Y}(x, y) = -1 + F_X(x) + F_Y(y) + \bar{F}_{X,Y}(x, y) \quad (**)$$

$$P(X>x, Y>y) = \bar{F}(x,y) = \left[1 + \frac{x}{a} + \frac{y}{b}\right]^{-c}$$

$$P(X>x) = \bar{F}(x,0) = \left[1+\frac{x}{a}\right]^{-c}$$

$$\shortparallel$$

$$1-P(X<x) \Rightarrow P(X<x) = 1-\left[1+\frac{x}{a}\right]^{-c}$$

Similarly, $P(Y<y) = 1-\left[1+\frac{y}{b}\right]^{-c}$

Using (**),

$$F(x,y) = -1 + \left[1-\left(1+\frac{x}{a}\right)^{-c}\right] + \left[1-\left(1+\frac{y}{b}\right)^{-c}\right]$$
$$+\left[1+\frac{x}{a}+\frac{y}{b}\right]^{-c}$$
$$= 1-\left(1+\frac{x}{a}\right)^{-c} - \left(1+\frac{y}{b}\right)^{-c}$$
$$+\left[1+\frac{x}{a}+\frac{y}{b}\right]^{-c}$$

1) $P[\max(X,Y)<x] = P(X<x, Y<x)$
$$= F(x,x)$$
$$= 1-\left(1+\frac{x}{a}\right)^{-c} - \left(1+\frac{x}{b}\right)^{-c}$$
$$+\left(1+\frac{x}{a}+\frac{x}{b}\right)^{-c}$$

LECTURE

23 OCTOBER

9:00-10:00AM

MATH3/4/68181

eg 3

$X_1, X_2, \ldots, X_M$ IID $N(\mu, \sigma^2)$

& M is a RV

$T = X_1 + \ldots + X_M \sim N(M\mu, M\sigma^2)$

$$F_T(t) = \Phi\left(\frac{t - M\mu}{\sqrt{M}\,\sigma}\right)$$

$\Phi(\cdot)$ CDF of $N(0,1)$

Total Prob Rule

$$= \sum_{m=1}^{\infty} \Phi\left(\frac{t - m\mu}{\sqrt{m}\,\sigma}\right) P(M=m)$$

$$f_T(t) = \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}\,\sigma}\,\phi\left(\frac{t - m\mu}{\sqrt{m}\,\sigma}\right) P(M=m)$$

ϕ PDF of $N(0,1)$

$\phi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$

$$E(T) = \sum_{m=1}^{\infty} m\mu \cdot P(M=m) = \mu \cdot \sum_{m=1}^{\infty} m \cdot P(M=m)$$

$$= \mu\, E(M)$$

$$E(T^2) = \sum_{m=1}^{\infty} \left[(m\mu)^2 + m\sigma^2\right] P(M=m)$$

$$= \mu^2 \sum_{m=1}^{\infty} m^2 P(M=m) + \sigma^2 \sum_{m=1}^{\infty} m\, P(M=m)$$

$$= \mu^2 E(M^2) + \sigma^2 E(M)$$

$$Var(T) = \mu^2 E(M^2) + \sigma^2 E(M) - \mu^2 (E(M))^2$$

eg 4

$$X_i \sim N(\mu_i, \sigma_i^2), \quad i = 1, 2, \ldots, M$$

M is a RV

$$T = X_1 + \ldots + X_M$$

$$\sim N\left(\sum_{i=1}^{M} \mu_i, \sum_{i=1}^{M} \sigma_i^2\right)$$

$$F_T(t) = \Phi\left(\frac{t - \sum_{i=1}^{M} \mu_i}{\sqrt{\sum_{i=1}^{M} \sigma_i^2}}\right)$$

Total Prob Rule

$$= \sum_{m=1}^{\infty} \Phi\left(\frac{t - \sum_{i=1}^{m} \mu_i}{\sqrt{\sum_{i=1}^{m} \sigma_i^2}}\right) P(M=m)$$

$$f_T(t) = \sum_{m=1}^{\infty} \phi\left(\frac{t - \sum_{i=1}^{m} \mu_i}{\sqrt{\sum_{i=1}^{m} \sigma_i^2}}\right) \cdot \frac{1}{\sqrt{\sum_{i=1}^{m} \sigma_i^2}} P(M=m)$$

$$E(T) = \sum_{m=1}^{\infty} \left(\sum_{i=1}^{m} \mu_i\right) P(M=m)$$

$$E(T^2) = \sum_{m=1}^{\infty} \left[\sum_{i=1}^{m} \sigma_i^2 + \left(\sum_{i=1}^{m} \mu_i\right)^2\right] P(M=m)$$

$$\boxed{X \sim N(\mu, \sigma^2) \Rightarrow E(X^2) = \mu^2 + \sigma^2}$$

$$\mathrm{Var}(T) = E(T^2) - (E(T))^2$$

eg5

$X_1, \ldots, X_m \sim$ IID $\text{Exp}(\lambda)$ & m fixed
"Exponetial distn"

$T = X_1 + \ldots + X_m$

$$E[e^{tT}] = E\left[e^{t(X_1 + \ldots + X_m)}\right]$$
$$= E[e^{tX_1}] \ldots E[e^{tX_m}]$$
$$= \frac{\lambda}{\lambda - t} \ldots \frac{\lambda}{\lambda - t}$$
$$= \left(\frac{\lambda}{\lambda - t}\right)^m,$$

MGF of a gamma RV with parameter m & λ

$T = X_1 + \ldots + X_m \sim \text{Gamma}(m, \lambda)$

$$F_T(t) = \mathbb{P}[\text{Gamma}(m, \lambda) < t]$$

$$f_T(t) = \frac{\lambda^m}{\Gamma(m)} t^{m-1} e^{-\lambda t}$$

$$E(T) = \frac{m}{\lambda}$$

eg 6

$\boxed{m = 2}$ X_1 & X_2 are dep RVs with Joint

PDF $f(x_1, x_2) = c(c+1)(1 + x_1 + x_2)^{-c-2}$, $x_1 > 0$, $x_2 > 0$

What is the distn of $T = X_1 + X_2$?

$$f_T(t) = \int_0^t f(x_1, t - x_1)\, dx_1$$

$$= \int_0^t c(c+1)(1+t)^{-c-2}\, dx_1$$

$$= c(c+1)\, t(1+t)^{-c-2}, \quad t > 0$$

$$F_T(t) = c(c+1)\int_0^t x(1+x)^{-c-2}\, dx$$

by parts

$$= c(c+1)\left\{\left[x\frac{(1+x)^{-c-1}}{-c-1}\right]_0^t + \frac{1}{c+1}\int_0^t (1+x)^{-c-1}\, dx\right\}$$

$$= c(c+1)\left\{-\frac{t(1+t)^{-c-1}}{c+1} + \frac{1}{c+1}\left[\frac{(1+x)^{-c}}{(-c)}\right]_0^t\right\}$$

$$= c(c+1)\left\{-\frac{t(1+t)^{-c-1}}{c+1} + \frac{1}{c+1}\left[\frac{(1+t)^{-c}}{(-c)} + \frac{1}{c}\right]\right\}.$$

$$E(T) = \int_0^\infty c(c+1)\, t^2 (1+t)^{-c-2}\, dt$$

$$\boxed{x = \frac{1}{1+t} \Rightarrow t = \frac{1}{x} - 1 \Rightarrow \frac{dt}{dx} = -\frac{1}{x^2}}$$

$$= c(c+1) \int_1^0 \left(\frac{1}{x} - 1\right)^2 x^{c+2} \left(-\frac{1}{x^2}\right) dx$$

$$= c(c+1) \boxed{\int_0^1 x^{c-2} (1-x)^2\, dx}$$

$$= c(c+1)\, B(c-1, 3)$$

Beta function

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1}\, dt$$

$$E(T^2) = \int_0^\infty c(c+1)\, t^3 (1+t)^{-c-2}\, dt$$

$$= c(c+1) \int_0^1 x^{c-3} (1-x)^3\, dx$$

$$= c(c+1)\, B(c-2, 4)$$

$$Var(T) = c(c+1)\, B(c-2, 4) - (c(c+1))^2 B^2(c-1, 3)$$

"Financial Risk Measures"

Topic for next week

LECTURE

27 OCTOBER

9:00-10:00AM

MATH3/4/68181

VaR: The number that killed us

By Pablo Triana

December 1, 2010 • Reprints

FROM THE ARCHIVES

On Sept. 10, 2009 former trader and bestselling author Nassim Taleb did something that he very seldom does: he wore a tie. Taleb has oftentimes publicly expressed his distaste for the blood-constraining artifacts, as well as for those who tend to don them, so the Lebanese-American let the world know that was a very special day for him by betraying a sacred personal disposition.

So what prompted the composer of "The Black Swan" to button his shirt all the way up on that fall date? He had been invited to a very solemn venue by very distinguished hosts. And that was an invitation that Taleb had every intention of accepting. In fact, he had been waiting and expecting it for more than a decade. The raison d'etre of the event for which his company was now being required had been close to Taleb´s heart for most of his professional and intellectual life. It represented a central theme in his actions and ideas, close to an obsession. He had through the years incessantly warned as to the havoc that might be wreaked should others massively act in a manner counter to his convictions. Such concerns typically went unheeded (to the detriment, it turned out, of society), but now he was being offered a pulpit that seemed irresistible. This time, the world would have no option but to listen attentively.

As Taleb entered the Rayburn Building of the U.S. House of Representatives on Capitol Hill that September morning, he must have felt vindication. As he approached the sober room where several men and women awaited the start of the House Committee on Science and Technology´s hearing on the responsibility of mathematical model Value at Risk (VaR) for the terrible economic and financial crisis that had caused so much misery, Taleb probably reflected proudly on all those times when, indefatigably and in the face of harsh opposition, he alerted us of the lethal threat to the system posed by the widespread use of VaR in finance. Now that the damage wrought by VaR seemed so inescapably obvious that lawmakers had been motivated into investigating the device, Taleb no longer seemed like a lone wolf howling at the moon.

What is so wrong about VaR, and why was Taleb so concerned about its impact? More importantly, why should VaR be held responsible for the crisis? VaR is a number that purports to estimate future losses derived from a portfolio of financial assets, and presents two major problems: 1) it is doomed to being a very wrong estimate, because of its analytical foundations and the realities of real-life markets; 2) in spite of such (well-known) deficiencies, it has for the past two decades become an ubiquitously influential force in the financial world, capable of directing decision-making inside the most important banks. In other words, by letting trading activity be guided by VaR, we have essentially exposed our economic fate to a deeply flawed mechanism. Such flawedness, as was the case not only in this crisis but also before, can yield untold malaise.

FINANCIAL TIMES

Last updated: October 15, 2013 3:49 am

Value at risk is very likely to cause a market crash

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From Mr James Tew.

Sir, Steve Johnson's report "Var calculation a 'time bomb' " (FTfm October 14) suggests that value at risk could cause a severe market crash. Of course it will. Jeremy Monk of Akro Investicni Spolecnost, who is quoted in the report, argues that as markets fall portfolios would be required to reduce their riskier assets (equities), which would make the markets implode and fall further in a downward spiral. Theorists and market practitioners may well discuss the virtues of Mr Monk's ideas, no doubt dependent on the precise method of calculation of Var, but they are intuitively plausible.

I have two more general points to make about Var and in particular its requirement as a tool demanded by the regulatory authorities.

The first issue is not about the tool itself, but the requirement for it to be interpreted and used in a particular manner. Its enforced use by the regulator makes everyone think the same way or at least behave the same way. If everyone is taught to think and behave in the same way, then Mr Monk's analysis is likely to be accurate.

Second, Var is based on historic data and does not look forward to the future. It makes no sense to rely on the past to determine the future. The use of Var does this. You can by all means interpret the past, along with all other knowledge at your disposal to make an educated guess (or have an educated view) of the future. But you cannot rely on it, which is what the current use of Var is required to do.

It is easier to the human mind, and safer to the human's career prospects, to rely on a "scientific" number to make a decision than to go through a hugely complex cognitive process to assess the implications of a number and then make a judgment. Using judgment means the decision maker is at risk. Var is certainly value at risk, but it should be the decision maker's value that is at risk, not the markets' or the investors'.

James Tew, Winchester, Hants, UK

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Financial Risk Measures

Suppose £1 million is invested into a property.

$Pr(\text{Loss}) > 0.9 \Rightarrow$ Not to invest

$Pr(\text{Loss}) < 10^{-6} \Rightarrow$ ok to invest

Mathematical Definition: A risk measure is a function ρ taking values in $(0, \infty)$ and satisfies

- $\rho(0) = 0$ "normalized property"
- $\rho(X + c) = \rho(X) + c$ "translative property"
- $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$ "monotone property"

where X & Y are RVs representing loss.

X = Loss with CDF $F(\cdot)$

The first risk measure (introduced by J P Morgan): Value at Risk defined by

$$\mathrm{VaR}_p(X) = \inf\{x : F(x) \geq p\}$$

" amount of loss exceeded with prob p "

eg

$\mathrm{VaR}_{0.99}(X)$ = " loss exceeded with prob 0.99 "

$\mathrm{VaR}_{0.01}(X)$ = " loss " " " 0.01 "

The second risk measure: Expected Shortfall defined by

$$\begin{aligned} ES_p(X) = \frac{1}{p}\Big[& E\left(X I\{X \leq \mathrm{VaR}_p(X)\}\right) \\ & + p\,\mathrm{VaR}_p(X) \\ & - \mathrm{VaR}_p(X)\Pr\left(X \leq \mathrm{VaR}_p(X)\right)\Big], \end{aligned}$$

$$I\{A\} = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false} \end{cases}$$

" average loss given it has exceeded $\mathrm{VaR}_p(X)$ "

Coherent (good) risk measure is one that ~~sati~~ is a risk measure and satisfies the 2 further conditions:

- $\rho(cX) = c\,\rho(X)$, $c > 0$ "positive homogeneity"
- $\rho(X+Y) \leq \rho(X) + \rho(Y)$ "sub-additivity"

Suppose X is an absolutely continuous RV.
In this case,

$$\left.\begin{aligned} VaR_p(X) &= F^{-1}(p) \\ ES_p(X) &= \frac{1}{p}\int_0^p F^{-1}(t)\,dt \end{aligned}\right\} \text{much simpler}$$

VaR & ES are valid risk measures.

- VaR <u>does</u> <u>not</u> satisfy the sub-additivity condition
- ES does satisfy all conditions, it is a coherent risk measure.

VaR

eg

$X \sim N(\mu, \sigma^2)$

$$\Rightarrow F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right), \quad \Phi \text{ is the cdf of } N(0,1)$$

$$\Rightarrow F^{-1}(p) = \mu + \sigma \Phi^{-1}(p)$$

$$\Rightarrow \boxed{VaR_p(X) = \mu + \sigma \Phi^{-1}(p)}$$

$$ES_p(X) = \frac{1}{p} \int_0^p \left[\mu + \sigma \Phi^{-1}(t)\right] dt$$

$$\Rightarrow \boxed{ES_p(X) = \mu + \frac{\sigma}{p} \cdot \int_0^p \Phi^{-1}(t) \, dG}$$

eg 2

$$F(x) = x^{\alpha}, \quad 0 < x < 1$$

$$VaR_p(X) = F^{-1}(p) = p^{1/\alpha}$$

$$ES_p(X) = \frac{1}{p} \int_0^p t^{\frac{1}{\alpha}} \, dt$$

$$= \frac{1}{p} \left[\frac{t^{\frac{1}{\alpha}+1}}{\frac{1}{\alpha}+1} \right]_0^p$$

$$= \frac{p^{\frac{1}{\alpha}}}{\frac{1}{\alpha}+1}.$$

Properties of VaR

i) $VaR_p(X+c) = VaR_p(X) + c$

ii) $VaR_p(cX) = c\, VaR_p(X)$

iii) $VaR_p(X) = -VaR_{1-p}(-X)$

iv) $X \geq 0 \Rightarrow VaR_p(X) \geq 0$

v) $X \geq Y \Rightarrow VaR_p(X) \geq VaR_p(Y)$.

Proof of (i): Suppose it holds.

$VaR_p(X+c) = VaR_p(X) + c.$

Let $Z = X + c.$

$$F_Z^{-1}(p) = F_X^{-1}(p) + c$$

$$\Leftrightarrow F_Z(F_Z^{-1}(p)) = F_Z(F_X^{-1}(p) + c)$$

$$\Leftrightarrow p = P(Z \leq F_X^{-1}(p) + c)$$

$$\Leftrightarrow p = P(Z - c \leq F_X^{-1}(p))$$

$$\Leftrightarrow p = P(X \leq F_X^{-1}(p))$$

$$\Leftrightarrow p = F_X(F_X^{-1}(p)) = p.$$

<u>Homework</u> : Prove (ii) - (v).

EXAMPLE CLASS

27 OCTOBER

10:00-11:00AM

MATH3/4/68181

Q1

$$F(x) = 1 - e^{-\lambda x} = p$$

$$\Leftrightarrow \quad e^{-\lambda x} = 1 - p$$

$$\Leftrightarrow \quad -\lambda x = \log(1-p)$$

$$\Leftrightarrow \quad x = -\frac{1}{\lambda}\log(1-p) = F^{-1}(p)$$

$$VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \left(-\frac{1}{\lambda}\log(1-t)\right) dt$$

$$= -\frac{1}{\lambda p}\int_0^p \log(1-t)\, dt$$

$$= -\frac{1}{\lambda p}\left\{\Big[t \cdot \log(1-t)\Big]_0^p + \int_0^p \frac{t}{1-t}\, dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\log(1-p) + \int_0^p \frac{t-1+1}{1-t}\, dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\log(1-p) + \Big[-t - \log(1-t)\Big]_0^p\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\log(1-p) - p - \log(1-p)\right\}.$$

Q3

$X \sim \mathrm{Uni}[a, b]$

$$F(x) = \frac{x-a}{b-a} = p \quad \Rightarrow \quad x = a + p(b-a)$$

$$\Rightarrow \quad VaR_p(X) = a + p(b-a)$$

$$\Rightarrow \quad ES_p(X) = \frac{1}{p} \int_0^p \left[a + t(b-a)\right] dt$$

$$= \frac{1}{p} \left[at + \frac{t^2}{2}(b-a)\right]_0^p$$

$$= a + \frac{p}{2}(b-a).$$

$\boxed{Q4}$

$$F(x) = 1 - \left(\frac{K}{x}\right)^{a} = p$$

$$\Rightarrow \left(\frac{K}{x}\right)^{a} = 1 - p$$

$$\Rightarrow \frac{K}{x} = (1-p)^{\frac{1}{a}}$$

$$\Rightarrow x = K\,(1-p)^{-\frac{1}{a}} = F^{-1}(p)$$

$$VaR_p(X) = K\,(1-p)^{-\frac{1}{a}}$$

$$ES_p(X) = \frac{1}{p} \cdot \int_0^p K\,(1-t)^{-\frac{1}{a}}\,dt$$

$$= \frac{K}{p}\left[\frac{(1-t)^{1-\frac{1}{a}}}{(-1)\left(1-\frac{1}{a}\right)}\right]_0^p$$

$$= \frac{K}{p\left(1-\frac{1}{a}\right)}\left[1 - (1-p)^{1-\frac{1}{a}}\right].$$

Q6

$$F(x) = \left[1 + \left(\frac{x}{a}\right)^{-b}\right]^{-1} = p$$

$$\Rightarrow \quad 1 + \left(\frac{x}{a}\right)^{-b} = p^{-1}$$

$$\Rightarrow \quad \left(\frac{x}{a}\right)^{-b} = p^{-1} - 1$$

$$\Rightarrow \quad \frac{x}{a} = \left(\frac{1-p}{p}\right)^{-\frac{1}{b}}$$

$$\Rightarrow \quad x = a\left(\frac{1-p}{p}\right)^{-\frac{1}{b}} \equiv F^{-1}(p)$$

$$VaR_p(X) = a\left(\frac{1-p}{p}\right)^{-\frac{1}{b}}$$

$$ES_p(X) = \frac{1}{p} \cdot \int_0^p a \cdot \left(\frac{1-t}{t}\right)^{-\frac{1}{b}} dt$$

$$= \frac{a}{p} \cdot \boxed{\int_0^p t^{\frac{1}{b}} (1-t)^{-\frac{1}{b}} dt}$$

$$\boxed{B_x(a,b) = \int_0^x t^{a-1} (1-t)^{b-1} dt}$$ Incomplete Beta Function

$$\Rightarrow = \frac{a}{p} \cdot B_p\left(\frac{1}{b}+1, 1-\frac{1}{b}\right).$$

Q7

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = p$$

$$\Rightarrow \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = 1 - p$$

$$\Rightarrow 1 + \frac{x}{\lambda} = (1-p)^{-\frac{1}{\alpha}}$$

$$\Rightarrow x = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right]$$

$$VaR_p(X) = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right]$$

$$ES_p(X) = \frac{\lambda}{p}\int_0^p \left[(1-t)^{-\frac{1}{\alpha}} - 1\right] dt$$

$$= \frac{\lambda}{p}\left[\frac{(1-t)^{1-\frac{1}{\alpha}}}{(-1)(1-\frac{1}{\alpha})} - t\right]_0^p$$

$$= \frac{\lambda}{p}\left[\frac{(1-p)^{1-\frac{1}{\alpha}}}{\frac{1}{\alpha}-1} - p + \frac{1}{1-\frac{1}{\alpha}}\right].$$

$\boxed{Q8}$

$$F(x) = e^{-\left(\frac{\sigma}{x}\right)^{\alpha}} = p$$

$$\Rightarrow -\left(\frac{\sigma}{x}\right)^{\alpha} = \log p$$

$$\Rightarrow \frac{\sigma}{x} = (-\log p)^{\frac{1}{\alpha}}$$

$$\Rightarrow x = \sigma(-\log p)^{-\frac{1}{\alpha}}$$

$$VaR_p(X) = \sigma(-\log p)^{-\frac{1}{\alpha}}$$

$$ES_p(X) = \frac{\sigma}{p}\int_0^p (-\log t)^{-\frac{1}{\alpha}}\, dt$$

$$\boxed{x = -\log t \Rightarrow t = e^{-x} \Rightarrow dt = -e^{-x}}$$

$$= \frac{\sigma}{p}\int_{\infty}^{-\log p} x^{-\frac{1}{\alpha}}(-e^{-x})\, dx$$

$$= \frac{\sigma}{p}\boxed{\int_{-\log p}^{\infty} x^{-\frac{1}{\alpha}} e^{-x}\, dx}$$

$$\boxed{\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t}\, dt}$$ Complementary Incomplete gamma function

$$= \frac{\sigma}{p}\,\Gamma\left(1-\frac{1}{\alpha}, -\log p\right)$$

LECTURE

29 OCTOBER

12:00-13:00PM

MATH4/68181

$X =$ Loss

If X is an absolutely continuous RV

$$VaR_p(X) = F^{-1}(p)$$

$$ES_p(X) = \frac{1}{p} \int_0^p F^{-1}(t)\, dt$$

Estimation methods :

- Parametric estimation methods
  (3rd yr)
- Non-parametric " "
  (3rd yr)
- Semi-parametric " "
  (4/6th yrs)

Semi-parametric estimation methods

i) Extreme value theroy method

Data: $X_1, X_2, \ldots\ldots$ on losses

$r_i = X_i - X_{i-1}$ "returns"

Ordered returns; $r_{(1)} > r_{(2)} > \ldots\ldots > r_{(n)}$

$$\widehat{VaR}_{1-p} = r_{(k+1)} \left(\frac{k}{np}\right)^{\hat{\xi}}$$

where

$$\hat{\xi} = \frac{1}{k} \sum_{i=1}^{k} \log \frac{r_{(i)}}{r_{(i+1)}}$$ "Hill's estimator"

OR

$$\hat{\xi} = \frac{1}{\log 2} \log \frac{r_{(k+1)} - r_{(2k+1)}}{r_{(2k+1)} - r_{(4k+1)}}$$ "Pickands' estimator"

and k is a number betwee 1 & n.

This is a semi-parametric estimator because: i) ξ is a parameter (of the GEV), ii) the remaing terms are purely functions of the data.

2) returns follow the GEV distn with CDF

$$e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} = p$$

$$\Rightarrow \quad \left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} = -\log p$$

$$\Rightarrow \quad 1+\xi\frac{x-\mu}{\sigma} = (-\log p)^{-\xi}$$

$$\Rightarrow \quad x = \mu + \frac{\sigma}{\xi}\left[(-\log p)^{-\xi} - 1\right]$$

$$\Rightarrow \quad \widehat{VaR}_p \cancel{(x)} = \hat{\mu} + \frac{\hat{\sigma}}{\hat{\xi}}\left[(-\log p)^{-\hat{\xi}} - 1\right] \quad (*)$$

take $(\hat{\mu}, \hat{\sigma})$ to be the MLEs.

take $\hat{\xi}$ to be the Hills' or Pickands' estimator

Then $(*)$ is a semi parametric estimator.

3) Generalized Pareto distribution

Data : $X_1, X_2, \ldots, X_n$ follow the GP distribution with CDF

$P(X > u)$

$$1 - \left[1 + \xi \frac{X - u}{\sigma}\right]^{-\frac{1}{\xi}}$$

In this case,

$$\widehat{VaR}_{1-p} = r_{(n-k+1)} + \frac{1}{1-2^{-\hat{\xi}}}\left[\left(\frac{k}{(n+1)p}\right)^{\hat{\xi}} - 1\right] \cdot \left(r_{(n-k+1)} - r_{(n-2k+1)}\right)$$

where $\hat{\xi} = \frac{1}{\log 2} \log \frac{r_{(n-k+1)} - r_{(n-2k+1)}}{r_{(n-2k+1)} - r_{(n-4k+1)}}$, (**)

$r_i = X_i - X_{i-1}$,

$r_{(1)} \le r_{(2)} \le \cdots \le r_{(n)}$,

k is number between 1 & n.

Due to Pickands (1975)

4) With notation as in (3),

$$\widehat{VaR}_{1-p} = r_{(n-k)} + \frac{\hat{a}}{\hat{\xi}}\left[\left(\frac{k}{np}\right)^{\hat{\xi}} - 1\right]$$

where

$$\hat{\xi} = M_{k+1}^{(1)} + 1 - \frac{1}{2}\left[1 - \frac{\left(M_{k+1}^{(1)}\right)^2}{M_{(k+1)}^{(2)}}\right]^{-1}, \qquad (***)$$

$$M_{(k+1)}^{\ell} = \frac{1}{k}\sum_{i=1}^{k}\left[\log r_{(n-i+1)} - \log r_{(n-i)}\right]^{\ell}, \quad \ell = 1, 2$$

$$\hat{a} = \frac{r_{(n-k)}\, M_{(k+1)}^{(1)}}{\rho_1},$$

$\rho_1 = 1$ if $\hat{\xi} \geq 0$, $\rho_1 = \frac{1}{1-\hat{\xi}}$ if $\hat{\xi} < 0$.

Due to Dekkers et al (1989).

5) With notation as in (3)

$$1 - P(X>u)\left[1+\xi\,\frac{X-u}{\sigma}\right]^{-\frac{1}{\xi}} = p$$

$$\Rightarrow \quad \left[1+\xi\,\frac{X-u}{\sigma}\right]^{-\frac{1}{\xi}} = \frac{1-p}{P(X>u)}$$

$$\Rightarrow \quad 1+\xi\,\frac{X-u}{\sigma} = \left[\frac{1-p}{P(X>u)}\right]^{-\xi}$$

$$\Rightarrow \quad X = u + \frac{\sigma}{\xi}\left\{\left[\frac{1-p}{P(X>u)}\right]^{-\xi} - 1\right\}$$

$$\Rightarrow \quad \widehat{VaR}_p = u + \frac{\hat{\sigma}}{\hat{\xi}}\left\{\left[\frac{1-p}{\widehat{P(X>u)}}\right]^{-\hat{\xi}} - 1\right\}$$

Take $\hat{\sigma}$ to be the MLE of σ

Take $\widehat{P(X>u)} = \dfrac{\text{No of obsns exceeding } u}{n}$

Take $\hat{\xi}$ to be (**) or (***).

LECTURE

30 OCTOBER

9:00-10:00AM

MATH3/4/68181

X = Loss

If X is an absolutely cont RV

$$VaR_p(X) = F^{-1}(p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

where F is the CDF of X.

<u>Estimation</u> <u>methods</u>:

- Parametric estimation methods
- Non-parametric " "

Parametric Estimation methods for VaR

1) Suppose $X_1, X_2, \ldots, X_n$ are observed losses. Suppose too $X_1, \ldots, X_n$ are IID $N(\mu, \sigma^2)$. We ~~known~~ know

$$VaR_p(X) = \mu + \sigma \Phi^{-1}(p).$$

$$\Rightarrow \widehat{VaR}_p(X) = \hat{\mu} + \hat{\sigma} \Phi^{-1}(p)$$

$$= \bar{X} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}\ \Phi^{-1}(p)$$

is the MLE of $VaR_p(X)$.

Is $\widehat{VaR}_p(X)$ unbiased for $VaR_p(X)$?

It is not unbiased.

2) Variance Covariance method

Portfolio with N investments.

Let X_i = loss for i^{th} investment

The total portfolio loss

$$T = \sum_{i=1}^{N} X_i$$

If $X_i \sim N(\mu_i, \sigma_i^2)$ are intep RVs then

$$T \sim N\left(\sum_{i=1}^{N} \mu_i , \sum_{i=1}^{N} \sigma_i^2\right).$$

$$VaR_p(T) = \sum_{i=1}^{N} \mu_i + \Phi^{-1}(p)\sqrt{\sum_{i=1}^{N} \sigma_i^2}$$

$$\Rightarrow \boxed{\widehat{VaR}_p(T) = \sum_{i=1}^{N} \hat{\mu}_i + \Phi^{-1}(p)\sqrt{\sum_{i=1}^{N} \hat{\sigma}_i^2}}$$

where $\hat{\mu}_i, \hat{\sigma_i^2}$ are the MLEs of μ_i, σ_i^2.

3) With the same set up as in 2), assume $X_i \sim N(\mu_i, \sigma_i^2)$ are dep RVs with $Cor(X_i, X_j) = \rho_{ij}$. Then

$$T \sim N\left(\sum_{i=1}^{N} \mu_i, \sum_{i=1}^{N} \sigma_i^2 + \sum_{i \neq j} \sigma_i \sigma_j \rho_{ij}\right).$$

$$\Rightarrow VaR_p(T) = \sum_{i=1}^{N} \mu_i + \Phi^{-1}(p) \sqrt{\sum_{i=1}^{N} \sigma_i^2 + \sum_{i \neq j} \sigma_i \sigma_j \rho_{ij}}$$

$$\Rightarrow \boxed{\widehat{VaR}_p(T) = \sum_{i=1}^{N} \hat{\mu}_i + \Phi^{-1}(p) \cdot \sqrt{\sum_{i=1}^{N} \hat{\sigma}_i^2 + \sum_{i \neq j} \hat{\sigma}_i \hat{\sigma}_j \hat{\rho}_{ij}}},$$

where $\hat{\mu}_i, \hat{\sigma}_i^2, \hat{\rho}_{ij}$ are MLEs of $\mu_i, \sigma_i^2, \rho_{ij}$

4) The losses follow the Weibull distribution with CDF

$$F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}} = p$$

$$\Rightarrow \quad e^{-\left(\frac{X}{\theta}\right)^{\beta}} = 1 - p$$

$$\Rightarrow \quad \left(\frac{X}{\theta}\right)^{\beta} = -\log(1-p)$$

$$\Rightarrow \quad X = \theta\left[-\log(1-p)\right]^{\frac{1}{\beta}}$$

$$\Rightarrow \quad \widehat{VaR_p(X)} = \hat{\theta}\left[-\log(1-p)\right]^{\frac{1}{\hat{\beta}}}.$$

It can be shown that $\hat{\beta}$ is the root of

$$\frac{\bar{X}^2}{s^2} = \frac{\left(\Gamma\left(1+\frac{1}{\beta}\right)\right)^2}{\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)},$$

where $\bar{X}$ = sample mean & s^2 = sample var.

Further,

$$\hat{\theta} = \frac{\bar{X}}{\Gamma\left(1+\frac{1}{\hat{\beta}}\right)}.$$

Non-parametric estimation methods for VaR

1) Historical method

Data on losses: $X_1, X_2, \ldots, X_n$

Order the data: $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$.

Estimate VaR by

$$\widehat{VaR}_p = X_{(i)} \quad \text{if} \quad p \in \left(\frac{i-1}{n}, \frac{i}{n}\right].$$

e.g.

Losses: 2 4 1 −3 5 $(n=5)$

Order: −3 1 2 4 5

$$\widehat{VaR}_{0.8} = 4$$

$$\widehat{VaR}_{0.1} = X_{(1)} = -3$$

2) <u>Bootstrap method</u>

Suppose $X_1, X_2, \ldots, X_n$ are observed losses

- Construct the <u>empirical CDF</u>

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} I\{X_i \leq x\}$$

- Simulate B random samples each of size n from $\hat{F}(\cdot)$.

- Compute $\widehat{VaR}_p$ for each of the B samples by using the historical method, resulting in say $\widehat{VaR}_p^{(1)}, \widehat{VaR}_p^{(2)}, \ldots, \widehat{VaR}_p^{(B)}$.

- $\widehat{VaR}_p = \text{Median}\left(\widehat{VaR}_p^{(1)}, \ldots, \widehat{VaR}_p^{(B)}\right)$.

3) <u>Jackknife method</u>

Suppose $X_1, X_2, \dots, X_n$ are observed losses.

- ~~Construct~~ Estimate $\widehat{VaR}_p$ by the historical method for

$$
\begin{array}{lll}
X_2, X_3, \dots, X_n & \text{resulting in} & \widehat{VaR}_p^{(1)} \\
X_1, X_3, \dots, X_n & \text{" \quad "} & \widehat{VaR}_p^{(2)} \\
X_1, X_2, X_4, \dots, X_n & \text{" \quad "} & \widehat{VaR}_p^{(3)} \\
\vdots & \vdots \quad \vdots & \vdots \\
X_1, X_2, \dots, X_{n-1} & \text{" \quad "} & \widehat{VaR}_p^{(n)}
\end{array}
$$

- $\widehat{VaR}_p = \text{Median}\left(\widehat{VaR}_p^{(1)}, \dots, \widehat{VaR}_p^{(n)}\right).$

4) Kernel method

Suppose $X_1, X_2, \dots, X_n$ are observed losses. The CDF F can be estimated by

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} G\left(\frac{x - x_i}{h}\right)$$

bandwidth

kernel function

Estimate VaR_p as the root of

$$\hat{F}(x) = p.$$

An alternative estimator is

$$\hat{VaR}_p = \frac{\sum_{i=1}^{n} \hat{F}\left(\frac{i-\frac{1}{2}}{n} - p\right) X_{(i)}}{\sum_{i=1}^{n} \hat{F}\left(\frac{i-\frac{1}{2}}{n} - p\right)},$$

where $X_{(i)}$ are the ordered losses in increasing order.

LECTURE

10 NOVEMBER

9:00-10:00AM

MATH3/4/68181

Expected Shortfall

X = Loss

If X is absoutely cont RV then

$$ES_p(X) = \frac{1}{p} \int_0^p VaR_p(X)\, dt$$

Due to Artzner (1997, 1999).

ES is a coherent risk measure.

Properties of ES

i) $X > Y \Rightarrow ES_p(X) > ES_p(Y)$

ii) $X \geq 0 \Rightarrow ES_p(X) \geq 0$

iii) $ES_p(\lambda X) = \lambda \, ES_p(X) \, , \lambda > 0$

iv) $ES_p(X + c) = ES_p(X) + c$

v) $ES_p(X+Y) \leq ES_p(X) + ES_p(Y)$

"sub-additivity"

Proof of (iv)

We know that

$$VaR_p(X+c) = VaR_p(X) + c$$

$$\Rightarrow \int_0^p VaR_t(X+c)\,dt = \int_0^p [VaR_t(X)+c]\,dt$$

$$\Rightarrow \frac{1}{p}\int_0^p VaR_t(X+c)\,dt = \frac{1}{p}\int_0^p [VaR_t(X)+c]\,dt$$

$$\Rightarrow ES_p(X+c) = \frac{1}{p}\int_0^p VaR_t(X)\,dt + \frac{1}{p}[ct]_0^p$$

$$= ES_p(X) + c$$

Proof of (iii)

$$VaR_p(\lambda X) = \lambda\, VaR_p(X)$$

$$\Rightarrow \frac{1}{p}\int_0^p VaR_t(\lambda X)\,dt = \frac{\lambda}{p}\int_0^p VaR_t(X)\,dt$$

$$\Rightarrow \quad ES_p(\lambda X) = \lambda \cdot ES_p(X).$$

- Parametric estimation methods for ES
  3rd
- Non-parametric " " " ~~ES~~ ES
  3rd
- Semi-parametric " " " ES
  4th / 6th

Parametric estimation methods for ES

1) $X = \text{Loss} \sim N(\mu, \sigma^2)$.

$$VaR_p(X) = \mu + \sigma \Phi^{-1}(p),$$

Φ is the CDF of $N(0,1)$

$$ES_p(X) = \mu + \frac{\sigma}{p} \int_0^p \Phi^{-1}(t)\, dt$$

Suppose $X_1, X_2, \ldots, X_n$ is a random sample on X. Then

$$\widehat{ES_p}(X) = \hat{\mu} + \frac{\hat{\sigma}}{p} \int_0^p \Phi^{-1}(t)\, dt$$

where

$$\hat{\mu} = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2}$$

is the MLE of $ES_p(X)$.

2) $X =$ Loss $\sim$ Weibull (α, σ).

$$F(x) = 1 - e^{-\left(\frac{x}{\sigma}\right)^{\alpha}} = p$$

$$\Rightarrow \quad e^{-\left(\frac{X}{\sigma}\right)^{\alpha}} = 1 - p$$

$$\Rightarrow \quad \left(\frac{X}{\sigma}\right)^{\alpha} = -\log(1-p)$$

$$\Rightarrow \quad X = \sigma\left[-\log(1-p)\right]^{\frac{1}{\alpha}} = VaR_p(X)$$

$$ES_p(X) = \frac{\sigma}{p}\int_0^p \left[-\log(1-t)\right]^{\frac{1}{\alpha}} dt$$

$$\boxed{\begin{array}{l} y = -\log(1-t) \Rightarrow 1-t = e^{-y} \Rightarrow t = 1 - e^{-y} \\ \Rightarrow \frac{dt}{dy} = e^{-y} \end{array}}$$

$$= \frac{\sigma}{p}\int_0^{-\log(1-p)} y^{\frac{1}{\alpha}} e^{-y}\, dy$$

$$= \frac{\sigma}{p}\,\gamma\left(\frac{1}{\alpha}+1, -\log(1-p)\right)$$

$$\left[\gamma(\alpha, a) = \int_0^a t^{\alpha-1} e^{-t}\, dt\right]$$
incomp gamma function

Suppose $X_1, X_2, \ldots, X_n$ is a random sample on X. Then the MLE for $ES_p(X)$ is

$$\widehat{ES}_p(X) = \frac{\hat{\sigma}}{p}\,\gamma\left(\frac{1}{\hat{\alpha}}+1, -\log(1-p)\right),$$

where $\hat{\sigma}$ & $\hat{\alpha}$ are the MLES of σ & α.

Non-parametric estimation methods for ES

1) Historical method

Data: $X_1, X_2, \ldots, X_n$

Ordered data: $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$

Then
$$\boxed{\widehat{ES}_p(X) = \frac{\sum_{i=1}^{[np]} X_{(i)}}{[np]}}$$

$[x]$ is the largest integer less than or equal to x.

eg $[1.5] = 1$, $[2.9] = 2$

2)

$$\widehat{ES_p(X)} = \frac{1}{p}\int_0^p \hat{F}^{-1}(u)\,du$$

where $\hat{F}(\circ)$ is the empirical CDF of $\{X_1, X_2, \ldots, X_n\}$.

$$\hat{F}(x) = \frac{1}{n}\sum_{i=1}^{n} I\{X_i \leq x\}$$

3) Kernel method

$$\widehat{ES}_p(X) = \frac{1}{np} \sum_{i=1}^{n} X_i A_h(\hat{q}_p - X_i)$$

where $A_h(u) = A\left(\frac{u}{h}\right)$,

$$A(x) = \int_{-\infty}^{x} K(u)\,du$$

$$\hat{q}_p = \sum_{i=1}^{n} \left[\int_{i-\frac{1}{n}}^{i/n} \frac{1}{h} K\left(\frac{t-p}{h}\right) dt \right] X_i$$

h = bandwith

K = kernel

Due to Yu et al (2010).

4) Richardson's method
(was an applied math professor at Manchester Uni)

i) Compute the empirical CDF $\hat{F}$ from the data

ii) simulate $X_1, \dots, X_N$ from $\hat{F}$

iii) use historical method to estimate ES for the sample in step ii)

iv) repeat ii) & iii) ~~10000~~ 1000 times and compute

$$m_N = \frac{1}{1000} \sum_{i=1}^{1000} \widehat{ES}_{N,i}$$

where $\widehat{ES}_{N,i}$ are the 1000 estimates for ES obtained in step iii)

v) Set $s_n = m_{N_n}$, $n = 1, 2, \dots, k+1$ for some k & $N_1, N_2, \dots, N_{k+1}$

vi) estimate ES by

$$\widehat{ES}_p(X) = \frac{\begin{vmatrix} s_1 & s_2 & \cdots & s_{k+1} \\ 1 & \frac{1}{2} & & \frac{1}{k+1} \\ & & & \\ 1^k & \left(\frac{1}{2}\right)^k & & \left(\frac{1}{k+1}\right)^k \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \frac{1}{2} & & \frac{1}{k+1} \\ & & & \\ 1^k & \left(\frac{1}{2}\right)^k & & \left(\frac{1}{k+1}\right)^k \end{vmatrix}}$$

EXAMPLE CLASS

10 NOVEMBER

10:00-11:00AM

MATH3/4/68181

Q1

$$\mathrm{VaR}_p(X) = -\frac{1}{\lambda}\log(1-p)$$

$$\mathrm{ES}_p(X) = -\frac{1}{p\lambda}\left[p\log(1-p) - p - \log(1-p)\right]$$

The likelihood is

$$L(\lambda) = \left[\prod_{i=1}^{n} \lambda e^{-\lambda x_i}\right] = \lambda^n e^{-\lambda\sum_{i=1}^{n} x_i}$$

The log is

$$\log L(\lambda) = n\log\lambda - \lambda\sum_{i=1}^{n} x_i$$

$$\Rightarrow \frac{d\log L(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0 \Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{X}}$$

So, the MLEs of VaR_p & ES_p are

$$\widehat{\mathrm{VaR}}_p(X) = -\bar{X}\log(1-p)$$

$$\widehat{\mathrm{ES}}_p(X) = -\frac{\bar{X}}{p}\left[p\log(1-p) - p - \log(1-p)\right].$$

Q2

$$f(x) = a x^{a-1}, \quad 0 < x < 1$$

$$F(x) = \int_0^x a y^{a-1} dy = [y^a]_0^x = x^a$$

$$F^{-1}(x) = x^{1/a} \Rightarrow VaR_p(X) = p^{\frac{1}{a}}$$

$$ES_p(X) = \frac{1}{p}\int_0^p t^{\frac{1}{a}} dt = \frac{1}{p}\left[\frac{t^{\frac{1}{a}+1}}{\frac{1}{a}+1}\right]_0^p$$

$$= \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1}.$$

The likelihood is

$$L(a) = \prod_{i=1}^n \left[a x_i^{a-1}\right] = a^n \left(\prod_{i=1}^n x_i\right)^{a-1}$$

The log is

$$\log L(a) = n \log a + (a-1)\sum_{i=1}^n \log x_i$$

$$\Rightarrow \frac{d \log L(a)}{da} = \frac{n}{a} + \sum_{i=1}^n \log x_i = 0 \Rightarrow \hat{a} = -\frac{n}{\sum_{i=1}^n \log x_i}$$

$$\Rightarrow \widehat{VaR_p}(X) = p^{-\frac{\sum \log x_i}{n}}$$

$$\widehat{ES_p}(X) = \frac{p^{-\frac{\sum \log x_i}{n}}}{-\frac{\sum \log x_i}{n} + 1}.$$

Q4

$X \sim LN(\mu, \sigma^2)$

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right)$$ from Stat Methods

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma x} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}}$$

$$\Phi\left(\frac{\log X - \mu}{\sigma}\right) = p$$

$$\Rightarrow \frac{\log X - \mu}{\sigma} = \Phi^{-1}(p) \Rightarrow X = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$= VaR_p(X)$$

$$ES_p(X) = \frac{1}{p} \int_0^p e^{\mu + \sigma \Phi^{-1}(t)}\, dt$$

$$= \frac{e^{\mu}}{p} \int_0^p e^{\sigma \Phi^{-1}(t)}\, dt$$

The likelihood is

$$L(\mu, \sigma) = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi}\,\sigma\, x_i} e^{-\frac{(\log x_i - \mu)^2}{2\sigma^2}} \right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} \left(\prod_{i=1}^{n} \frac{1}{x_i} \right) e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (\log x_i - \mu)^2}$$

The log is

$$\log L = -\frac{n}{2} \log(2\pi) - n \log \sigma - \sum_{i=1}^{n} \log x_i$$

$$- \frac{1}{2\sigma^2} \sum_{i=1}^{n} (\log x_i - \mu)^2$$

$$\frac{\partial \log L}{\partial \mu} = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (-2)(\log x_i - \mu) = 0 \quad -(1)$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (\log x_i - \mu)^2 = 0 \quad -(2)$$

$$(1) \Rightarrow \quad \hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \log x_i$$

$$(2) \Rightarrow \quad \widehat{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} (\log x_i - \hat{\mu})^2$$

$$\Rightarrow \quad \widehat{VaR}_p(X) = e^{\hat{\mu} + \hat{\sigma}\,\Phi^{-1}(p)}$$

$$\widehat{ES}_p(X) = \frac{e^{\hat{\mu}}}{p} \int_0^p e^{\hat{\sigma}\,\Phi^{-1}(t)} \, dt$$

LECTURE

12 NOVEMBER

12:00-13:00PM

MATH4/68181

1)

Data: $X_1, \ldots, X_n$

Returns: $r_i = X_i - X_{i-1}$

Ordered: $r_{(1)} \leq r_{(2)} \leq \cdots \leq r_{(n)}$

$$\widehat{ES}_p = \frac{1}{p} \int_0^p \exp\left[\left(\frac{k\,[n(p+0.05)]}{nq} \right)^{\frac{1}{\hat{\alpha}}} \cdot r_{[n(p+0.05)]} \right] dq - 1$$

where $k \geq 1$ &

$$\hat{\alpha} = \left[\frac{1}{[n(p+0.05)]} \sum_{i=1}^{[n(p+0.05)]} \log\left(\frac{r_{(i)}}{r_{([n(p+0.05)])}} \right) \right]^{-1}.$$

$[x]$ denotes the ~~tht~~ largest integer less than or equal to x.

2) Data: $x_1, x_2, \ldots, x_n$

Ordered: $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$

$$\widehat{ES}_p = \frac{1}{p} \int_{\frac{k}{n}}^{p} \widehat{F}^{-1}(t)\,dt + \frac{k\, X_{(n-k)}}{np(1-\hat{\gamma})}$$

where $\widehat{F}$ is the empirical CDF,

$$\hat{\gamma} = \frac{1}{k} \sum_{i=1}^{k} \log \frac{X_{(n-i+1)}}{X_{(n-k)}}$$

k is a number between 1 & n.

Financial Ratios – this will not be in the test.

X_1 $\qquad$ X_2

$$F_{X_1, X_2}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)$$

"Joint CDF"

$$\bar{F}_{X_1, X_2}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$$

"Joint survival function"

$F_{X_1}(x_1)$, $F_{X_2}(x_2)$ – marginal CDFs of X_1, X_2

$\bar{F}_{X_1}(x_1)$, $\bar{F}_{X_2}(x_2)$ – marginal survival functions of X_1 & X_2

$$\bar{F}_{X_1, X_2}(x_1, x_2) = 1 - F_{X_1}(x_1) - F_{X_2}(x_2) + F_{X_1, X_2}(x_1, x_2)$$

$$F(x_1, x_2) = 1 - \bar{F}_{X_1}(x_1) - \bar{F}_{X_2}(x_2) + \bar{F}_{X_1, X_2}(x_1; x_2)$$

$$\bar{F}_{X_1}(x_1) = 1 - F_{X_1}(x_1)$$

$$\bar{F}_{X_2}(x_2) = 1 - F_{X_2}(x_2)$$

$X_1 \quad X_2 \quad X_3$

$$F_{X_1, X_2, X_3}(x_1, x_2, x_3) = P(X_1 < x_1, X_2 < x_2, X_3 < x_3)$$

Joint CDF of (X_1, X_2, X_3)

$$\bar{F}_{X_1, X_2, X_3}(x_1, x_2, x_3) = P(X_1 > x_1, X_2 > x_2, X_3 > x_3)$$

Joint survival function of (X_1, X_2, X_3)

$$\bar{F}_{X_1, X_2, X_3}(x_1, x_2, x_3) = 1 - F_{X_1}(x_1) - F_{X_2}(x_2) - F_{X_3}(x_3) + F_{X_1, X_2}(x_1, x_2) + F_{X_1, X_3}(x_1, x_3) + F_{X_2, X_3}(x_2, x_3) - F_{X_1, X_2, X_3}(x_1, x_2, x_3)$$

$$F_{X_1, X_2, X_3}(x_1, x_2, x_3) = 1 - \bar{F}_{X_1}(x_1) - \bar{F}_{X_2}(x_2) - \bar{F}_{X_3}(x_3) + \bar{F}_{X_1, X_2}(x_1, x_2) + \bar{F}_{X_1, X_3}(x_1, x_3) + \bar{F}_{X_2, X_3}(x_2, x_3) - \bar{F}_{X_1, X_2, X_3}(x_1, x_2, x_3).$$

$X_1, X_2, \ldots, X_p$

$$F_{X_1, X_2, \ldots, X_p}(x_1, x_2, \ldots, x_p)$$

$$= P(X_1 < x_1, X_2 < x_2, \ldots, X_p < x_p)$$

Joint CDF of $(X_1, X_2, \ldots, X_p)$.

$$\bar{F}_{X_1, X_2, \ldots, X_p}(x_1, x_2, \ldots, x_p)$$

$$= P(X_1 > x_1, X_2 > x_2, \ldots, X_p > x_p)$$

Joint survival function of $(X_1, X_2, \ldots, X_p)$.

What is the relationship between F & $\bar{F}$?

LECTURE

13 NOVEMBER

9:00-10:00AM

MATH3/4/68181

Revision for the Test

Q11, Sheet 1

i) G belongs to Gumbel domain $\Rightarrow$ F also belongs to Gumbel domain

ii) G belongs to Frechet domain $\Rightarrow$ F also belongs to Frechet domain

iii) G belongs to Weibull domain $\Rightarrow$ F also belongs to Weibull domain

i) Assume G belongs to Gumbel domain.

$\exists \; h(t) > 0$ such that

$$\lim_{t \uparrow w(G)} \frac{1 - G(t + x\,h(t))}{1 - G(t)} = e^{-x}.$$

We need to find the limit of

$$\lim_{t \uparrow w(F)} \frac{1 - F(t + x\,h(t))}{1 - F(t)}$$

$$\overset{LH}{=} \lim_{t \uparrow w(F)} \frac{-f(t + x h(t)) \cdot (1 + x h'(t))}{-f(t)}$$

$$= \lim_{t \uparrow w(F)} \frac{\cancel{K}\, g(t + x h(t)) \left[G(t + x h(t))\right]^{a-1} \left\{1 - G(t + x h(t))\right\}^{b-1} e^{-cG(t + x h(t))}}{\cancel{K}\, g(t) \left[G(t)\right]^{a-1} \left\{1 - G(t)\right\}^{b-1} e^{-c G(t)}}$$

$$\cdot \, (1 + x h'(t))$$

$$= \lim_{t \uparrow w(F)} \frac{g(t + x h(t))}{g(t)} \cdot (1 + x h'(t)) \left[\frac{G(t + x h(t))}{G(t)}\right]^{a-1}$$

$$\cdot \left[\frac{1 - G(t + x h(t))}{1 - G(t)}\right]^{b-1} e^{cG(t) - cG(t + x h(t))}$$

$$= \lim_{t \uparrow w(G)} \frac{g(t+xh(t))\,(1+xh'(t))}{g(t)} \cdot \left[\frac{\overbrace{G(t+xh(t))}^{\to 1}}{\underbrace{G(t)}_{\to 1}}\right]^{a-1}$$

$$\cdot \left[\frac{1-G(t+xh(t))}{1-G(t)}\right]^{b-1} e^{\,c\underbrace{G(t)}_{\to 1} - c\underbrace{G(t+xh(t))}_{\to 1}}$$

$$= \lim_{t \uparrow w(G)} \frac{g(t+xh(t))\,(1+xh'(t))}{g(t)} \cdot \left[\frac{1-G(t+xh(t))}{1-G(t)}\right]^{b-1}$$

$$\overset{L'H}{=} \lim_{t \uparrow w(G)} \frac{1-G(t+xh(t))}{1-G(t)} \cdot \left[\frac{1-G(t+xh(t))}{1-G(t)}\right]^{b-1}$$

$$= \lim_{t \uparrow w(G)} \left[\frac{1-G(t+xh(t))}{1-G(t)}\right]^{b} = \left[e^{-x}\right]^{b} = e^{-bx}$$

$\Rightarrow$ F also belongs to the Gumbel domain

Sheet 6

$$VaR_p(X) = F^{-1}(p)$$

$$= \frac{p^{\beta} - (1-p)^{\gamma}}{\delta}$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{1}{p}\int_0^p \left[\frac{t^{\beta} - (1-t)^{\gamma}}{\delta}\right] dt$$

$$= \frac{1}{p\delta}\left[\frac{t^{\beta+1}}{\beta+1} + \frac{(1-t)^{\gamma+1}}{\gamma+1}\right]_0^p$$

$$= \frac{1}{p\delta}\left[\frac{p^{\beta+1}}{\beta+1} + \frac{(1-p)^{\gamma+1}}{\gamma+1} - \frac{1}{\gamma+1}\right]$$

Q2, Sheet 2/3

X is a discrete RV with PMF

$$p(x) = \begin{cases} 1 & k = k_0 \\ 0 & k \neq k_0 \end{cases}$$

What is $w(F)$?

$$F(x) = \begin{cases} 0 & k < k_0 \\ 1 & k \geq k_0 \end{cases}$$

$$F(x) = 1 \Rightarrow x = k_0 \Rightarrow w(F) = k_0$$

$$\lim_{k \uparrow w(F)} \frac{P(X=k)}{1-F(k-1)} = \frac{P(X=k_0)}{1-F(k_0-1)}$$

$$= \frac{1}{1-0} = 1 \neq 0$$

Last Q, Sheet 2

$$F(x) = 1 - q^{(x+1)^a}$$

$$\boxed{p(x) = F(x) - F(x-1)}$$

$$p(x) = 1 - q^{(x+1)^a} - 1 + q^{x^a}$$
$$= q^{x^a} - q^{(x+1)^a}$$

$$F(x) = 1 \Rightarrow 1 - q^{(x+1)^a} = 1$$
$$\Rightarrow q^{(x+1)^a} = 0$$
$$\Rightarrow (x+1)^a \log q = \log 0 = -\infty$$
$$\Rightarrow (x+1)^a = +\infty$$
$$\Rightarrow x+1 = +\infty \Rightarrow x = +\infty$$
$$\Rightarrow w(F) = +\infty$$

$$\lim_{x \uparrow +\infty} \frac{p(x)}{1 - F(x-1)} = \lim_{x \uparrow +\infty} \frac{q^{x^a} - q^{(x+1)^a}}{1 - (1 - q^{x^a})}$$

$$= \lim_{x \uparrow +\infty} 1 - q^{(x+1)^a - x^a}$$

$$= \lim_{x \uparrow +\infty} 1 - q^{x^a\left[\left(1+\frac{1}{x}\right)^a - 1\right]}$$

$$\approx \lim_{x \uparrow +\infty} 1 - q^{x^a\left[\cancel{1} + \frac{a}{x} - \cancel{1}\right]}$$

$$= \lim_{x \uparrow +\infty} 1 - q^{a x^{a-1}}$$

$$= \begin{cases} 1 - q & a = 1 \\ 0 & a < 1 \\ 1 & a > 1 \end{cases}$$

ETT holds only if $a < 1$.

ETT does not hold if $a \geq 1$

LECTURE

19 NOVEMBER

12:00-13:00PM

MATH4/68181

Q3 In-class test

$X_1, X_2, \ldots, X_\alpha$ IID Pareto with

CDF $F(x) = 1 - \left(\frac{k}{x}\right)^a, \quad x > k$

(i) Find CDF of $Y = \max(X_1, \ldots, X_\alpha)$

$$F_Y(y) = P(Y \leq y)$$

$$= P(\max(X_1, \ldots, X_\alpha) \leq y)$$

$$= P(X_1 \leq y, \ldots, X_\alpha \leq y)$$

$$= P(X_1 \leq y) \cdots P(X_\alpha \leq y)$$

$$= \left[1 - \left(\frac{k}{y}\right)^a\right] \cdots \left[1 - \left(\frac{k}{y}\right)^a\right]$$

$$= \left[1 - \left(\frac{k}{y}\right)^a\right]^\alpha$$

(ii) $$f_Y(y) = \alpha\left[1 - \left(\frac{k}{y}\right)^a\right]^{\alpha-1} \frac{(-1)k^a(-a)}{y^{a+1}}$$

$$= \frac{\alpha a k^a}{y^{a+1}} \left[1 - \left(\frac{k}{y}\right)^a\right]^{\alpha-1}$$

(iii)

$$E(Y^n) = \int_k^{\infty} y^n \frac{\alpha a k^a}{y^{a+1}} \cdot \left[1 - \left(\frac{k}{y}\right)^a\right]^{\alpha-1} dy$$

Set $x = \left(\frac{k}{y}\right)^a$

$$\Rightarrow \frac{k}{y} = x^{\frac{1}{a}} \Rightarrow y = k x^{-\frac{1}{a}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{k}{a} x^{-\frac{1}{a}-1}$$

$$= \alpha \cancel{a} k^a \int_1^0 \left(k \cancel{x}^{-\frac{1}{a}}\right)^{n-a-1} (1-x)^{\alpha-1} \cdot \left(-\frac{k}{\cancel{a}}\right) x^{-\frac{1}{a}-1} dx$$

$$= \alpha k^n \int_0^1 x^{-\frac{n}{a}} (1-x)^{\alpha-1} dx$$

$$= \alpha k^n \, B\left(1 - \frac{n}{a}, \alpha\right)$$

$$E(Y) = \alpha k \, B\left(1 - \frac{1}{a}, \alpha\right)$$

$$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2$$

$$= \alpha k^2 B\left(1 - \frac{2}{a}, \alpha\right) - \left(\alpha k \, B\left(1 - \frac{1}{a}, \alpha\right)\right)^2.$$

(iv)

$$\left[1-\left(\frac{K}{y}\right)^{a}\right]^{\alpha} = p$$

$$\Rightarrow \quad 1-\left(\frac{K}{y}\right)^{a} = p^{\frac{1}{\alpha}}$$

$$\Rightarrow \quad \left(\frac{k}{y}\right)^{a} = 1-p^{\frac{1}{\alpha}}$$

$$\Rightarrow \quad \frac{k}{y} = \left[1-p^{\frac{1}{\alpha}}\right]^{\frac{1}{a}}$$

$$\Rightarrow \quad y = K\left[1-p^{\frac{1}{\alpha}}\right]^{-\frac{1}{a}} = VaR_p(Y)$$

(v) $$ES_p(Y) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{K}{p}\int_0^p \left[1-t^{\frac{1}{\alpha}}\right]^{-\frac{1}{a}} dt$$

$$= \frac{K}{p}\int_0^p \sum_{k=0}^{\infty}\binom{-\frac{1}{a}}{k}(-1)^k\, t^{\frac{k}{\alpha}}\,dt$$

$$= \frac{K}{p}\sum_{k=0}^{\infty}\binom{-1/a}{k}(-1)^k\int_0^p t^{\frac{k}{\alpha}}\,dt$$

$$= \frac{K}{p}\sum_{k=0}^{\infty}\binom{-1/a}{k}(-1)^k\left[\frac{t^{1+\frac{k}{a}}}{1+\frac{k}{a}}\right]_0^p$$

$$= \frac{K}{p}\sum_{k=0}^{\infty}\binom{-1/a}{k}(-1)^k\,\frac{p^{1+\frac{k}{a}}}{1+\frac{k}{a}}.$$

Portfolios

consists of more than one investment.

X_1 = Loss on 1st investment

X_2 = " " 2nd "

$\vdots$

X_k = " " kth " .

One would like to minimize the loss, or identify the investments having smaller loss: what is the prob of

$$X_1 < X_2$$

$$X_1 < X_2 < X_3$$

$$\vdots$$

$$X_1 < X_2 < \cdots < X_k \text{ ?}$$

eg 1

$X_1 \sim N(\mu_1, \sigma_1^2)$ & $X_2 \sim N(\mu_2, \sigma_2^2)$.
Suppose X_1 & X_2 are indep.
What is $P(X_1 < X_2)$?

$$P(X_1 < X_2) = P(X_1 - X_2 < 0)$$

$$= P(N(\mu_1, \sigma_1^2) - N(\mu_2, \sigma_2^2) < 0)$$

$$= P(N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2) < 0)$$

$$= P\left(\frac{N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}} < \frac{0 - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)$$

$$= P\left(N(0, 1) < \frac{\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)$$

$$= \Phi\left(\frac{\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right).$$

Suppose X_1 & X_2 are indep RVs with PDFs f_1, f_2 & CDFs F_1, F_2.

$$P(X_1 < X_2)$$

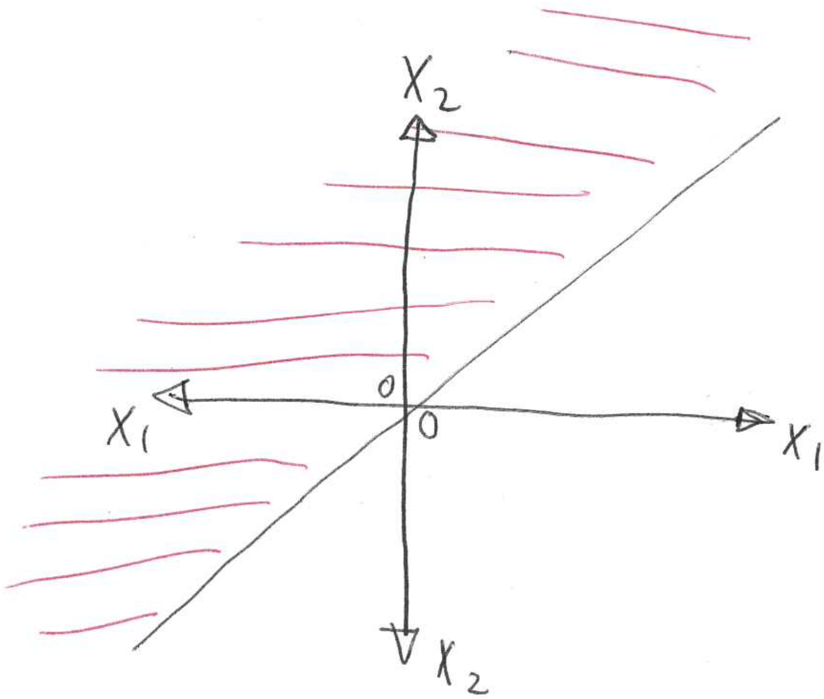
$$= \int_{-\infty}^{+\infty} F_1(x)\, f_2(x)\, dx$$

$$= \int_{-\infty}^{+\infty} \left[1 - F_2(x)\right] f_1(x)\, dx$$

Suppose X_1 & X_2 are dependent RVs with joint PDF $\cancel{f}\,f(x_1, x_2)$. Then

$$P(X_1 < X_2)$$

$$= \int_{-\infty}^{+\infty} \int_{x_1}^{\infty} f(x_1, x_2)\, dx_2\, dx_1$$



Homework: Suppose

$$X_i \sim N(\mu_i, \sigma_i^2), \quad i = 1, 2, \ldots, k$$

are indep RVs.

Find

$$P(X_1 < X_2 < \cdots < X_k)$$

LECTURE

20 NOVEMBER

9:00-10:00AM

MATH3/4/68181

In-class test feedback

Q1 a) book work

Q1 b) book work

Q1 c)

$$F(x) = \left\{1 - \left[1 - G^{\theta}(x)\right]^4\right\}^{\alpha}$$

i) Assume G belongs to the Gumbel domain, that is

$$\lim_{t \uparrow w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x}.$$

$$\lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \uparrow w(F)} \frac{1 - \left\{1 - \left[1 - G^{\theta}(t + x\gamma(t))\right]^4\right\}^{\alpha}}{1 - \left\{1 - \left[1 - G^{\theta}(t)\right]^4\right\}^{\alpha}}$$

$(1-y)^{\alpha} \approx 1 - \alpha y$

$$= \lim_{t \uparrow w(F)} \frac{\cancel{1} - \left\{\cancel{1} - \alpha \cdot \left[1 - G^{\theta}(t + x\gamma(t))\right]^4\right\}}{\cancel{1} - \left\{\cancel{1} - \alpha \left[1 - G^{\theta}(t)\right]^4\right\}}$$

$$= \lim_{t \uparrow w(F)} \left[\frac{1 - G^{\theta}(t + x\gamma(t))}{1 - G^{\theta}(t)}\right]^4$$

$$\overset{LH}{=} \lim_{t \uparrow w(F)} \left[\frac{-\cancel{\theta}\, \overbrace{G^{\theta-1}(t+x\gamma(t))}^{\to 1}\, g(t+x\gamma(t))\,(1+x\gamma'(t))}{-\cancel{\theta}\, \underbrace{G^{\theta-1}(t)}_{\to 1}\, g(t)} \right]^4$$

$$= \lim_{t \uparrow w(G)} \left[\frac{g(t+x\gamma(t))\,(1+x\gamma'(t))}{g(t)} \right]^4$$

$$\overset{LH}{=} \lim_{t \uparrow w(G)} \left[\frac{1-G(t+x\gamma(t))}{1-G(t)} \right]^4$$

$$= \left[e^{-x}\right]^4 = e^{-4x}$$

$\Rightarrow$ F also belongs to Gumbel domain.

Q2 (i) most did this ok.

Q2 (ii) $f(x) = 6x(1-x), \quad 0 < x < 1$

$$F(x) = \int_0^x (6x - 6x^2)\, dx$$

$$= 3x^2 - 2x^3$$

$$F(x) = 1 \Rightarrow w(F) = 1$$

$$\lim_{t \uparrow 1} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \lim_{t \uparrow 1} \frac{1 - \left(3(t + x\gamma(t))^2 - 2(t + x\gamma(t))^3\right)}{1 - (3t^2 - 2t^3)}$$

$$\overset{LH}{=} \lim_{t \uparrow 1} \frac{-6(t + x\gamma(t)) \cdot (1 + x\gamma'(t)) + 6(t + x\gamma(t))^2 \cdot (1 + x\gamma'(t))}{-6t + 6t^2}$$

$$\overset{LH}{=} \lim_{t \uparrow 1} \frac{-6(t + x\gamma(t)) \cdot x\gamma''(t) - 6(1 + x\gamma'(t))^2 + 6(t + x\gamma(t))^2 \cdot x\gamma''(t) + 12(t + x\gamma(t))(1 + x\gamma'(t))^2}{-6 + 12t}$$

$\neq e^{-x} \Rightarrow$ does not belong to Gumbel domain

Also does not belong to Frechet domain because $\omega(F) = 1 < \infty$.

$$\lim_{t \downarrow 0} \frac{1 - F(1-tx)}{1 - F(1-t)} = \lim_{t \downarrow 0} \frac{1-\left[3(1-tx)^2 - 2(1-tx)^3\right]}{1-\left[3(1-t)^2 - 2(1-t)^3\right]}$$

$$= \lim_{t \downarrow 0} \frac{\cancel{1} - \left[\cancel{3} - 6tx + 3t^2x^2 - \cancel{2} + 6tx - 6(tx)^2 + 2(tx)^3\right]}{\cancel{1} - \left[\cancel{3} - 6t + 3t^2 - \cancel{2} + 6t - 6t^2 + 2t^3\right]}$$

$$= \lim_{t \downarrow 0} \frac{\cancel{6tx} - 3t^2x^2 \cancel{- 6tx} + 6(tx)^2 - 2(tx)^3}{\cancel{6t} - 3t^2 \cancel{+ 6t} + 6t^2 - 2t^3}$$

$$\boxed{= \cancel{\lim_{t \downarrow 0}} \frac{6x - \underbrace{3tx^2}_{\to 0} - 6x + \underbrace{6tx^2}_{\to 0} + \underbrace{2t^2x^3}_{\to 0}}{6 - \underbrace{3t}_{\to 0} - 6 + \underbrace{6t}_{\to 0} + \underbrace{2t^2}_{\to 0}}}$$

$$= \lim_{t \downarrow 0} \frac{3t^2x^2 - 2(tx)^3}{3t^2 - 2t^3}$$

$$= \lim_{t \downarrow 0} \frac{3x^2 - \underbrace{2tx^3}_{\to 0}}{3 - \underbrace{2t}_{\to 0}} = x^2$$

$\Rightarrow$ F belongs to Weibull domain.

Q2 iv) $F(x) = \Phi^2(x)$

$F(x) = 1 \Rightarrow w(F) = +\infty$,

$$\lim_{t \uparrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{1 - \Phi^2(t + x\gamma(t))}{1 - \Phi^2(t)}$$

$$\overset{LH}{=} \lim_{t \uparrow \infty} \frac{-\cancel{2}\,\Phi(t + x \cdot \gamma(t)) \cdot \phi(t + x\gamma(t))(1 + x\gamma'(t))}{-\cancel{2}\,\Phi(t) \cdot \phi(t)}$$

$\Phi(t + x\gamma(t)) \to 1$, $\Phi(t) \to 1$

$$= \lim_{t \uparrow \infty} \frac{\cancel{\frac{1}{\sqrt{2\pi}}}\, e^{-\frac{(t + x\gamma(t))^2}{2}}}{\cancel{\frac{1}{\sqrt{2\pi}}}\, e^{-\frac{t^2}{2}}} \cdot (1 + x\gamma'(t))$$

$$= \lim_{t \uparrow \infty} e^{\frac{t^2 - (t + x\gamma(t))^2}{2}} \cdot (1 + x\gamma'(t))$$

$$= \lim_{t \uparrow \infty} e^{\frac{\cancel{t^2} - \cancel{t^2} - 2tx\gamma(t) - x^2\gamma^2(t)}{2}} \cdot (1 + x\gamma'(t))$$

$$= \lim_{t \uparrow \infty} e^{-tx\gamma(t) - \frac{x^2\gamma^2(t)}{2}} \cdot (1 + x\gamma'(t))$$

Choose $\gamma(t) = \frac{1}{t}$.

$$= \lim_{t \to \infty} e^{-x - \underbrace{\frac{x^2}{2t^2}}_{\to 0}} \cdot \left(1 + \underbrace{x \cdot \left(-\frac{1}{t^2}\right)}_{\to 0}\right)$$

$$= e^{-x}$$

$\Rightarrow$ F belongs to Gumbel.

X = Income

Sometimes the values of X are over/under reported.

eg.

25,000 reported income

25, 125 actual "

Suppose X = True Income

Then under-reported income can be written as

$$Z = XY$$

a RV taking values in $(0,1)$

Suppose X = True Income.

Then over-reported income can be written as

$$Z = X / r$$

a RV taking values in $(0,1)$

Underreported/Overreported Income

In the economic literature, the under reported income is commonly expressed by the multiplicative relationship $Z = XY$, where Y is a multiplicative error and X denotes the true income. It is known that if Y has the power function distribution then X is Pareto distributed if and only if Z is also, see Krishnaji (1970).

The over reported income is commonly expressed by the multiplicative relationship $Z = X/Y$, where X and Y are independent random variables with X denoting the true income and Y a multiplicative error taking values in the interval $(0, 1)$. It is known that if Y has the power function distribution then X is Pareto distributed if and only if Z is also, see Krishnaji (1970).

A Pareto random variable has cdf specified by $F(x) = 1 - (K/x)^a$ for $x > K$. A power function random variable has cdf specified by $F(x) = x^c$ for $0 < x < 1$.

Proofs will be given

References

[1] Krishnaji, N. (1970). Characterization of the Pareto distribution through a model of under-reported incomes. *Econometrica*, **38**, 251-255.

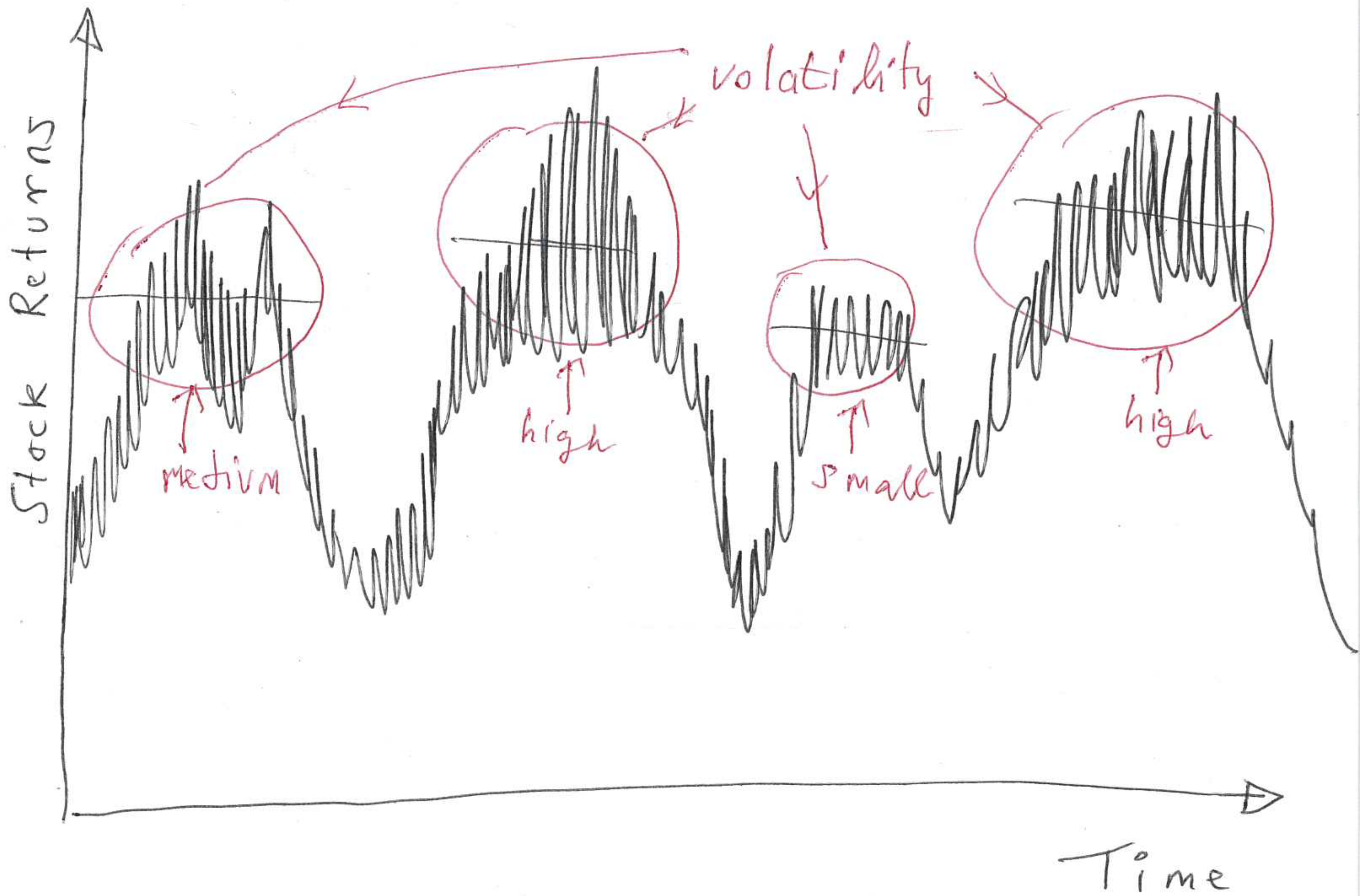
LECTURE

24 NOVEMBER

9:00-10:00AM

MATH3/4/68181

Models for Stock Returns



V = Volatility

It is reasonable to suppose that V is a RV.

Let X = Stock Returns.

Suppose $X \mid V$ has PDF $f_{X|V}$.
(Cond distn of X given V)

Then the unconditional (actual) distribution of X will have the PDF

$$\boxed{f_X(x) = \int_0^\infty f_{X|V}(x|v)\, f_V(v)\, dv}$$

Cond PDF of $X|V$

PDF of V

The corresponding CDF is

$$\boxed{F_X(x) = \int_0^\infty F_{X|V}(x|v)\, f_V(v)\, dv}$$

Cond CDF of $X|V$

PDF of V

Moments

$$E[X] = E[E[X|V]]$$

$$E[X^2] = E[E[X^2|V]]$$

$$\vdots$$

$$E[X^n] = E[E[X^n|V]]$$

$$Var[X] = E[X^2] - (E[X])^2$$

$$= E[E[X^2|V]] - (E[E[X|V]])^2.$$

eg.1

$$X \mid V \sim N(0, \sigma^2)$$

Variance

$$V = \sigma^2$$

V has PDF $\dfrac{2\lambda^2}{\sigma^3} e^{-\left(\frac{\lambda}{\sigma}\right)^2}$

(PDF of σ) $\lambda > 0$

$$f_X(x) = \int_0^\infty \boxed{f_{X|V}(x|v)}\, \boxed{f_V(v)}\, dv$$

$$= \int_0^\infty \left[\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}\right] \cdot \frac{2\lambda^2}{\sigma^3} \cdot e^{-\frac{\lambda^2}{\sigma^2}} d\sigma$$

$$= \frac{2\lambda^2}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sigma^4} e^{-\left(\frac{x^2}{2}+\lambda^2\right)\frac{1}{\sigma^2}} d\sigma$$

Set $y = \frac{1}{\sigma^2} \Rightarrow \sigma = \frac{1}{\sqrt{y}}$

$$\Rightarrow \frac{d\sigma}{dy} = -\frac{1}{2} y^{-\frac{3}{2}}$$

$$= \frac{2\lambda^2}{\sqrt{2\pi}} \int_\infty^0 y^2 e^{-\left(\frac{x^2}{2}+\lambda^2\right)y} \cdot \left(-\frac{1}{2} y^{-\frac{3}{2}}\right) \cdot dy$$

$$= \frac{\lambda^2}{\sqrt{2\pi}} \int_0^\infty y^{\frac{1}{2}} e^{-\left(\frac{x^2}{2}+\lambda^2\right)y} \, dy$$

$$\boxed{\begin{aligned} &\text{Set} \quad z = \left(\frac{x^2}{2}+\lambda^2\right) y \\ &\qquad y = \frac{z}{\frac{x^2}{2}+\lambda^2} \Rightarrow dy = \frac{dz}{\frac{x^2}{2}+\lambda^2} \end{aligned}}$$

$$= \frac{\lambda^2}{\sqrt{2\pi}} \int_0^\infty \sqrt{\frac{z}{\frac{x^2}{2}+\lambda^2}} \, e^{-z} \cdot \frac{dz}{\frac{x^2}{2}+\lambda^2}$$

$$= \frac{\lambda^2}{\sqrt{2\pi}} \left(\frac{x^2}{2}+\lambda^2\right)^{-\frac{3}{2}} \boxed{\int_0^\infty \sqrt{z}\, e^{-z} \, dz}$$

$$= \frac{\lambda^2}{\sqrt{2\pi}} \left(\frac{x^2}{2}+\lambda^2\right)^{-\frac{3}{2}} \quad \boxed{\Gamma\left(\frac{3}{2}\right)} = \frac{\sqrt{\pi}}{2}$$

$$= \frac{\lambda^2}{2\sqrt{2}} \left(\frac{x^2}{2}+\lambda^2\right)^{-\frac{3}{2}}.$$

$$\boxed{\underline{f_X(x)} = \frac{\lambda^2}{2\sqrt{2}} \left(\frac{x^2}{2}+\lambda^2\right)^{-\frac{3}{2}}}$$

eg 2

$$X \mid V \sim N(0, \sigma^2)$$

PDF of σ is Uni $[a, b]$.

$$f_X(x) = \int_a^b \left[\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}\right] \cdot \frac{1}{b-a} \, d\sigma$$

$$= \frac{1}{\sqrt{2\pi}(b-a)} \int_a^b \frac{1}{\sigma} e^{-\frac{x^2}{2\sigma^2}} \, d\sigma$$

$$\boxed{\begin{aligned} &\text{Set } y = \frac{x^2}{2\sigma^2} \Rightarrow \sigma = \frac{x}{\sqrt{2y}} \\ &\Rightarrow \frac{d\sigma}{dy} = -\frac{x}{\sqrt{2}} \cdot \frac{1}{2} \cdot y^{-\frac{3}{2}} \end{aligned}}$$

$$= \frac{1}{\sqrt{2\pi}(b-a)} \int_{\frac{\cancel{x}^2}{2a^2}}^{\frac{x^2}{2b^2}} \frac{\cancel{\sqrt{2}}\, y}{\cancel{x}} \cdot e^{-y} \cdot \left(-\frac{\cancel{x}}{2\cancel{\sqrt{2}}}\right) \cdot y^{-\frac{3}{2}} \, dy$$

$$= \frac{1}{2\sqrt{2\pi}(b-a)} \int_{\frac{x^2}{2b^2}}^{\frac{x^2}{2a^2}} y^{-1} e^{-y} \, dy$$

$$= \frac{1}{2\sqrt{2\pi}(b-a)} \left[\Gamma\left(0, \frac{x^2}{2b^2}\right) - \Gamma\left(0, \frac{x^2}{2a^2}\right)\right]$$

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} \, dt$$

$$E[X] = E[E[X|V]]$$

$$= E[0]$$

$$= 0$$

$$E[X^2] = E[\underline{\underline{E[X^2|V]}}]$$

$$= E[\sigma^2]$$

$$= \int_a^b \sigma^2 \cdot \frac{1}{b-a} \, d\sigma$$

$$= \frac{1}{b-a} \left[\frac{\sigma^3}{3} \right]_a^b = \frac{b^3 - a^3}{3(b-a)}$$

$$\mathrm{Var}[X] = \frac{b^3 - a^3}{3(b-a)} .$$

EXAMPLE CLASS

24 NOVEMBER

10:00-11:00AM

MATH3/4/68181

Q1.

X = stock returns

$X \mid \lambda \sim Exp(\lambda)$

$\lambda \sim Exp(a)$.

$$f_X(x) = \int_0^\infty \boxed{f_{X|\lambda}(x|\lambda)} \, \boxed{f_\lambda(\lambda)} \, d\lambda$$

$$= \int_0^\infty \lambda e^{-\lambda x} \quad a e^{-\lambda a} \, d\lambda$$

$$= a \int_0^\infty \lambda \, e^{-\lambda(x+a)} \, d\lambda$$

$$\boxed{\text{Set } y = \lambda(x+a) \Rightarrow \lambda = \frac{y}{x+a} \Rightarrow d\lambda = \frac{dy}{x+a}}$$

$$= a \int_0^\infty \frac{y}{x+a} \cdot e^{-y} \cdot \frac{dy}{x+a}$$

$$= \frac{a}{(x+a)^2} \cdot \boxed{\int_0^\infty y e^{-y} \, dy} = \Gamma(2) = 1! = 1$$

$$= \frac{a}{(x+a)^2}.$$

Suppose $X_1, X_2, \ldots, X_n$ is a random sample of actual stock returns.

$$L(a) = \prod_{i=1}^{n} \frac{a}{(X_i + a)^2} = a^n \left[\prod_{i=1}^{n} (X_i + a) \right]^{-2}$$

$$\log L(a) = n \log a - 2 \sum_{i=1}^{n} \log (X_i + a)$$

$$\frac{d \log L(a)}{d a} = \frac{n}{a} - 2 \sum_{j=1}^{n} \frac{1}{X_i + a} .$$

The MLE of a is the root of

$$\boxed{\frac{n}{a} = 2 \sum_{i=1}^{n} \frac{1}{X_i + a}}$$

$\boxed{Q2}$

$$X \mid \lambda \sim \text{Exp}(\lambda)$$

$$\lambda \sim \text{Uni}[a, b]$$

$$f_X(x) = \int_a^b \lambda e^{-\lambda x} \cdot \frac{1}{b-a} \, d\lambda$$

$$= \frac{1}{b-a} \int_a^b \lambda e^{-\lambda x} \, d\lambda$$

$$= \frac{1}{b-a} \left\{ \left[\lambda \cdot \frac{e^{-\lambda x}}{(-x)} \right]_a^b + \frac{1}{x} \int_a^b e^{-\lambda x} \, d\lambda \right\}$$

$$= \frac{1}{b-a} \left\{ \frac{b e^{-bx} - a e^{-ax}}{(-x)} + \frac{1}{x} \left[\frac{e^{-\lambda x}}{(-x)} \right]_a^b \right\}$$

$$= \frac{1}{b-a} \left\{ \frac{b e^{-bx} - a e^{-ax}}{(-x)} - \frac{e^{-bx} - e^{-ax}}{x^2} \right\}$$

$\boxed{Q3}$

$X \mid \lambda \sim Exp(\lambda)$

λ has PDF $a\lambda^{a-1}$, $0 < \lambda < 1$

$$f_X(x) = \int_0^1 \lambda e^{-\lambda x} \cdot a\lambda^{a-1} \, d\lambda$$

$$= a \int_0^1 \lambda^a e^{-\lambda x} \, d\lambda$$

$$\boxed{\text{Set } y = \lambda x \Rightarrow d\lambda = \frac{dy}{x}}$$

$$= a \int_0^x \left(\frac{y}{x}\right)^a e^{-y} \frac{dy}{x}$$

$$= a x^{-a-1} \boxed{\int_0^x y^a e^{-y} \, dy}$$

$$= a x^{-a-1} \gamma(a+1, x)$$

$$\boxed{\gamma(a, x) = \int_0^x t^{a-1} e^{-t} \, dt}$$

Incomp gamma function

$\boxed{Q4}$

$X \mid \lambda \sim \text{Exp}(\lambda)$

$\lambda \sim$ has PDF $\dfrac{a k^a}{\lambda^{a+1}}, \quad \lambda > k$

$$f_X(x) = \int_k^\infty \lambda e^{-\lambda x} \cdot \frac{a k^a}{\lambda^{a+1}} \, d\lambda$$

$$= a k^a \int_k^\infty \lambda^{-a} e^{-\lambda x} \, d\lambda$$

$$\boxed{\text{Set } \quad y = \lambda x \Rightarrow \quad d\lambda = \frac{dy}{x}}$$

$$= a k^a \int_{kx}^\infty \left(\frac{y}{x}\right)^{-a} e^{-y} \frac{dy}{x}$$

$$= a k^a x^a \int_{kx}^\infty y^{-a} e^{-y} \frac{dy}{x}$$

$$= a k^a x^{a-1} \Gamma(1-a, kx)$$

$$\boxed{\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} \, dt}$$

LECTURE

26 NOVEMBER

12:00-13:00PM

MATH4/68181

$\boxed{\text{Copulas}}$

- A copula is a joint CDF of Uniform [0, 1] random variables.

- In the bivariate case a copula is denoted by $C(u, v)$, $0 < u < 1$, $0 < v < 1$

- Formally, $C(u,v)$ is a joint CDF of two Uniform [0, 1] RVs satisfying

i) $C(u, 0) = C(0, v) = 0$

ii) $C(u, 1) = u$

iii) $C(1, v) = v$

iv) $\dfrac{\partial C(u, v)}{\partial u} \geq 0$

v) $\dfrac{\partial C(u, v)}{\partial v} \geq 0$

- For every CDF F there is a corresponding copula.

Suppose (X, Y) with joint CDF F.

Let $F_X(x) = F(x, \infty)$ marginal CDF of X

$F_Y(y) = F(\infty, y)$ " " " Y

Let

$U = F_X(X) \sim \text{Uni}[0,1]$

$V = F_Y(Y) \sim \text{Uni}[0,1]$

" Prob Integral Transform "

Then $P(U \le u, V \le v)$

$= P(F_X(X) \le u, F_Y(Y) \le v)$

$= P(X \le F_X^{-1}(u), Y \le F_Y^{-1}(v))$

$= \boxed{F(F_X^{-1}(u), F_Y^{-1}(v))}$

↑ is a copula

RS

Suppose (X, Y) has CDF

$$F(x,y) = 1 - \left[1+\frac{x}{a}\right]^{-c} - \left[1+\frac{y}{b}\right]^{-c}$$

$$+ \left[1+\frac{x}{a}+\frac{y}{b}\right]^{-c}$$

$$F_X(x) = F(x,\infty) = 1-\left[1+\frac{x}{a}\right]^{-c}$$

$$F_Y(y) = F(\infty, y) = 1-\left[1+\frac{y}{b}\right]^{-c}$$

$$F_X^{-1}(u) = a\left[(1-u)^{-\frac{1}{c}} - 1\right]$$

$$F_Y^{-1}(v) = b\left[(1-v)^{-\frac{1}{c}} - 1\right]$$

So,

$$C(u,v) = F(F_X^{-1}(u), F_Y^{-1}(v))$$

$$= 1-\left[1+\frac{a\left[(1-u)^{-\frac{1}{c}}-1\right]}{a}\right]^{-c}$$

$$-\left[1+\frac{b\left[(1-v)^{-\frac{1}{c}}-1\right]}{b}\right]^{-c}$$

$$+\left[1+\frac{a\left[(1-u)^{-\frac{1}{c}}-1\right]}{a}+\frac{b\left[(1-v)^{-\frac{1}{c}}-1\right]}{b}\right]^{-c}$$

$$= \boxed{u+v-1+\left[(1-u)^{-\frac{1}{c}}+(1-v)^{-\frac{1}{c}}-1\right]^{-c}}$$

is a copula

Copula was discovered by Sklar (1968).

Most popular copula is the <u>Gaussian copula</u>:

$$C(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^{\Phi^{-1}(u)} \int_0^{\Phi^{-1}(v)} e^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}} \, dy\, dx$$

$-1 < \rho < 1$, Φ CDF of $N(0,1)$

<u>t Copula</u>:

$$C(u,v) = \frac{\Gamma\left(1+\frac{\theta}{2}\right)}{\pi\theta\sqrt{1-\rho^2}\,\Gamma\left(\frac{\theta}{2}\right)} \int_{-\infty}^{t_\theta^{-1}(u)} \int^{t_\theta^{-1}(v)} \left[1+\frac{x^2+y^2-2\rho xy}{\theta(1-\rho^2)}\right]^{-1-\frac{\theta}{2}} dy\, dx$$

$\theta > 0$ = degree of freedom

$-1 < \rho < 1$, $t_\theta(\cdot)$ CDF of Student't RV with df $= \theta$.

eg

1) $C(u,v) = uv$

"independence copula"

Show that this is a copula:

i) $C(u,0) = u \cdot 0 = 0$ ✓

$C(0,v) = 0 \cdot v = 0$ ✓

ii) $C(u,1) = u \cdot 1 = u$ ✓

iii) $C(1,v) = 1 \cdot v = v$ ✓

iv) $\dfrac{\partial C(u,v)}{\partial u} = v \geq 0$ ✓

v) $\dfrac{\partial C(u,v)}{\partial v} = u \geq 0$ ✓

$\Rightarrow$ C is a copula.

2.) $C(u,v) = \min(u,v)$

"Complete dependence copula"

i) $C(u,0) = \min(u,0) = 0$ ✓

$C(0,v) = \min(0,v) = 0$ ✓

ii) $C(u,1) = \min(u,1) = u$ ✓

iii) $C(1,v) = \min(1,v) = v$ ✓

iv) $$\frac{\partial C(u,v)}{\partial u} = \begin{cases} \frac{\partial}{\partial u} u & \text{if } u \leq v \\ \frac{\partial}{\partial u} v & \text{if } u > v \end{cases}$$

$$= \begin{cases} 1 & \text{if } u \leq v \\ 0 & \text{if } u > v \end{cases}$$

≥ 0 ✓

v) $$\frac{\partial C(u,v)}{\partial v} = \begin{cases} \frac{\partial}{\partial v} u & \text{if } u \leq v \\ \frac{\partial}{\partial v} v & \text{if } u > v \end{cases}$$

$$= \begin{cases} 0 & \text{if } u \leq v \\ 1 & \text{if } u > v \end{cases}$$

$$\geq 0 \checkmark$$

$\Rightarrow$ C is a copula.

3)

$$C(u,v) = uv e^{-\theta \log u \log v}, \quad 0 < \theta \leq 1$$

i) $C(u,0) = 0$ ✓

$C(0,v) = 0$ ✓

ii) $C(u,1) = u \cdot 1 \cdot e^{-\theta(\log u)(\log 1)}$

$= u$ ✓

iii) $C(1,v) = 1 \cdot v \cdot e^{-\theta(\log 1)(\log v)}$

$= v$ ✓

iv) $$\frac{\partial C(u,v)}{\partial u} = v e^{-\theta \log u \log v} + \cancel{u} v e^{-\theta \log u \log v} \left(-\theta \cdot \frac{1}{u} \log v\right)$$

$$= v(1-\theta \overset{<0}{\log v}) e^{-\theta \log u \log v} \geq 0$$ ✓

v) $$\frac{\partial C(u,v)}{\partial v} = u(1-\theta \overset{<0}{\log u}) \cdot e^{-\theta \log u \log v} \geq 0$$ ✓

$\Rightarrow$ C is a copula.

LECTURE

27 NOVEMBER

9:00-10:00AM

MATH3/4/68181

$\boxed{\text{UNDER}}$ / $\boxed{\text{OVER}}$ $\boxed{\text{REPORTED}}$ $\boxed{\text{INCOME}}$

X = True Income

Exact Values of X are usually unknown. Often what is reported are less than/ greater than the true income.

eg

£ 24, 998.56 – True Income

£ 25, 000 – Reported "

Over reported income

eg

£ 50, 100.99 – True Income

£ 50, 000 – Reported income

Under reported income.

Mathematical Representations

$$Z = X Y$$

Under Reported Income

True Income

a RV in (0,1)

$$Z = \frac{X}{Y}$$

Over Reported Income

The most popular model for X (true income) is the Pareto distribution.

But we have no observations of X.

We have observations on only Z.

If X is a Pareto RV what distribution does Z follow?

Theorem 1 Suppose Y is a power function RV with CDF $F_Y(y) = y^c$, $0 < y < 1$. Then X is a Pareto RV if and only if $(Z = XY)$ is also a Pareto RV.

Theorem 2 Suppose Y is a power function RV with CDF $F_Y(y) = y^c$, $0<y<1$. Then X is a Pareto RV if and only if $Z = \frac{X}{Y}$ is also a Pareto RV.

$\boxed{\text{Proof of Thm 1}}$ Assume X is a Pareto RV with $F_X(x) = 1 - \left(\frac{K}{x}\right)^a$, $x > K$. Then

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(XY < z) \\
&= P\left(X < \frac{z}{Y}\right)
\end{aligned}$$

$$= \int_0^1 P\left(X < \frac{z}{y}\right) f_Y(y)\,dy$$

Total Prob Rule

$$= \int_0^1 F_X\left(\frac{z}{y}\right) f_Y(y)\,dy$$

$$= \int_0^1 \left[1 - \left(\frac{Ky}{z}\right)^a\right] \cdot c \cdot y^{c-1}\,dy$$

$$= c\int_0^1 y^{c-1}\,dy - (K/z)^a \cdot c \cdot \int_0^1 y^{c-1+a}\,dy$$

$$= c\left[\frac{y^c}{c}\right]_0^1 - \left(\frac{K}{z}\right)^a \cdot c \cdot \left[\frac{y^{c+a}}{c+a}\right]_0^1$$

$$= 1 - \left(\frac{K}{z}\right)^a \cdot c \cdot \frac{1}{c+a}$$

$$= 1 - \left(\frac{\frac{K c^{\frac{1}{a}}}{(c+a)^{\frac{1}{a}}}}{z}\right)^a = 1 - \left(\frac{K^*}{z}\right)^a,$$

where $K^* = \dfrac{K c^{\frac{1}{a}}}{(c+a)^{\frac{1}{a}}}$.

$\Rightarrow$ Z is a Pareto RV.

Assume Z is a Pareto RV with CDF $F_Z(z) = 1 - \left(\frac{L}{z}\right)^b, \; z > L.$

Then

$$F_X(x) = P(X < x)$$
$$= P(XY < xY)$$
$$= P(Z < xY)$$
$$= \int_0^1 P(Z < xy)\, f_Y(y)\, dy$$

Total Prob Rule

$$= \int_0^1 F_Z(xy)\, f_Y(y)\, dy$$
$$= \int_0^1 \left[1 - \left(\frac{L}{xy}\right)^b\right] \cdot c \cdot y^{c-1}\, dy$$
$$= c \cdot \int_0^1 y^{c-1}\, dy - \left(\frac{L}{x}\right)^b \cdot c \cdot \int_0^1 y^{c-1-b}\, dy$$
$$= c \left[\frac{y^c}{c}\right]_0^1 - \left(\frac{L}{x}\right)^b \cdot c \cdot \left[\frac{y^{c-b}}{c-b}\right]_0^1$$
$$= 1 - \left(\frac{L}{x}\right)^b \cdot \frac{c}{c-b}$$
$$= 1 - \left(\frac{L^*}{x}\right)^b, \quad \text{where } L^* = \frac{L c^{\frac{1}{b}}}{(c-b)^{\frac{1}{b}}}$$

$\Rightarrow$ X is a Pareto RV.
Hence Theorem 1 is proved.

$\boxed{\text{Proof of Thm 2}}$ Assume X is a Pareto RV with CDF $F_X(x) = 1 - \left(\frac{K}{x}\right)^a$, $x > K$.

Then

$$F_Z(z) = P(Z < z)$$
$$= P\left(\frac{X}{Y} < z\right)$$
$$= P(X < z \cdot Y)$$
$$= \int_0^1 P(X < z \cdot y) \cdot f_Y(y)\, dy$$
$$= \int_0^1 F_X(z \cdot y)\, f_Y(y)\, dy$$
$$= \int_0^1 \left[1 - \left(\frac{K}{z \cdot y}\right)^a\right] \cdot c \cdot y^{c-1}\, dy$$
$$= c \int_0^1 y^{c-1}\, dy - \left(\frac{K}{z}\right)^a \cdot c \int_0^1 y^{c-1-a}\, dy$$
$$= c \cdot \left[\frac{y^c}{c}\right]_0^1 - \left(\frac{K}{z}\right)^a \cdot c \cdot \left[\frac{y^{c-a}}{c-a}\right]_0^1$$
$$= 1 - \left(\frac{K}{z}\right)^a \cdot \frac{c}{c-a}$$
$$= 1 - \left(\frac{K^*}{z}\right)^a, \quad K^* = \frac{K c^{\frac{1}{a}}}{(c-a)^{\frac{1}{a}}}$$

$\Rightarrow$ Z is a Pareto RV.

Assume Z is a Pareto RV with CDF $F_Z(z) = 1-\left(\frac{L}{z}\right)^b$, $z > L$. Then

$$F_X(x) = P(X < x)$$
$$= P\left(\frac{X}{Y} < \frac{x}{Y}\right)$$
$$= P\left(Z < \frac{x}{Y}\right)$$
$$= \int_0^1 P\left(Z < \frac{x}{y}\right) f_Y(y)\,dy$$

Total Prob Rule

$$= \int_0^1 F_Z\left(\frac{x}{y}\right) f_Y(y)\,dy$$
$$= \int_0^1 \left[1-\left(\frac{Ly}{x}\right)^b\right] \cdot c \cdot y^{c-1}\,dy$$
$$= c \cdot \int_0^1 y^{c-1}\,dy - \left(\frac{L}{x}\right)^b \cdot c \cdot \int_0^1 y^{b+c-1}\,dy$$
$$= c \cdot \left[\frac{y^c}{c}\right]_0^1 - \left(\frac{L}{x}\right)^b \cdot c \left[\frac{y^{b+c}}{b+c}\right]_0^1$$
$$= 1 - \left(\frac{L}{x}\right)^b \cdot \frac{c}{b+c} = 1-\left(\frac{L^*}{x}\right)^b,$$

where $L^* = \frac{L c^{\frac{1}{b}}}{(b+c)^{\frac{1}{b}}}$.

$\Rightarrow$ X is a Pareto RV.

Hence Thm 2 is proved.

LECTURE

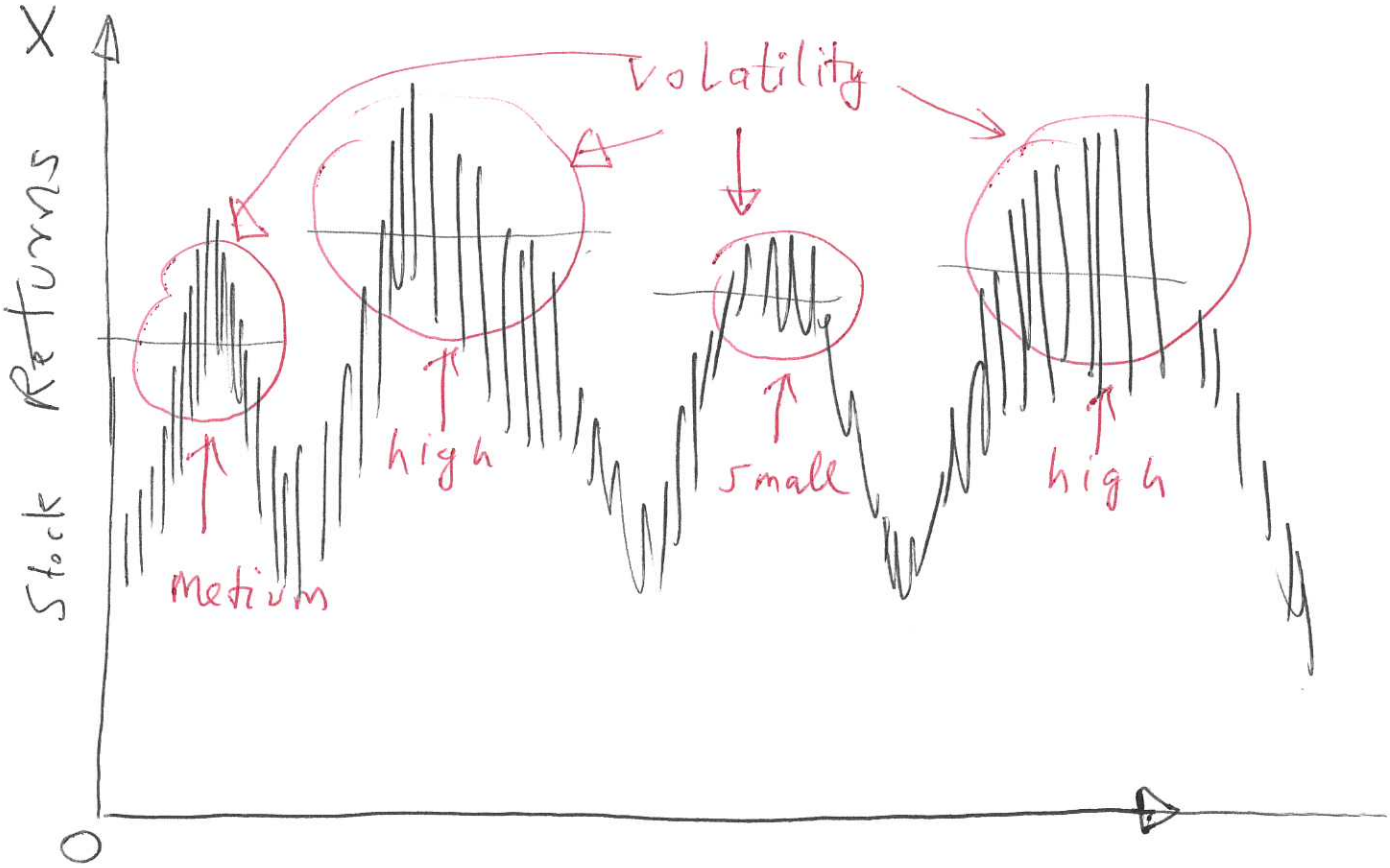
1 DECEMBER

9:00-10:00AM

MATH3/4/68181

Models for Stock Returns

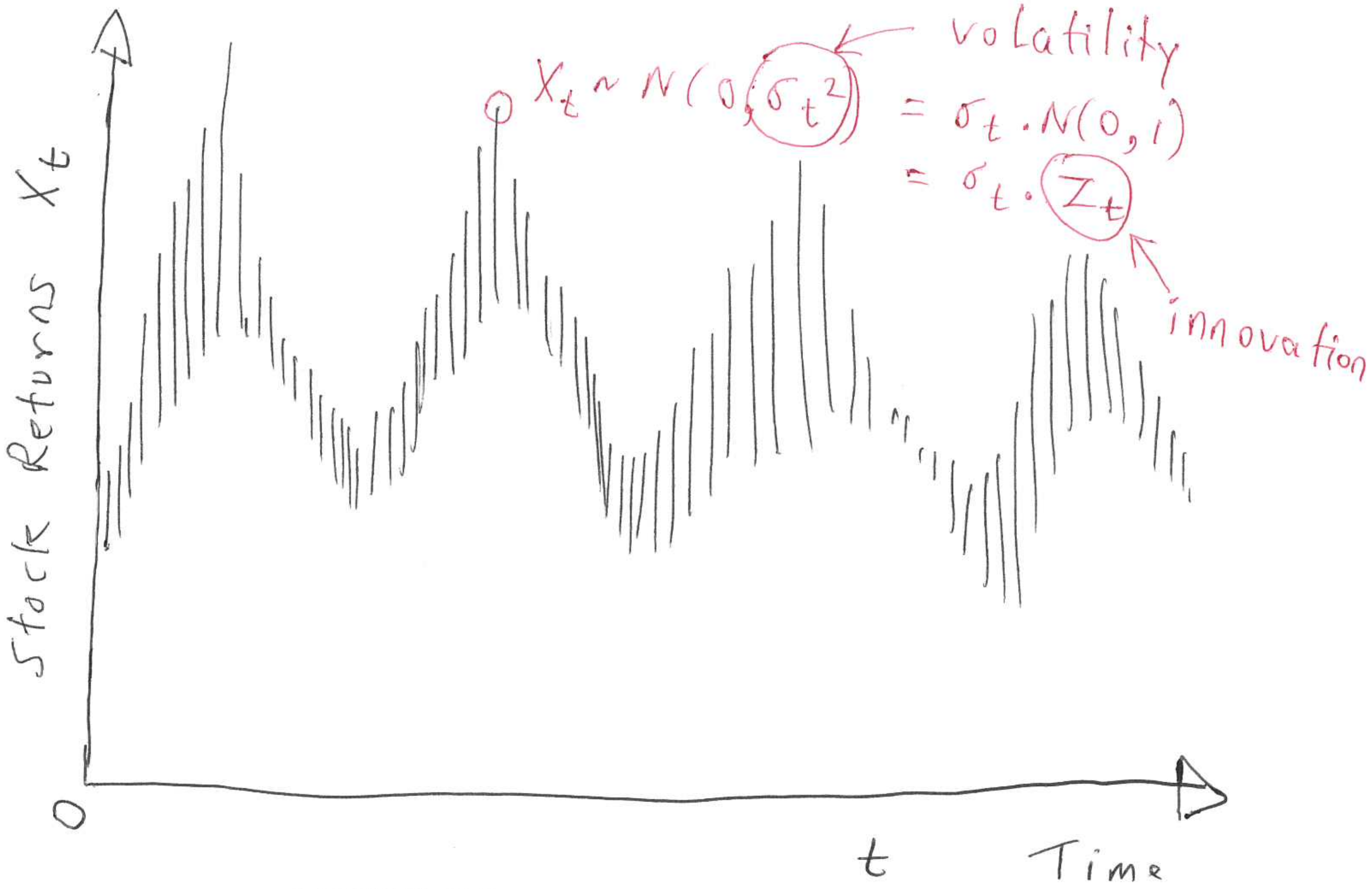
(covered last week)



$$X \mid V \sim N(0, \sigma^2)$$

$$V = \sigma^2 \sim f_V(\cdot)$$

Stock Models II



$$\boxed{X_t = \sigma_t \cdot Z_t}, \quad Z_t \sim N(0,1)$$

i) ARCH(q) model : $X_t = \sigma_t Z_t$

where $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2$,

$Z_t \sim N(0,1)$

$$\Rightarrow \quad Z_t = \frac{X_t}{\sigma_t} = \frac{X_t}{\sqrt{\alpha_0 + \alpha_1 X_{t-1}^2 + \dots + \alpha_q X_{t-q}^2}}$$

$$\sim N(0,1)$$

The likelihood is

$$L(\alpha_0, \alpha_1, \dots, \alpha_q) = \prod_{t=1}^{n} \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{Z_t^2}{2}} \right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} \sum_{t=1}^{n} Z_t^2}$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} \sum_{t=1}^{n} \frac{X_t^2}{\alpha_0 + \alpha_1 X_{t-1}^2 + \dots + \alpha_q X_{t-q}^2}}.$$

The log likelihood is

$$\log L = -\frac{n}{2} \log(2\pi) - \frac{1}{2} \sum_{t=1}^{n} \frac{X_t^2}{\alpha_0 + \alpha_1 X_{t-1}^2 + \dots + \alpha_q X_{t-q}^2}.$$

$$\frac{\partial \log L}{\partial \alpha_0} = \frac{1}{2}\sum_{t=1}^{n} \frac{X_t^2}{\left(\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2\right)^2} = 0$$

$$\frac{\partial \log L}{\partial \alpha_1} = \frac{1}{2}\sum_{t=1}^{n} \frac{X_t^2 X_{t-1}^2}{\left(\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2\right)^2} = 0$$

$$\vdots$$

$$\frac{\partial \log L}{\partial \alpha_q} = \frac{1}{2}\sum_{t=1}^{n} \frac{X_t^2 X_{t-q}^2}{\left(\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2\right)^2} = 0$$

The MLEs of $\alpha_0, \alpha_1, \ldots, \alpha_q$ are the simultaneous ~~equatio~~ solutions of these equations.

ii) GARCH (p, q) model:

$$X_t = \sigma_t Z_t, \quad Z_t \sim N(0, 1)$$

where
$$\sigma_t^2 = \alpha_0 + \underbrace{\alpha_1 X_{t-1}^2 + \dots + \alpha_q X_{t-q}^2}_{\text{previous } q \text{ stock returns}} + \underbrace{\beta_1 \sigma_{t-1}^2 + \dots + \beta_p \sigma_{t-p}^2}_{\text{previous } p \text{ volatility}}$$

$$\Rightarrow \quad Z_t = \frac{X_t}{\sigma_t} = \frac{X_t}{\sqrt{\alpha_0 + \alpha_1 X_{t-1}^2 + \dots + \alpha_q X_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \dots + \beta_p \sigma_{t-p}^2}}$$

The likelihood is

$$L(\alpha_0, \alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_p)$$

$$= \prod_{t=1}^{n} \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{Z_t^2}{2}} \right]$$

$$= (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2} \sum_{t=1}^{n} Z_t^2}$$

$$= (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2} \sum_{t=1}^{n} \frac{X_t^2}{\alpha_0 + \alpha_1 X_{t-1}^2 + \dots + \alpha_q X_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \dots + \beta_p \sigma_{t-p}^2}}$$

The log likelihood is

$$\log L = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n} \frac{X_t^2}{\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2}$$

The ML equations are:

$$\frac{\partial \log L}{\partial \alpha_0} = \frac{1}{2}\sum_{t=1}^{n} \frac{X_t^2}{(\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2)^2} = 0$$

$$\frac{\partial \log L}{\partial \alpha_1} = \frac{1}{2}\sum_{t=1}^{n} \frac{X_t^2 X_{t-1}^2}{(\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2)^2} = 0$$

$\vdots$

$$\frac{\partial \log L}{\partial \alpha_q} = \frac{1}{2}\sum_{t=1}^{n} \frac{X_t^2 X_{t-q}^2}{(\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2)^2} = 0$$

$$\frac{\partial \log L}{\partial \beta_1} = \frac{1}{2}\sum_{t=1}^{n} \frac{X_t^2 \sigma_{t-1}^2}{(\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2)^2}$$

$\vdots$ $= 0$

$$\frac{\partial \log L}{\partial \beta_p} = \frac{1}{2}\sum_{t=1}^{n} \frac{X_t^2 \sigma_{t-p}^2}{(\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2)^2}$$

$$= 0$$

The MLEs of $\alpha_0, \alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p$ are the simultaneous solutions of ~~theses~~ these equations.

iii) NGARCH model : $X_t = \sigma_t Z_t$

where $$\sigma_t^2 = \omega + \alpha(X_{t-1} - \theta\sigma_{t-1})^2 + \beta\sigma_{t-1}^2$$

$$Z_t \sim N(0,1)$$

quadratic dependence of σ_t^2 on σ_{t-1}

" " of σ_t^2 on X_{t-1}

iv) QGARCH model : $X_t = \sigma_t Z_t$

where $$\sigma_t^2 = K + \alpha X_{t-1}^2 + \beta\sigma_{t-1}^2 + \phi X_{t-1}$$

$$Z_t \sim N(0,1)$$

quadratic dependence of σ_t^2 on X_{t-1}

linear dependence of σ_t^2 on σ_{t-1}^2.

v) GJR - GARCH model: $X_t = \sigma_t Z_t$

Initials of the authors

where

$$\sigma_t^2 = K + \delta \sigma_{t-1}^2 + \alpha X_{t-1}^2 + \phi X_{t-1}^2 I_{t-1},$$

$$I_{t-1} = \begin{cases} 0 & \text{if } X_{t-1} \cancel{Z_{t-1}} \geq 0 \\ 1 & \text{if } X_{t-1} \cancel{Z_{t-1}} < 0 \end{cases}$$

$$Z_t \sim N(0,1)$$

σ_t^2 takes different forms depending on whether the returns are positive/negative.

EXAMPLE CLASS

1 DECEMBER

10:00-11:00AM

MATH3/4/68181

$\boxed{Q1}$

$$X_t = \sigma_t Z_t$$

where $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2$

$$Z_t \sim N(0, 1)$$

$$E(X_t) = E(\sigma_t Z_t) = E(\sigma_t) E(Z_t) = 0$$

$$E(X_t^2) = E(\sigma_t^2 Z_t^2) = E(\sigma_t^2) E(Z_t^2)$$
$$= E(\sigma_t^2)$$
$$= E(\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2)$$
$$= \alpha_0 + \alpha_1 E(X_{t-1}^2) + \cdots + \alpha_q E(X_{t-q}^2)$$
$$= \alpha_0 + \alpha_1 E(\sigma_{t-1}^2) + \cdots + \alpha_q E(\sigma_{t-q}^2).$$

If we assume stationarity, then

$$E(\sigma_t^2) = \alpha_0 + \alpha_1 E(\sigma_t^2) + \cdots + \alpha_q E(\sigma_t^2).$$

$$\Rightarrow (1 - \alpha_1 - \cdots - \alpha_q) E(\sigma_t^2) = \alpha_0$$

$$\Rightarrow E(\sigma_t^2) = \frac{\alpha_0}{1 - \alpha_1 - \cdots - \alpha_q}$$

$$\Rightarrow \boxed{Var(X_t) = \frac{\alpha_0}{1 - \alpha_1 - \cdots - \alpha_q}}$$

$\boxed{Q2}$

$$X_t = \sigma_t Z_t$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2$$

$$E(X_t) = E(\sigma_t Z_t) = E(\sigma_t) E(Z_t) = 0$$

$$E(X_t^2) = E(\sigma_t^2 Z_t^2) = E(\sigma_t^2) E(Z_t^2)$$

$$= E(\sigma_t^2)$$

$$= \alpha_0 + \alpha_1 E(X_{t-1}^2) + \cdots + \alpha_q E(X_{t-q}^2) + \beta_1 E(\sigma_{t-1}^2) + \cdots + \beta_p E(\sigma_{t-p}^2)$$

$$= \alpha_0 + \alpha_1 E(\sigma_{t-1}^2) + \cdots + \alpha_q E(\sigma_{t-q}^2) + \beta_1 E(\sigma_{t-1}^2) + \cdots + \beta_p E(\sigma_{t-p}^2).$$

Assuming stationarity of σ_t,

$$E(\sigma_t^2) = \alpha_0 + \alpha_1 E(\sigma_t^2) + \cdots + \alpha_q E(\sigma_t^2) + \beta_1 E(\sigma_t^2) + \cdots + \beta_p E(\sigma_t^2)$$

$$E(\sigma_t^2) = \frac{\alpha_0}{1 - \alpha_1 - \cdots - \alpha_q - \beta_1 - \cdots - \beta_p}.$$

$$= Var(X_t)$$

Q3

$$X_t = \sigma_t Z_t \quad , \quad Z_t \sim N(0,1)$$

$$\sigma_t^2 = \omega + \alpha(X_{t-1} - \theta\sigma_{t-1})^2 + \beta\sigma_{t-1}^2$$

$$E(X_t) = E(\sigma_t Z_t) = E(\sigma_t)E(Z_t) = 0$$

$$E(X_t^2) = E(\sigma_t^2 Z_t^2) = E(\sigma_t^2)$$

$$= E[\omega + \alpha(X_{t-1} - \theta\sigma_{t-1})^2 + \beta\sigma_{t-1}^2]$$

$$= \omega + \alpha E(X_{t-1}^2) - 2\alpha\theta E(X_{t-1}\sigma_{t-1}) + \alpha\theta^2 E(\sigma_{t-1}^2) + \beta E(\sigma_{t-1}^2)$$

$$= \omega + \alpha E(\sigma_{t-1}^2) - \underbrace{2\alpha\theta E(X_{t-1})E(\sigma_{t-1})}_{=0} + \alpha\theta^2 E(\sigma_{t-1}^2) + \beta E(\sigma_{t-1}^2)$$

Assuming stationarity of σ_t,

$$E(\sigma_t^2) = \omega + \alpha E(\sigma_t^2) + \alpha\theta^2 E(\sigma_t^2) + \beta E(\sigma_t^2).$$

$$\Rightarrow E(\sigma_t^2) = Var(X_t) = \frac{\omega}{1-\alpha-\alpha\theta^2-\beta}.$$

LECTURE

3 DECEMBER

12:00-13:00PM

MATH4/68181

Univariate extremal type theorem

extreme value

Suppose $X_1, X_2, \ldots$ are independent and identically distributed (iid) random variables with common cumulative distribution function (cdf) F. Let $\boxed{M_n = \max\{X_1, \ldots, X_n\}}$ denote the maximum of the first n random variables and let $w(F) = \sup\{x : F(x) < 1\}$ denote the upper end point of F. If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \to G(x)$$

as $n \to \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax + b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$I \;:\; \Lambda(x) = \exp\{-\exp(-x)\}, \qquad x \in \Re;$$

$$II \;:\; \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\{-x^{-\alpha}\} & \text{if } x \geq 0 \end{cases}$$

for some $\alpha > 0$;

$$III \;:\; \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases}$$

for some $\alpha > 0$.

BIVARIATE EXTREME VALUES

(X, Y)

eg X = oil price

Y = gold "

Suppose (X, Y) has the joint CDF F.
That is, $F(x, y) = P(X \leq x, Y \leq y)$

For later use, $F_X(x) = F(x, \infty) = P(X \leq x)$
$F_Y(y) = F(\infty, y) = P(Y \leq y)$.

Suppose $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ is a random sample on (X, Y).

How to define <u>bivariate</u> <u>exteme</u> <u>value</u>?

$$(L_n, M_n) = (\max(X_1, X_2, \ldots, X_n), \max(Y_1, Y_2, \ldots, Y_n)).$$

In parallel with the ETT, we derive

$$P\left(\frac{L_n - b_n}{a_n} \leqslant x, \ \frac{M_n - d_n}{c_n} \leqslant y\right)$$

$$= P(L_n \leqslant a_n x + b_n, \ M_n \leqslant c_n y + d_n)$$

$$= P(X_1 \leqslant a_n x + b_n, \ldots, X_n \leqslant a_n x + b_n, \ Y_1 \leqslant c_n y + d_n, \ldots, Y_n \leqslant c_n y + d_n)$$

$$= P(X_1 \leqslant a_n x + b_n, Y_1 \leqslant c_n y + d_n, \ldots X_n \leqslant a_n x + b_n, Y_n \leqslant c_n y + d_n)$$

$$\overset{\text{indep}}{=} P(X_1 \leqslant a_n x + b_n, Y_1 \leqslant c_n y + d_n) \cdots P(X_n \leqslant a_n x + b_n, Y_n \leqslant c_n y + d_n)$$

$$\overset{\text{identical}}{=} F(a_n x + b_n, c_n y + d_n) \cdots F(a_n x + b_n, c_n y + d_n)$$

$$= F^n(a_n x + b_n, c_n y + d_n)$$

If a non-degenerate G is obtained as the limit, what are the formulas for a_n, b_n, C_n & d_n?

~~Let~~ Recall $F_X(\cdot)$ & $F_Y(\cdot)$ are the ~~marginals~~ marginal CDFs of X & Y.

i) if F_X & F_Y belong to the Gumbel domain then

$$a_n = \gamma\left(F_X^{-1}\left(1-\tfrac{1}{n}\right)\right), \quad b_n = F_X^{-1}\left(1-\tfrac{1}{n}\right)$$

$$C_n = \gamma\left(F_Y^{-1}\left(1-\tfrac{1}{n}\right)\right), \quad d_n = F_Y^{-1}\left(1-\tfrac{1}{n}\right).$$

ii) if F_X & F_Y belong to the Frechet domain then

$$a_n = F_X^{-1}\left(1-\tfrac{1}{n}\right), \quad b_n = 0$$

$$C_n = F_Y^{-1}\left(1-\tfrac{1}{n}\right), \quad d_n = 0$$

iii) if F_X & F_Y belong to the Weibull domain then

$$a_n = w(F_X) - F_X^{-1}\left(1-\frac{1}{n}\right), \quad b_n = w(F_X)$$

$$c_n = w(F_Y) - F_Y^{-1}\left(1-\frac{1}{n}\right), \quad d_n = w(F_Y).$$

If there is a non-degenerate CDF G such that

$$F^{n}(a_n x + b_n, c_n y + d_n) \rightarrow G(x, y)$$

as $n \rightarrow \infty$ then the possible forms for G can be uncountably infinite.

i) Suppose F_X & F_Y belong to the Gumbel domain. In this case, G can be expressed as

$$G(x,y) = e^{-\int_0^1 \min\left[\frac{f_1(s)}{e^x}, \frac{f_2(s)}{e^y}\right] ds}$$

where f_1, f_2 are non-negative functions with $\int_0^1 f_1(t)\, dt = \int_0^1 f_2(t)\, dt = 1$.

ii) Suppose F_X & F_Y still belong to the Gumbel domain. In this case, an alternative expression for G is

$$G(x, y) = e^{-(e^{-x}+e^{-y})\,k(y-x)}$$

where $k(\cdot)$ satisfies

a) $\lim_{t \to +\infty} k(t) = \lim_{t \to -\infty} k(t) = 1$

b) $\frac{d}{dt}\left[(1+e^{-t})\,k(t)\right] \leq 0$

c) $\frac{d}{dt}\left[(1+e^{t})\,k(t)\right] \geq 0$

d) $(1+e^{-t})\,k''(t) + (1-e^{-t})\,k'(t) \geq 0$.

$F_X(x) = 1 - e^{-X}, \quad F_Y(y) = 1 - e^{-Y}$

iii) Suppose ~~F_X & F_Y belong to the Fréchet domain.~~ In this case, possible forms for G can be expressed as

$$\overline{G}(x, y) = e^{-(x+y) A\left(\frac{y}{x+y}\right)}$$

where $A(\circ)$ satisfies

a) $A(0) = A(1) = 1$

b) $\max(w, 1-w) \leq A(w) \leq 1, \ \forall \ w \in [0, 1]$

c) A is convex.

Note: $\cancel{G(x,y) = 1 - (1 - e^{-\frac{1}{x}}) - (1 - e^{-\frac{1}{y}}) + \overline{G}(x,y)}$

$\cancel{= e^{-\frac{1}{x}} + e^{-\frac{1}{y}} - 1 + \overline{G}(x,y)}$.

$$G(x, y) = 1 - e^{-X} - e^{-Y} + \overline{G}(x, y)$$

iv) Suppose F_X & F_Y belong to the Fréchet domain. In this case, possible forms for G can be expressed as

$$G(x, y) = e^{-\left(\frac{1}{x} + \frac{1}{y}\right) A\left(\frac{x}{x+y}\right)}$$

where $A(\cdot)$ satisfies

a) $A(0) = A(1) = 1$

b) $\max(w, 1-w) \leq A(w) \leq 1 \quad \forall w \in [0,1]$

c) $A(\cdot)$ is convex.

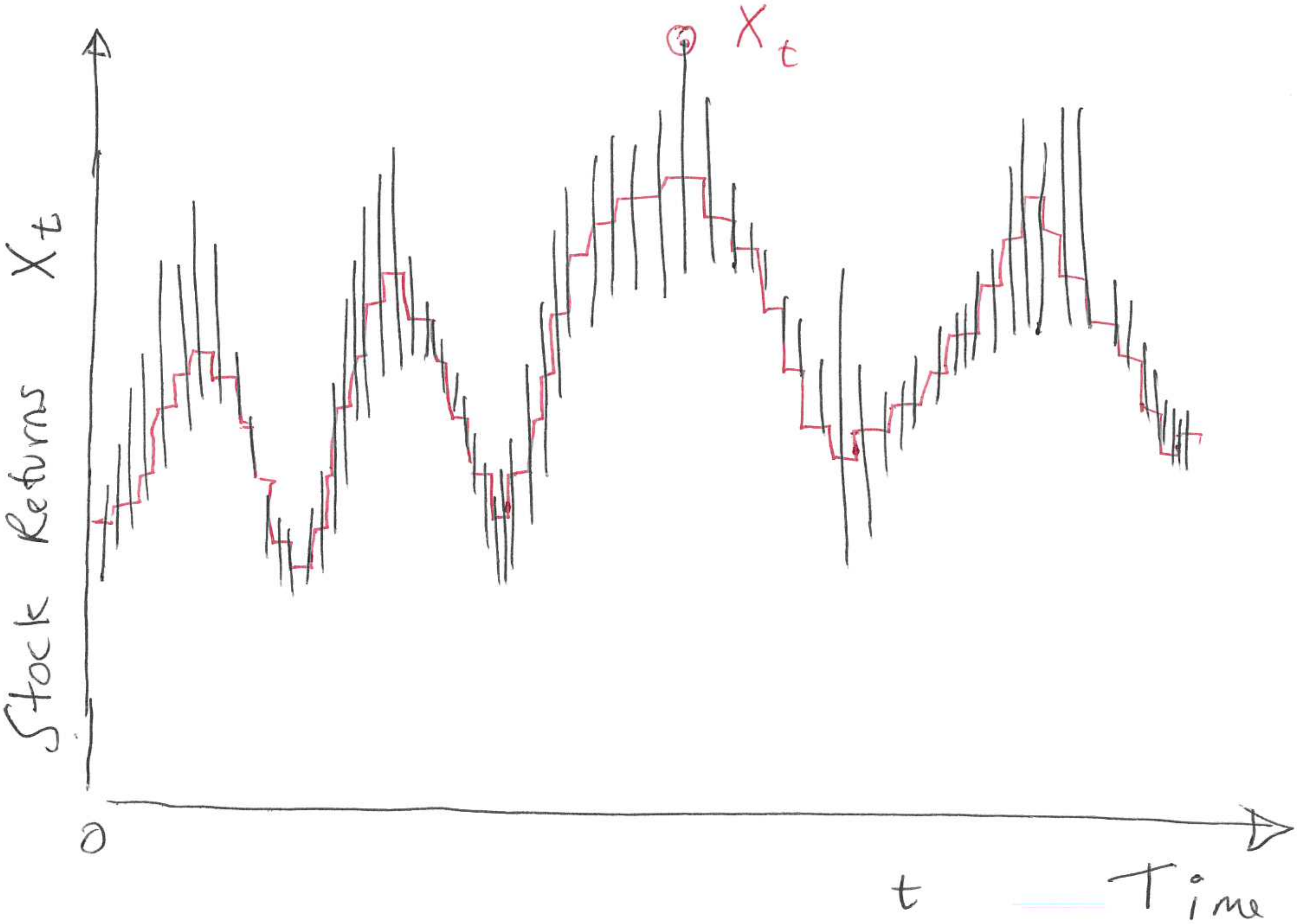
LECTURE

4 DECEMBER

9:00-10:00AM

MATH3/4/68181

Stock Models III



"Random Walk"

$$X_t - X_0 = \underbrace{(X_t - X_{t-1})} + \underbrace{(X_{t-1} - X_{t-2})} + \cdots + \underbrace{(X_1 - X_0)}$$

increments of the random walk

$$= Y_t + Y_{t-1} + \cdots + Y_1$$

$$\boxed{X_t - X_0 = \sum_{i=1}^{t} Y_i} \quad \text{---} \quad (1)$$

$$\frac{X_t}{X_0} = \frac{X_t}{X_{t-1}} \frac{X_{t-1}}{X_{t-2}} \cdots \frac{X_1}{X_0}$$

$$= Z_t \cdot Z_{t-1} \cdots Z_1$$

$$\boxed{\frac{X_t}{X_0} = \prod_{i=1}^{t} Z_i} \quad \text{— } (2)$$

(1) is an additive random walk

(2) is a multiplicative " "

We want to predict values of X_t.

$\Rightarrow$ We need to derive the distribution of X_t

eg Suppose $Y_i \sim N(\mu, \sigma^2)$ IID

$$\Rightarrow \sum_{i=1}^{t} Y_i \sim N(t\mu, t\sigma^2)$$

$$\Rightarrow X_t - X_0 \sim N(t\mu, t\sigma^2)$$

$$\Rightarrow X_t \sim N(X_0 + t\mu, t\sigma^2)$$

[assuming X_0 is known]

eg

Suppose $Z_i \sim LN(\mu, \sigma^2)$ IID

$$\Rightarrow \prod_{i=1}^{t} Z_i \sim LN(t\mu, t\sigma^2)$$

$$\Rightarrow \frac{X_t}{X_0} \sim LN(t\mu, t\sigma^2)$$

$$\Rightarrow \log \frac{X_t}{X_0} \sim N(t\mu, t\sigma^2)$$

$$\Rightarrow \log X_t \sim N(t\mu + \log X_0, t\sigma^2)$$

$$\Rightarrow \boxed{X_t \sim LN(t\mu + \log X_0, t\sigma^2)}$$

[assuming X_0 is known]

Normal & log-normal are not good models for stock data.

In the general case, prediction can be based on moments, i.e derive expressions for

$$E[X_t - X_0]$$
$$E[(X_t - X_0)^2]$$
$$E[(X_t - X_0)^3]$$
$$E[(X_t - X_0)^4]$$

and use them for prediction.

Suppose Y_i are IID.

$$E[X_t - X_0] = E\left[\sum_{i=1}^{t} Y_i\right]$$

$$= \sum_{i=1}^{t} E[Y_i] = t \cdot E(Y)$$

$$E[(X_t - X_0)^2] = E\left[\left(\sum_{i=1}^{t} Y_i\right)^2\right]$$

$$= \sum_{i=1}^{t} E(Y_i^2) + \sum_{i \neq j} E(Y_i Y_j)$$

$$= \sum_{i=1}^{t} E(Y_i^2) + \sum_{i \neq j} E(Y_i) E(Y_j)$$

$$= t \cdot E(Y^2) + t(t-1) (E(Y))^2$$

$$E[(X_t - X_0)^3] = E\left[\left(\sum_{i=1}^{t} Y_i\right)^3\right]$$

$$= \sum_{i=1}^{t} E(Y_i^3) + \sum_{i=j \neq k} E(Y_i Y_j Y_k)$$

$$+ \sum_{i=k \neq j} E(Y_i Y_j Y_k)$$

$$+ \sum_{i \neq j = k} E(Y_i Y_j Y_k) + \sum_{i \neq j \neq k} E(Y_i Y_j Y_k)$$

$$= t \cdot E(Y^3) + t(t-1) E(Y^2) E(Y)$$

$$+ t(t-1) E(Y^2) E(Y)$$

$$+ t(t-1) E(Y^2) E(Y) + t(t-1)(t-2) (E(Y))^3$$

$$= t E(Y^3) + 3t(t-1) E(Y^2) E(Y) + t(t-1)(t-2) (E(Y))^3$$

$$E\left((X_t - X_0)^4\right) = t\,E(Y^4) + 4t(t-1)E(Y^3)E(Y)$$

show this

$$+ 6t(t-1)(t-2)\,E(Y^2)\left(E(Y)\right)^2$$

$$+ 3t(t-1)\left(E(Y^2)\right)^2$$

$$+ t(t-1)(t-2)(t-3)\left(E(Y)\right)^4$$

These expressions can be applied to predict future moments for any t.

All one needs are estimates for $E(Y)$, $E(Y^2)$, $E(Y^3)$, $E(Y^4)$.

These can be based on past data.

Model 1 (2)

$$\frac{X_t}{X_0} = \prod_{i=1}^{t} Z_i$$

Suppose Z_i are IID.

$$E\left(\frac{X_t}{X_0}\right) = E\left[\prod_{i=1}^{t} Z_i\right] = E[Z^t]$$

$$E\left(\left(\frac{X_t}{X_0}\right)^2\right) = E\left(Z^{2t}\right)$$

$$E\left(\left(\frac{X_t}{X_0}\right)^3\right) = E\left(Z^{3t}\right)$$

$$E\left(\left(\frac{X_t}{X_0}\right)^4\right) = E\left(Z^{4t}\right)$$

Again these expressions can be used to predict moments for future t.

LECTURE

8 DECEMBER

9:00-10:00AM

MATH3/4/68181

REVISION FOR EXAM

Prob Sheet 7

X_i IID Exp (λ)

$X = \max(X_1, X_2, \ldots, X_\alpha)$

(i) Find the CDF of X.

$$F_X(x) = P(X < x)$$
$$= P(\max(X_1, \ldots, X_\alpha) < x)$$
$$= P(X_1 < x, \ldots, X_\alpha < x)$$
$$= P(X_1 < x) \ldots P(X_\alpha < x)$$
$$= (1 - e^{-\lambda x}) \ldots (1 - e^{-\lambda x})$$
$$= (1 - e^{-\lambda x})^\alpha$$

(ii) $f_X(x) = \alpha \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha - 1}$

(iii)

$$E(X^n) = \int_0^\infty x^n \cdot \alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}\,dx$$

$$= \alpha\lambda \int_0^\infty x^n e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}\,dx$$

$$\boxed{\begin{array}{l} \text{Set} \quad y = e^{-\lambda x} \Rightarrow x = -\dfrac{\log y}{\lambda} \\ \dfrac{dy}{dx} = -\lambda e^{-\lambda x} = -\lambda y \end{array}}$$

$$= \alpha\cancel{\lambda} \int_1^0 \left(-\frac{\log y}{\lambda}\right)^n \cdot \cancel{y}\,(1-y)^{\alpha-1}\left(-\frac{dy}{\cancel{\lambda}\cancel{y}}\right)$$

$$= \frac{\alpha}{\lambda^n}\int_0^1 (-\log y)^n (1-y)^{\alpha-1}\,dy$$

$$= \frac{\alpha}{\lambda^n}(-1)^n \int_0^1 (\log y)^n (1-y)^{\alpha-1}\,dy$$

$$\boxed{\begin{array}{l} \dfrac{dy^a}{da} = y^a \log y \\ \dfrac{d^n y^a}{da^n} = y^a (\log y)^n \end{array}}$$

$$= \frac{\alpha}{\lambda^n}(-1)^n \int_0^1 \frac{d^n}{da^n} y^a \Big|_{a=0} (1-y)^{\alpha-1}\,dy$$

$$= \frac{\alpha}{\lambda^n} (-1)^n \frac{d^n}{da^n} \boxed{\int_0^1 y^a (1-y)^{\alpha-1} \, dy} \Bigg|_{a=0}$$

$$= \frac{\alpha}{\lambda^n} (-1)^n \frac{d^n}{da^n} B(a+1, \alpha) \Bigg|_{a=0}$$

$$\Rightarrow \boxed{E(X^n) = \frac{\alpha (-1)^n}{\lambda^n} \frac{d^n}{da^n} B(a+1, \alpha) \Bigg|_{a=0}}$$

(iv) $E(X) = - \dfrac{\alpha}{\lambda} \dfrac{d}{da} B(a+1, \alpha) \Bigg|_{a=0}$

(v) $\mathrm{Var}(X) = E(X^2) - (E(X))^2$

$$= \frac{\alpha}{\lambda^2} \frac{d^2}{da^2} B(a+1, \alpha) \Bigg|_{a=0} - \frac{\alpha^2}{\lambda^2} \left[\frac{d}{da} B(a+1, \alpha) \Bigg|_{a=0} \right]^2$$

(vi)

$$\left(1-e^{-\lambda x}\right)^{\alpha} = p$$

$$\Rightarrow \quad 1-e^{-\lambda x} = p^{\frac{1}{\alpha}}$$

$$\Rightarrow \quad e^{-\lambda x} = 1-p^{\frac{1}{\alpha}}$$

$$\Rightarrow \quad x = -\frac{1}{\lambda}\log\left(1-p^{\frac{1}{\alpha}}\right)$$

$$\Rightarrow \quad VaR_p(X) = -\frac{1}{\lambda}\log\left(1-p^{\frac{1}{\alpha}}\right)$$

(v)

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{1}{p}\int_0^p -\frac{1}{\lambda}\log\left(1-t^{\frac{1}{\alpha}}\right)dt$$

$$= -\frac{1}{\lambda p}\int_0^p \log\left(1-t^{\frac{1}{\alpha}}\right)dt$$

$$= -\frac{1}{\lambda p}\int_0^p \sum_{i=1}^{\infty}\frac{\left(t^{\frac{1}{\alpha}}\right)^i}{i}\,dt$$

$$\boxed{\log(1-x) = -\sum_{i=1}^{\infty}\frac{x^i}{i}}$$

$$= +\frac{1}{\lambda p}\sum_{i=1}^{\infty}\frac{1}{i}\int_0^p t^{\frac{i}{\alpha}}\,dt = +\frac{1}{\lambda p}\sum_{i=1}^{\infty}\frac{p^{\frac{i}{\alpha}+1}}{i\left(\frac{i}{\alpha}+1\right)}.$$

(vi) $X_1, X_2, \ldots, X_n$ are IID on X.

$$L(\alpha, \lambda) = \prod_{i=1}^{n} \left[\alpha \lambda e^{-\lambda x_i} (1-e^{-\lambda x_i})^{\alpha-1}\right]$$

$$= (\alpha\lambda)^n e^{-\lambda \sum_{i=1}^{n} x_i} \left[\prod_{i=1}^{n} (1-e^{-\lambda x_i})\right]^{\alpha-1}$$

$$\log L = n \log(\alpha\lambda) - \lambda \sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n} \log(1-e^{-\lambda x_i})$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log(1-e^{-\lambda x_i}) = 0 \quad -(1)$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n} \frac{(-1)e^{-\lambda x_i}(-x_i)}{1-e^{-\lambda x_i}} = 0 \quad -(2)$$

$$(1) \Rightarrow \alpha = -\frac{n}{\sum_{i=1}^{n} \log(1-e^{-\lambda x_i})} \quad -(3)$$

Sub (3) into (2):

$$\frac{n}{\lambda} - \sum_{i=1}^{n} x_i - \left(\frac{n}{\sum_{i=1}^{n} \log(1-e^{-\lambda x_i})} + 1\right)\sum_{i=1}^{n} \frac{x_i e^{-\lambda x_i}}{1-e^{-\lambda x_i}} = 0 \quad -(4)$$

Solve eq (4) for α. Then sub into (3) and find the estimate for α.

Q14, Sheet 7

$$G(x) = 1 - \left[1 + \xi \frac{x - t}{\sigma}\right]^{-\frac{1}{\xi}} = p$$

$$\Rightarrow \left[1 + \xi \frac{x - t}{\sigma}\right]^{-\frac{1}{\xi}} = 1 - p$$

$$\Rightarrow 1 + \xi \frac{x - t}{\sigma} = (1-p)^{-\xi}$$

$$\Rightarrow \xi \frac{x - t}{\sigma} = (1-p)^{-\xi} - 1$$

$$\Rightarrow x = t + \frac{\sigma}{\xi}\left[(1-p)^{-\xi} - 1\right].$$

Prob sheet 1, Q13

$$G(x) = e^{-\left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} = p$$

$$\Rightarrow \quad -\left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} = \log p$$

$$\Rightarrow \quad \left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} = -\log p$$

$$\Rightarrow \quad 1+\xi \frac{x-\mu}{\sigma} = (-\log p)^{-\xi}$$

$$\Rightarrow \quad \xi \frac{x-\mu}{\sigma} = (-\log p)^{-\xi} - 1$$

$$\Rightarrow \quad x = \mu + \frac{\sigma}{\xi}\left[(-\log p)^{-\xi} - 1\right].$$

EXAMPLE CLASS

8 DECEMBER

10:00-11:00AM

MATH3/4/68181

(b) $$F(x) = 1 - \left[1 - e^{-\frac{\lambda}{x}}\right]^{\alpha}$$

$$F(x) = 1 \Rightarrow 1 - \left[1 - e^{-\frac{\lambda}{x}}\right]^{\alpha} = 1$$

$$\Rightarrow \left[1 - e^{-\frac{\lambda}{x}}\right]^{\alpha} = 0$$

$$\Rightarrow 1 - e^{-\frac{\lambda}{x}} = 0$$

$$\Rightarrow e^{-\frac{\lambda}{x}} = 1$$

$$\Rightarrow \frac{\lambda}{x} = 0 \Rightarrow x = +\infty$$

$$\Rightarrow w(F) = +\infty.$$

$$\frac{1 - F(tx)}{1 - F(t)} = \frac{\left[1 - e^{-\frac{\lambda}{tx}}\right]^{\alpha}}{\left[1 - e^{-\frac{\lambda}{t}}\right]^{\alpha}}$$

$$= \left[\frac{1 - e^{-\frac{\lambda}{tx}}}{1 - e^{-\frac{x}{t}}}\right]^{\alpha} = \left[\frac{1 - \left(1 - \frac{\lambda}{tx}\right)}{1 - \left(1 - \frac{\lambda}{t}\right)}\right]^{\alpha}$$

$$\boxed{e^{-y} \approx 1 - y}$$

$$= x^{-\alpha}$$

$\Rightarrow$ F belongs to Frechet domain.

(ii) $$F(x) = 1 - \frac{(1-p)^k e^{-k\beta x}}{(1-pe^{-\beta x})^k}$$

$$F(x) = 1 \Rightarrow \frac{(1-p)^k e^{-k\beta x}}{(1-pe^{-\beta x})^k} = 0$$

$$\Rightarrow \frac{(1-p)^k \cancel{e^{-k\beta x}}}{(e^{\beta x} - p)^k \cancel{e^{-k\beta x}}} = 0$$

$$\Rightarrow \left(\frac{1-p}{e^{\beta x} - p}\right)^k = 0 \Rightarrow \frac{1-p}{e^{\beta x} - p} = 0$$

$$\Rightarrow e^{\beta x} - p = +\infty \Rightarrow e^{\beta x} = +\infty$$

$$\Rightarrow x = +\infty \Rightarrow w(F) = +\infty.$$

$$\frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \frac{\dfrac{\cancel{(1-p)^k}\, e^{-k\beta(t + x\gamma(t))}}{\underbrace{\left[1 - p e^{-\beta(t + x\gamma(t))}\right]^k}_{\to 1}}}{\dfrac{\cancel{(1-p)^k}\, e^{-k\beta t}}{\underbrace{\left[1 - p e^{-\beta t}\right]^k}_{\to 1}}}$$

$$\to \quad e^{-k\beta x \gamma(t)} = e^{-x}$$

$$\text{if} \quad \gamma(t) = \frac{1}{k\beta}$$

Gumbel domain.

(ii) $$F(x) = \frac{\left(1-e^{-\beta x}\right)^{\alpha}}{1-p+p\left(1-e^{-\beta x}\right)^{\alpha}}$$

$$F(x) = 1$$

$$\Rightarrow \frac{\left(1-e^{-\beta x}\right)^{\alpha}}{1-p+p\left(1-e^{-\beta x}\right)^{\alpha}} = 1$$

$$\Rightarrow (1-p)\left(1-e^{-\beta x}\right)^{\alpha} = 1-p$$

$$\Rightarrow \left(1-e^{-\beta x}\right)^{\alpha} = 1$$

$$\Rightarrow 1-e^{-\beta x} = 1$$

$$\Rightarrow e^{-\beta x} = 0 \Rightarrow \beta x = +\infty \Rightarrow x = +\infty$$

$$\Rightarrow \omega(F) = +\infty.$$

$$\frac{1-F(t+x\gamma(t))}{1-F(t)} = \frac{1-\dfrac{\left(1-e^{-\beta(t+x\gamma(t))}\right)^{\alpha}}{1-p+p\left(1-e^{-\beta(t+x\gamma(t))}\right)^{\alpha}}}{1-\dfrac{\left(1-e^{-\beta t}\right)^{\alpha}}{1-p+p\left(1-e^{-\beta t}\right)^{\alpha}}}$$

$$\to \frac{1-\left(1-e^{-\beta(t+x\gamma(t))}\right)^{\alpha}}{1-\left(1-e^{-\beta t}\right)^{\alpha}}$$

$$\to \frac{\cancel{1}-(\cancel{1}-\cancel{\alpha}e^{-\beta(t+x\gamma(t))})}{\cancel{1}-(\cancel{1}-\cancel{\alpha}e^{-\beta t})} \qquad \boxed{(1-y)^{\alpha} \approx 1-\alpha y}$$

$$= e^{-\beta x\gamma(t)} = e^{-x} \text{ if } \gamma(t) = \frac{1}{\beta}.$$ Gumbel domain

(a) $F(x) = 1 - e^{1-(1+\lambda x)^{\alpha}}$

$$F(x) = 1$$

$$\Rightarrow \quad 1 - e^{1-(1+\lambda x)^{\alpha}} = 1$$

$$\Rightarrow \quad e^{1-(1+\lambda x)^{\alpha}} = 0$$

$$\Rightarrow \quad 1-(1+\lambda x)^{\alpha} = -\infty$$

$$\Rightarrow \quad (1+\lambda x)^{\alpha} = +\infty$$

$$\Rightarrow \quad 1+\lambda x = +\infty \Rightarrow x = +\infty$$

$$\Rightarrow \quad w(F) = +\infty$$

$$\frac{1-F(t+x\gamma(t))}{1-F(t)} = \frac{e^{1-(1+\lambda t+\lambda x\gamma(t))^{\alpha}}}{e^{1-(1+\lambda t)^{\alpha}}}$$

$$= e^{\left[(1+\lambda t)^{\alpha} - (1+\lambda t+\lambda x\gamma(t))^{\alpha}\right]}$$

$$= e^{(1+\lambda t)^{\alpha}\left[1-\left(1+\frac{\lambda x\gamma(t)}{1+\lambda t}\right)^{\alpha}\right]}$$

$$\boxed{(1+y)^{\alpha} \simeq 1+\alpha y}$$

$$= e^{(1+\lambda t)^\alpha \left[\cancel{1} - \left(\cancel{1} + \frac{\alpha \lambda x \gamma(t)}{1+\lambda t} \right) \right]}$$

$$= e^{-\alpha \lambda x \gamma(t) (1+\lambda t)^{\alpha - 1}}$$

$$\rightarrow e^{-x} \quad \text{as} \quad t \to \infty \quad \text{if} \quad \gamma(t) = \frac{1}{\alpha \lambda (1+\lambda t)^{\alpha-1}}$$

$\Rightarrow$ F belongs to the Gumbel domain.

Suppose $X_1, X_2, \ldots, X_n$ are IID from $F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^\alpha}$. Find the MLEs of θ & α.

The PDF is $f(x) = \frac{\alpha x^{\alpha-1}}{\theta^\alpha} e^{-\left(\frac{x}{\theta}\right)^\alpha}$.

So, the likelihood is

$$L(\theta, \alpha) = \prod_{i=1}^{n} \left[\frac{\alpha X_i^{\alpha-1}}{\theta^\alpha} e^{-\left(\frac{X_i}{\theta}\right)^\alpha} \right]$$

$$= \frac{\alpha^n}{\theta^{n\alpha}} \left(\prod_{i=1}^{n} X_i \right)^{\alpha-1} e^{-\theta^{-\alpha} \sum_{i=1}^{n} X_i^\alpha}$$

The log likelihood is

$$\log L = n \log \alpha - n\alpha \log \theta + (\alpha-1) \sum_{i=1}^{n} \log X_i - \theta^{-\alpha} \sum_{i=1}^{n} X_i^\alpha .$$

The partial derivatives are

$$\frac{\partial \log L}{\partial \theta} = -\frac{n\alpha}{\theta} + \alpha \theta^{-\alpha-1} \sum_{i=1}^{n} X_i^\alpha = 0 \quad \text{— (1)}$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - n \log \theta + \sum_{i=1}^{n} \log X_i - \sum_{i=1}^{n} \left(\frac{X_i}{\theta}\right)^\alpha \log \left(\frac{X_i}{\theta}\right) = 0 \quad \text{— (2)}$$

$$(1) \Rightarrow \quad n\,\theta^{\alpha} = \sum_{i=1}^{n} x_i^{\alpha}$$

$$\Rightarrow \quad \theta = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha}\right)^{\frac{1}{\alpha}} \quad - \quad (3)$$

Sub (3) into (2):

$$\frac{n}{\alpha} - \frac{n}{\alpha}\log\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha}\right) + \sum_{i=1}^{n}\log x_i$$

$$-n\sum_{i=1}^{n}\frac{x_i^{\alpha}}{\sum_{j=1}^{n} x_j^{\alpha}}\log\left(\frac{n^{\frac{1}{\alpha}}\,x_i}{\left(\sum_{j=1}^{n} x_j^{\alpha}\right)^{\frac{1}{\alpha}}}\right) = 0 \quad - (4)$$

Solve (4) for α to get its MLE.
Sub into (3) to find the MLE of θ.

LECTURE

10 DECEMBER

12:00-13:00PM

MATH4/68181

eg 1

$$F(x, y) = 1 - [1+x]^{-c} - [1+y]^{-c} + [1+x+y]^{-c}.$$

If $F^n(a_n x + b_n, c_n y + d_n) \to G(x, y)$

what is G?

$$F_X(x) = F(x, \infty) = 1 - [1+x]^{-c}$$

$$F_Y(y) = F(\infty, y) = 1 - [1+y]^{-c}$$

$F_X(x)$ belongs to the Fréchet domain because

$$\lim_{t\uparrow\infty} \frac{1 - F_X(tx)}{1 - F_X(t)} = \lim_{t\uparrow\infty} \frac{\cancel{1} - (\cancel{1} - [1+tx]^{-c})}{\cancel{1} - (\cancel{1} - [1+t]^{-c})}$$

$$= \lim_{t\uparrow\infty} \left(\frac{1+tx}{1+t}\right)^{-c} = x^{-c}.$$

Similarly, $F_Y(y)$ also belongs to the Fréchet domain because

$$\lim_{t\uparrow\infty} \frac{1 - F_Y(ty)}{1 - F_Y(t)} = \lim \frac{\cancel{1} - (\cancel{1} - [1+ty]^{-c})}{\cancel{1} - (\cancel{1} - [1+t]^{-c})}$$

$$= \lim_{t\uparrow\infty} \left(\frac{1+ty}{1+t}\right)^{-c} = y^{-c}.$$

$$F_X(x) = 1-[1+x]^{-c} \Rightarrow F_X^{-1}(x) = (1-x)^{-\frac{1}{c}} - 1$$

$$F_X^{-1}\left(1-\frac{1}{n}\right) = \left(1-\left(1-\frac{1}{n}\right)\right)^{-\frac{1}{c}} - 1 = \boxed{n^{\frac{1}{c}} - 1}$$

$$F_Y(y) = 1-[1+y]^{-c} \Rightarrow F_Y^{-1}(y) = (1-y)^{-\frac{1}{c}} - 1$$

$$F_Y^{-1}\left(1-\frac{1}{n}\right) = \left(1-\left(1-\frac{1}{n}\right)\right)^{-\frac{1}{c}} - 1 = \boxed{n^{\frac{1}{c}} - 1}$$

$$\Rightarrow \boxed{a_n = n^{\frac{1}{c}} - 1, \quad b_n = 0, \quad c_n = n^{\frac{1}{c}} - 1, \quad d_n = 0}$$

$$\lim_{n\to\infty} F^n(a_n X + b_n, c_n Y + d_n)$$

$$= \lim F^n\left(\left(n^{\frac{1}{c}}-1\right)X, \left(n^{\frac{1}{c}}-1\right)Y\right)$$

$$\left(n^{\frac{1}{c}}\right)^{-c} = n^{-1}$$

$$= \lim_{n\to\infty} \Big\{1 - \left[1+\left(n^{\frac{1}{c}}-1\right)X\right]^{-c} - \left[1+\left(n^{\frac{1}{c}}-1\right)Y\right]^{-c} + \left[1+\left(n^{\frac{1}{c}}-1\right)X + \left(n^{\frac{1}{c}}-1\right)Y\right]^{-c}\Big\}^n$$

$$= \lim_{n\to\infty} \Big\{1 - n^{-1}\left[X+(1-X)\overbrace{n^{-\frac{1}{c}}}^{0}\right]^{-c} - n^{-1}\left[Y+(1-Y)\overbrace{n^{-\frac{1}{c}}}^{0}\right]^{-c} + n^{-1}\left[X+Y+(1-X-Y)\overbrace{n^{-\frac{1}{c}}}^{0}\right]^{-c}\Big\}^n$$

$$= \lim_{n\to\infty} \left\{1 - n^{-1}X^{-c} - n^{-1}Y^{-c} + n^{-1}(X+Y)^{-c}\right\}^n$$

$$= \lim_{n \to \infty} \left\{ 1 + \frac{(x+y)^{-c} - x^{-c} - y^{-c}}{n} \right\}^n$$

$$= \boxed{e^{(x+y)^{-c} - x^{-c} - y^{-c}}} \qquad \boxed{\left(1 + \frac{z}{n}\right)^n \to e^z \text{ as } n \to \infty}$$

$$= G(x, y)$$

is a bivariate extreme value distribution.

eg 2 $\quad F(x,y) = 1 - e^{-x} - e^{-y} + e^{-(x^r + y^r)^{\frac{1}{r}}}$.

If $\quad F^n(a_n x + b_n, c_n y + d_n) \to G(x,y)$

as $\quad n \to \infty \quad$ what is G?

$$F_X(x) = F(x,\infty) = 1 - e^{-x}$$

$$F_Y(y) = F(\infty, y) = 1 - e^{-y}$$

F_X belongs to the Gumbel domain because

$$\lim_{t \uparrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \lim_{t \uparrow \infty} \frac{1 - (1 - e^{-t - x\gamma(t)})}{1 - (1 - e^{-t})}$$

$$= \lim_{t \uparrow \infty} e^{-x\gamma(t)} = e^{-x}$$

if $\gamma(t) = 1$.

Similarly, F_Y also belongs to the Gumbel domain with $\gamma(t) = 1$.

$F_X(x) = 1 - e^{-x}$, $\quad F_X^{-1}(x) = -\log(1-x)$

$a_n = \gamma(F_X^{-1}(1 - \frac{1}{n})) = 1$, $\quad b_n = -\log(1 - (1 - \frac{1}{n})) = \log n$

$F_Y(y) = 1 - e^{-y}$, $\quad F_Y^{-1}(y) = -\log(1-y)$

$c_n = \gamma(F_Y^{-1}(1 - \frac{1}{n})) = 1$, $\quad d_n = \log n$.

$$\lim_{n\to\infty} F^n(a_n x + b_n,\ c_n y + d_n)$$

$$= \lim_{n\to\infty} F^n(x + \log n,\ y + \log n)$$

$$= \lim \left[1 - e^{-x-\log n} - e^{-y-\log n} + e^{-((x+\log n)^r + (y+\log n)^r)^{\frac{1}{r}}}\right]^n$$

$$= \lim_{n\to\infty}\left[1 - \frac{e^{-x}}{n} - \frac{e^{-y}}{n} + e^{-(\log n)\left(\left(\frac{x}{\log n}+1\right)^r + \left(\frac{y}{\log n}+1\right)^r\right)^{\frac{1}{r}}}\right]^n$$

$$= \lim_{n\to\infty}\left[1 - \frac{e^{-x}+e^{-y}}{n} + n^{-\left(\left(\underbrace{\frac{x}{\log n}}_{\to 0}+1\right)^r + \left(\underbrace{\frac{y}{\log n}}_{\to 0}+1\right)^r\right)^{\frac{1}{r}}}\right]^n$$

$$\boxed{e^{-(\log n)a} = n^{-a}}$$

$$= \lim_{n\to\infty}\left[1 - \frac{e^{-x}+e^{-y}}{n} + n^{-2^{\frac{1}{r}}}\right]^n$$

$$= \lim_{n\to\infty}\left[1 + \frac{n^{1-2^{\frac{1}{r}}} - e^{-x} - e^{-y}}{n}\right]^n$$

$$= \lim_{n\to\infty} e^{\underbrace{n^{1-2^{\frac{1}{r}}}}_{\to 0} - e^{-x} - e^{-y}}, \quad r>0$$

$$= e^{-e^{-x}-e^{-y}} = G(x,y)$$

LECTURE

11 DECEMBER

9:00-10:00AM

MATH3/4/68181

REVISION FOR EXAM

Exam date: Tues 19 Jan 2016
2 pm

$$VaR_p(X) = F^{-1}(p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right), \quad \Phi \text{ CDF of } N(0,1)$$

$$= p$$

$$\Rightarrow \frac{x-\mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow x = \mu + \sigma\Phi^{-1}(p)$$

$$\Rightarrow \boxed{VaR_p(X) = \mu + \sigma\Phi^{-1}(p)}$$

$$ES_p(X) = \frac{1}{p}\cdot\int_0^p \left[\mu + \sigma\Phi^{-1}(t)\right]dt$$

$$= \boxed{\mu + \frac{\sigma}{p}\int_0^p \Phi^{-1}(t)\,dt}$$

(a) $$L(\mu, \sigma) = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(X_i-\mu)^2}{2\sigma^2}} \right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i-\mu)^2}$$

$$\log L = -\frac{n}{2} \log(2\pi) - n \log \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i-\mu)^2$$

(b) $$\frac{\partial \log L}{\partial \mu} = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} 2\cdot(X_i-\mu)(-1) = 0$$

$$\Rightarrow \sum_{i=1}^{n} (X_i-\mu) = 0$$

$$\Rightarrow \left(\sum_{i=1}^{n} X_i\right) - n\mu = 0$$

$$\Rightarrow \hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}$$

(c) $$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{\cancel{2}}{\cancel{2}\sigma^3} \sum_{i=1}^{n} (X_i-\mu)^2 = 0$$

$$\Rightarrow n\sigma^2 = \sum_{i=1}^{n} (X_i-\mu)^2$$

$$\Rightarrow \widehat{\sigma^2} = \frac{1}{n} \cdot \sum_{i=1}^{n} (X_i-\bar{X})^2$$

$$\Rightarrow \hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i-\bar{X})^2}$$

(d)

$$\widehat{VaR}_p(x) = \hat{\mu} + \hat{\sigma}\,\Phi^{-1}(p)$$

$$= \bar{X} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}\;\Phi^{-1}(p)$$

$$\widehat{ES}_p(x) = \hat{\mu} + \frac{\hat{\sigma}}{p}\int_0^p \Phi^{-1}(t)\,dt$$

$$= \bar{X} + \frac{1}{p}\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}\int_0^p \Phi^{-1}(t)\,dt$$

If $\hat{\theta}$ is an estimator of θ

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

We say $\hat{\theta}$ is biased if $\text{Bias}(\hat{\theta}) \neq 0$

We say $\hat{\theta}$ is unbiased if $\text{Bias}(\hat{\theta}) = 0$

Suppose $X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2)$ IID

Then

$$\sum_{i=1}^{n}(X_i - \bar{X})^2 \sim \sigma^2 \chi^2_{n-1}$$

(e) $E\left[\widehat{VaR}_p(X)\right]$

$$= E\left[\bar{X} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2}\;\Phi^{-1}(p)\right]$$

$$= E(\bar{X}) + E\left[\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2}\right]\Phi^{-1}(p)$$

$$= \mu + E\left[\sqrt{\frac{\sigma^2}{n}\chi^2_{n-1}}\right]\Phi^{-1}(p)$$

$$= \mu + \frac{\sigma}{\sqrt{n}}\,E\left[\sqrt{\chi^2_{n-1}}\right]\Phi^{-1}(p)$$

work this out

$$\neq \mu + \sigma\,\Phi^{-1}(p)$$

$\Rightarrow$ $\widehat{VaR}_p(X)$ is biased.

(f) $E\left[\widehat{ES}_p(X)\right]$

$$= E\left[\bar{X} + \frac{1}{p}\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2}\int_0^p \Phi^{-1}(t)\,dt\right]$$

$$= E[\bar{X}] + \frac{1}{p}E\left[\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2}\right]\int_0^p \Phi^{-1}(t)\,dt$$

$$= \mu + \frac{\sigma}{p\sqrt{n}}\cdot E\left[\sqrt{\chi^2_{n-1}}\right]\cdot\int_0^p \Phi^{-1}(t)\,dt$$

work this out

$$\neq \mu + \frac{\sigma}{p}\int_0^p \Phi^{-1}(t)\,dt$$

$\Rightarrow$ $\widehat{ES}_p(X)$ is biased.

$$E\left[\sqrt{\chi^2_{n-1}}\right]$$

$$= \int_0^\infty \sqrt{x} \cdot \underbrace{\frac{1}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)} x^{\frac{n-1}{2}-1} e^{-\frac{x}{2}}}_{\text{PDF of } \chi^2_{n-1}} dx$$

$$= \frac{1}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty x^{\frac{n}{2}-1} e^{-\frac{x}{2}} dx$$

$$\boxed{y = \frac{x}{2} \Rightarrow x = 2y \Rightarrow dx = 2\,dy}$$

$$= \frac{1}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)} \cdot \int_0^\infty (2y)^{\frac{n}{2}-1} e^{-y} \cdot 2\,dy$$

$$= \frac{\sqrt{2}}{\Gamma\left(\frac{n-1}{2}\right)} \boxed{\int_0^\infty y^{\frac{n}{2}-1} e^{-y}\,dy}$$

$$= \frac{\sqrt{2}\;\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}.$$

$X \sim \text{Uni}[a, b]$

$$F_X(x) = \frac{x-a}{b-a} = p$$

$$\Rightarrow \quad x = a + (b-a)p$$

$$\Rightarrow \quad \boxed{VaR_p(X) = a + (b-a)p}$$

$$ES_p(X) = \frac{1}{p}\int_0^p \left[a + (b-a)t\right] dt$$

$$= \frac{1}{p}\left[at + \frac{(b-a)t^2}{2}\right]_0^p$$

$$= \boxed{a + \frac{(b-a)p}{2}}$$

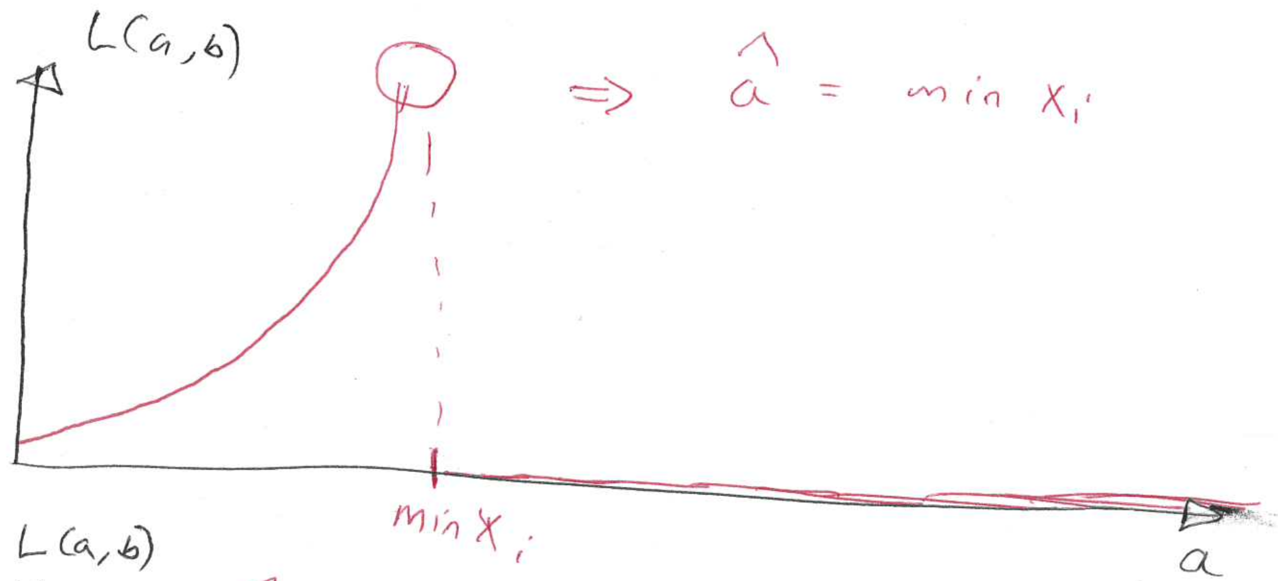
$X_1, X_2, \ldots, X_n$ IID $\mathrm{Uni}[a, b]$

(i) $$L(a, b) = \prod_{i=1}^{n} \left[\frac{1}{b-a} \cdot I\{a \le X_i \le b\} \right]$$

$$= \frac{1}{(b-a)^n} \left[\prod_{i=1}^{n} I\{a \le X_i \le b\} \right]$$

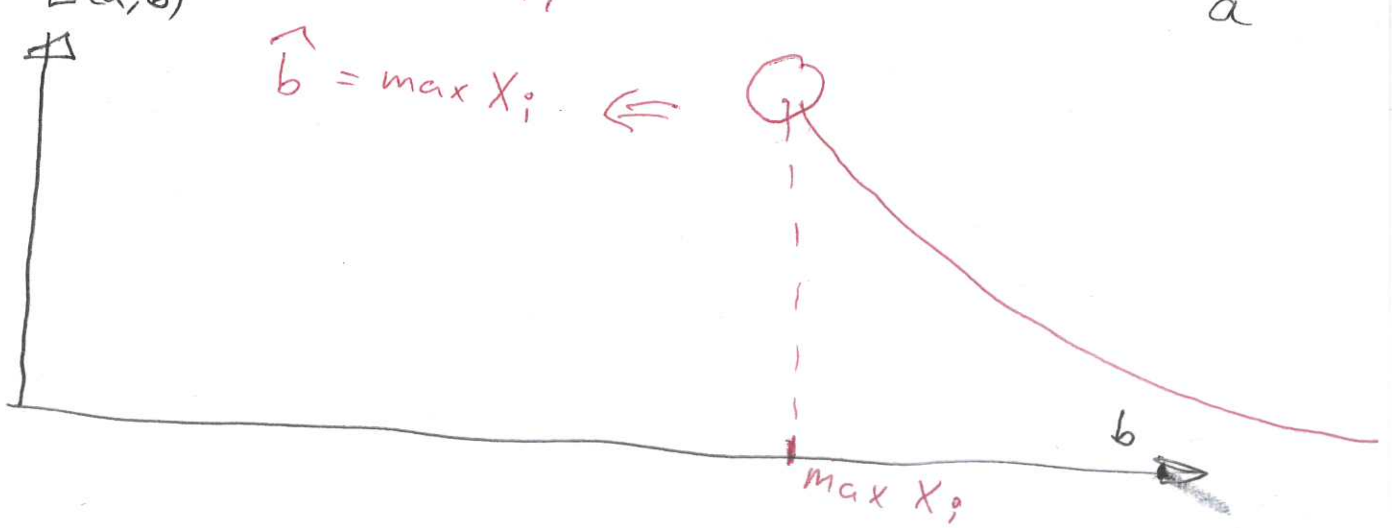
$$= \boxed{\frac{1}{(b-a)^n} \cdot I\{\min X_i \ge a \ \& \ \max X_i \le b\}}$$

(ii)



$\Rightarrow \hat{a} = \min X_i$

(iii)



$\hat{b} = \max X_i \Leftarrow$

LECTURE

15 DECEMBER

9:00-10:00AM

MATH3/4/68181

REVISION FOR EXAM

Q6, 2014 / 2015 Exam

(a) $F_Y(y) = P(Y < y)$

$= P(\min(X_1, \ldots, X_\alpha) < y)$

$= 1 - P(\min(X_1, \ldots, X_\alpha) \geq y)$

$= 1 - P(X_1 \geq y, \ldots, X_\alpha \geq y)$

$\overset{\text{indep}}{=} 1 - P(X_1 \geq y) \ldots P(X_\alpha \geq y)$

$= 1 - [1 - F(y)] \ldots [1 - F(y)]$

$= 1 - [1 - F(y)]^\alpha$

$= 1 - \left[1 - \left(1 - \left(\frac{k}{y}\right)^a\right)\right]^\alpha$

$\boxed{= 1 - \left(\frac{k}{y}\right)^{a\alpha}}$

(b) $$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= a\alpha\, k^{a\alpha}\, y^{-a\alpha-1}, \quad y \geq k$$

(c)
$$E(Y^n) = \int_k^\infty y^n \cdot a\alpha\, k^{a\alpha}\, y^{-a\alpha-1}\, dy$$

$$= a\alpha\, k^{a\alpha} \int_k^\infty y^{n-a\alpha-1}\, dy$$

$$= a\alpha\, k^{a\alpha} \left[\frac{y^{n-a\alpha}}{n-a\alpha} \right]_k^\infty$$

$$= a\alpha\, k^{a\alpha} \left[0 - \frac{k^{n-a\alpha}}{n-a\alpha} \right] \quad \underline{\text{if } n < a\alpha}$$

$$= \frac{a\alpha\, k^n}{a\alpha - n} \quad \text{if } n < a\alpha$$

$$E(Y) = \frac{a\alpha k}{a\alpha - 1} \quad \text{if } 1 < a\alpha$$

$$Var(Y) = E(Y^2) - (E(Y))^2$$

$$= \frac{a\alpha k^2}{a\alpha - 2} - \frac{(a\alpha)^2 k^2}{(a\alpha-1)^2} \quad \text{if } 2 < a\alpha.$$

(d) $F_Y(y) = 1 - \left(\frac{k}{y}\right)^{a\alpha} = p$

$$\Rightarrow \left(\frac{k}{y}\right)^{a\alpha} = 1 - p$$

$$\Rightarrow \frac{k}{y} = (1-p)^{\frac{1}{a\alpha}}$$

$$\Rightarrow y = k(1-p)^{-\frac{1}{a\alpha}}$$

$$\Rightarrow VaR_p(Y) = k(1-p)^{-\frac{1}{a\alpha}}$$

(e) $ES_p(Y) = \frac{1}{p}\int_0^p VaR_t(Y)\,dt$

$$= \frac{1}{p}\cdot\int_0^p k(1-t)^{-\frac{1}{a\alpha}}\,dt$$

$$= \frac{k}{p}\left[\frac{(1-t)^{1-\frac{1}{a\alpha}}}{(-1)\left(1-\frac{1}{a\alpha}\right)}\right]_0^p$$

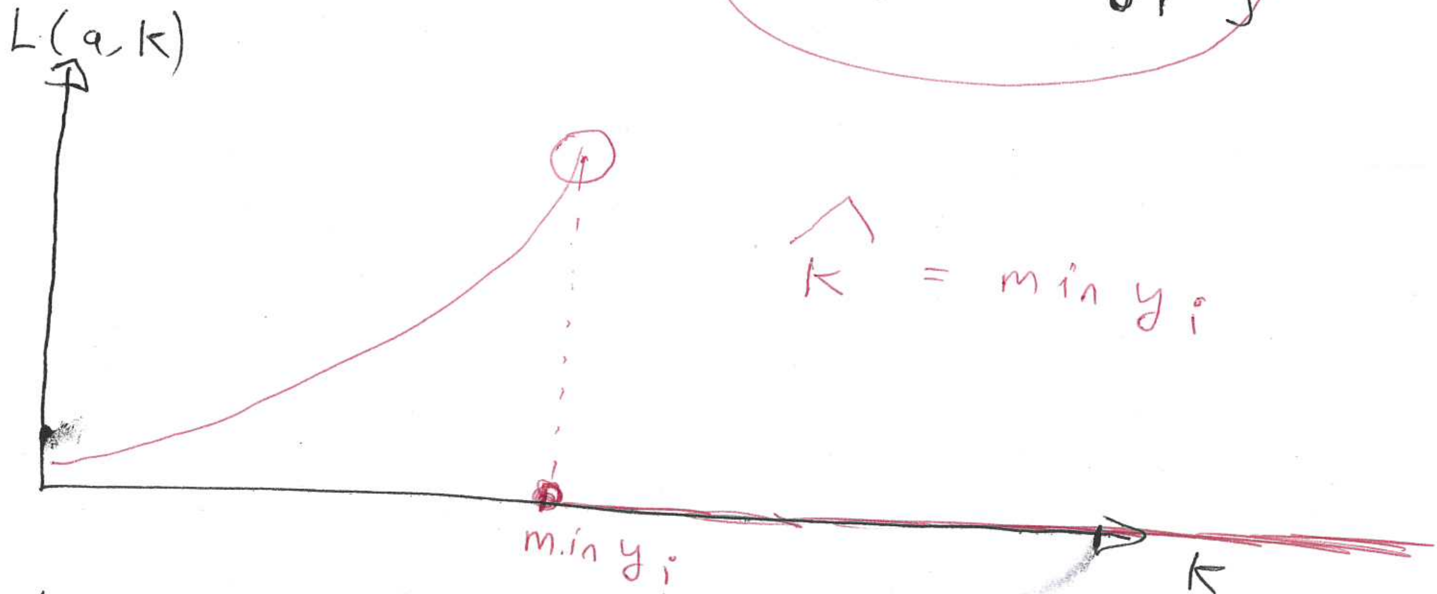
$$= \frac{k}{p}\cdot\frac{a\alpha}{1-a\alpha}\left[(1-p)^{1-\frac{1}{a\alpha}} - 1\right].$$

(f)

$$L(a, k) = \prod_{i=1}^{n} \left\{ \left[a\alpha\, k^{a\alpha}\, y_i^{-a\alpha-1} \right] \cdot I\{ y_i \geq k \} \right\}$$

$$= (a\alpha)^n k^{na\alpha} \left(\prod_{i=1}^{n} y_i \right)^{-a\alpha-1} \left(\prod_{i=1}^{n} I\{ y_i \geq k \} \right)$$

$$= (a\alpha)^n k^{na\alpha} \left(\prod_{i=1}^{n} y_i \right)^{-a\alpha-1} I\{ k \leq \min y_i \}$$



$$\hat{k} = \min y_i$$

$$\log L = n \log(a\alpha) + na\alpha \log k - (a\alpha+1) \sum_{i=1}^{n} \log y_i + \log I\{ k \leq \min y_i \}$$

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} + n\alpha \log k - \alpha \sum_{i=1}^{n} \log y_i = 0$$

$$\Rightarrow \hat{a} = \frac{n}{\alpha \sum_{i=1}^{n} \log y_i - n\alpha \log \hat{k}}$$

Q5, 2014/2015 Exam

(a)
$$VaR_p(X) = F^{-1}(p)$$
$$ES_p(X) = \frac{1}{p}\int_0^p F^{-1}(t)\,dt$$

(b)
$$f(x) = \frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}}, \quad -\infty < x < \infty$$

(i)
$$F(x) = \frac{1}{2\lambda}\int_{-\infty}^{x} e^{-\frac{|y|}{\lambda}}\,dy$$

$\boxed{\text{Suppose } x < 0}$:
$$= \frac{1}{2\lambda}\int_{-\infty}^{x} e^{-\frac{(-y)}{\lambda}}\,dy$$
$$= \frac{1}{2\lambda}\left[\frac{e^{\frac{y}{\lambda}}}{\frac{1}{\lambda}}\right]_{-\infty}^{x}$$
$$= \frac{1}{2\lambda}\cdot\lambda\cdot\left(e^{\frac{x}{\lambda}} - 0\right) = \frac{1}{2}e^{\frac{x}{\lambda}}$$

$\boxed{\text{Suppose } x \geq 0}$:
$$= \frac{1}{2\lambda}\left[\int_{-\infty}^{0} e^{-\frac{|y|}{\lambda}}\,dy + \int_0^x e^{-\frac{|y|}{\lambda}}\,dy\right]$$
$$= \frac{1}{2\lambda}\left[\int_{-\infty}^{0} e^{\frac{y}{\lambda}}\,dy + \int_0^x e^{-\frac{y}{\lambda}}\,dy\right]$$
$$= \frac{1}{2\lambda}\left\{\left[\frac{e^{\frac{y}{\lambda}}}{\frac{1}{\lambda}}\right]_{-\infty}^{0} + \left[\frac{e^{-\frac{y}{\lambda}}}{\left(-\frac{1}{\lambda}\right)}\right]_0^x\right\}$$
$$= 1 - \frac{1}{2}e^{-\frac{x}{\lambda}}$$

(i)

$$F(x) = \begin{cases} \frac{1}{2} e^{\frac{x}{\lambda}} & x < 0 \\ 1 - \frac{1}{2} e^{-\frac{x}{\lambda}}, & x \geq 0 \end{cases}$$

$$F(0) = \frac{1}{2}$$

(ii)

$$\frac{1}{2} e^{\frac{x}{\lambda}} = p$$

$$\Rightarrow e^{\frac{x}{\lambda}} = 2p$$

$$\Rightarrow \frac{x}{\lambda} = \log(2p) \Rightarrow \boxed{x = \lambda \log(2p)}$$

$$1 - \frac{1}{2} e^{-\frac{x}{\lambda}} = p$$

$$\Rightarrow \frac{1}{2} e^{-\frac{x}{\lambda}} = 1 - p$$

$$\Rightarrow e^{-\frac{x}{\lambda}} = 2(1-p)$$

$$\Rightarrow x = \boxed{-\lambda \log(2(1-p))}$$

$$VaR_p(X) = \begin{cases} \lambda \log(2p) & \text{if } p \leq \frac{1}{2} \\ -\lambda \log(2(1-p)) & \text{if } p > \frac{1}{2} \end{cases}$$

EXAMPLE CLASS

15 DECEMBER

10:00-11:00AM

MATH3/4/68181

(iii)

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$\boxed{\text{Suppose } p \le \tfrac{1}{2}}$:

$$= \frac{1}{p}\int_0^p \lambda \log(2t)\,dt$$

$$= \frac{\lambda}{p}\int_0^p (\log 2 + \log t)\,dt$$

$$= \frac{\lambda}{p}\left[(\log 2)\,p + \int_0^p \log t\,dt\right]$$

$$= \frac{\lambda}{p}\left[(\log 2)\,p + \left[t\cdot\log t\right]_0^p - \int_0^p t\cdot\frac{1}{t}\,dt\right]$$

$$= \frac{\lambda}{p}\left[(\log 2)\cdot p + p\cdot\log p - p\right]$$

$\boxed{\text{Suppose } p > \tfrac{1}{2}}$: $= \frac{1}{p}\int_0^{\frac{1}{2}} VaR_t(X)\,dt + \frac{1}{p}\int_{\frac{1}{2}}^p VaR_t(X)\,dt$

$$= \frac{1}{p}\int_0^{\frac{1}{2}} \lambda\log(2t)\,dt + \frac{1}{p}\int_{\frac{1}{2}}^p (-\lambda)\log\big(2(1-t)\big)\,dt$$

$$= \frac{\lambda}{p}\int_0^{\frac{1}{2}} (\log 2 + \log t)\,dt - \frac{\lambda}{p}\int_{\frac{1}{2}}^p \big(\log 2 + \log(1-t)\big)\,dt$$

$$= \frac{\lambda}{p}\left[\frac{(\log 2)}{2} + \int_0^{\frac{1}{2}} \log t\,dt\right] - \frac{\lambda}{p}\left[(\log 2)\left(p-\frac{1}{2}\right) + \int_{\frac{1}{2}}^p \log(1-t)\,dt\right]$$

$$= \frac{\lambda}{p}\left[\frac{\log 2}{2} + \left[t\cdot\log t\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} t\cdot\frac{1}{t}\,dt\right]$$

$$- \frac{\lambda}{p}\left[(\log 2)\cdot\left(p-\frac{1}{2}\right) + \left[t\cdot\log(1-t)\right]_{\frac{1}{2}}^p + \int_{\frac{1}{2}}^p \frac{t}{1-t}\,dt\right]$$

$$= \frac{\lambda}{p}\left[\cancel{\frac{\log 2}{2}} + \frac{1}{2}\cdot \cancel{\log \frac{1}{2}} - \frac{1}{2}\right]$$

$$- \frac{\lambda}{p}\left[(\log 2)\cdot\left(p - \frac{1}{2}\right) + p\cdot \log(1-p) - \frac{1}{2}\cdot \log\frac{1}{2} + \int_{\frac{1}{2}}^{p} \frac{t-1+1}{1-t}\, dt\right]$$

$$= -\frac{\lambda}{2p} - \frac{\lambda}{p}\left[(\log 2)\cdot\left(p - \frac{1}{2}\right) + p\cdot \log(1-p) - \frac{1}{2}\cdot \log\frac{1}{2} + \Big[-\log(1-t) - t\Big]_{\frac{1}{2}}^{p}\right]$$

$$= -\frac{\lambda}{2p} - \frac{\lambda}{p}\left[(\log 2)\cdot p + p\cdot \log(1-p) - \log(1-p) - p + \log\frac{1}{2} + \frac{1}{2}\right].$$

(c) (i)

$$L(\lambda) = \prod_{i=1}^{n} \left[\frac{1}{2\lambda} e^{-\frac{|x_i|}{\lambda}} \right]$$

$$= \frac{1}{(2\lambda)^n} e^{-\frac{1}{\lambda} \sum_{i=1}^{n} |x_i|}$$

(ii) $\log L(\lambda) = -n \log(2\lambda) - \frac{1}{\lambda} \sum_{i=1}^{n} |x_i|$

$$\frac{d \log L}{d\lambda} = -\frac{n}{\lambda} + \frac{1}{\lambda^2} \sum_{i=1}^{n} |x_i| = 0$$

$$\Rightarrow \hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} |x_i|$$

$$\frac{d^2 \log L}{d\lambda^2} = \frac{n}{\lambda^2} - \frac{2}{\lambda^3} \sum_{i=1}^{n} |x_i|$$

$$= \frac{1}{\lambda^3} \left[n\lambda - 2 \sum_{i=1}^{n} |x_i| \right]$$

$$= \frac{1}{\hat{\lambda}^3} \left[n\hat{\lambda} - 2 \sum_{i=1}^{n} |x_i| \right] \text{ at } \lambda = \hat{\lambda}$$

$$= \frac{1}{\hat{\lambda}^3} \left[\sum_{i=1}^{n} |x_i| - 2 \sum_{i=1}^{n} |x_i| \right] < 0$$

$\Rightarrow \hat{\lambda}$ is an MLE.

(iii)

$$\widehat{VaR_p}(X) = \begin{cases} \hat{\lambda} \log(2p) & p \leq \frac{1}{2} \\ -\hat{\lambda} \log(2(1-p)) & p > \frac{1}{2} \end{cases}$$

$$\widehat{ES_p}(X) = \begin{cases} \frac{\hat{\lambda}}{2}\left[(\log 2) + p \cdot \log p - p\right] \\ \qquad \text{if} \quad p \leq \frac{1}{2} \\ -\frac{\hat{\lambda}}{2p} - \frac{\hat{\lambda}}{p}\left[(\log 2) \cdot p + p \cdot \log(1-p) \right. \\ \qquad \left. - \log(1-p) - p + \log \frac{1}{2} + \frac{1}{2}\right] \\ \qquad \text{if} \quad p > \frac{1}{2} \end{cases}$$

where $\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} |x_i|$.

(iv) To show that $\widehat{VaR}_p$ & $\widehat{ES}_p$ are unbiased, it is sufficient that we show that $\boxed{Bias(\hat{\lambda}) = 0}$.

$$Bias(\hat{\lambda}) = E(\hat{\lambda}) - \lambda$$

$$= E\left(\frac{1}{n}\sum_{i=1}^{n} |X_i|\right) - \lambda$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(|X_i|) - \lambda$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{\infty} \boxed{|x| \cdot \frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}}}\, dx\right) - \lambda$$

even function

$$= \left(\frac{1}{n}\sum_{i=1}^{n} \cancel{\frac{2}{2\lambda}} \cdot \int_{0}^{\infty} x e^{-\frac{x}{\lambda}}\, dx\right) - \lambda$$

$$= \left(\frac{1}{n\lambda}\sum_{i=1}^{n}\int_{0}^{\infty} x e^{-\frac{x}{\lambda}}\, dx\right) - \lambda$$

$$\boxed{y = \frac{x}{\lambda} \Rightarrow x = \lambda y \Rightarrow dx = \lambda\, dy}$$

$$= \left(\frac{1}{n\lambda}\sum_{i=1}^{n}\int_{0}^{\infty} \lambda y e^{-y} \cdot \lambda\, dy\right) - \lambda$$

$$= \left(\frac{\lambda}{n}\sum_{i=1}^{n}\int_{0}^{\infty} y e^{-y} dy\right) - \lambda$$

$$= \frac{\lambda}{n}\left(\sum_{i=1}^{n} \Gamma(2)\right) - \lambda$$

$$= \frac{\lambda}{n} \cdot \sum_{i=1}^{n} 1 - \lambda$$

$$= \frac{\lambda}{n} \cdot n - \lambda = 0 \Rightarrow \mathrm{Bias}(\hat{\lambda}) = 0$$

$\Rightarrow \widehat{VaR}_p(X)$ & $\widehat{ES}_p(X)$

are unbiased.

LECTURE

17 DECEMBER

12:00-13:00PM

MATH4/68181

Q1, Sheet 12

$$\bar{G}(x,y) = e^{-\frac{\theta y^2}{x+y} + \theta y - x - y}$$

$$P(Y>y) = P(X>0, Y>y) = \bar{G}(0,y) = e^{-\theta y + \theta y - y} = e^{-y}$$

$$P(X>x) = P(X>x, Y>0) = \bar{G}(x,0) = e^{-x}$$

$$\Rightarrow \quad \left.\begin{aligned} P(Y \le y) &= 1 - e^{-y} \\ P(X \le x) &= 1 - e^{-x} \end{aligned}\right\} \text{Exponential margins}$$

$$\bar{G}(x,y) = e^{-(x+y)\left[\frac{\theta y^2}{(x+y)^2} - \frac{\theta y}{x+y} + 1\right]}$$

$$= e^{-(x+y)\, A\left(\frac{y}{x+y}\right)}$$

where $A(w) = \theta w^2 - \theta w + 1.$

(a) $A(0) = \theta \cdot 0^2 - \theta \cdot 0 + 1 = 1$ ✓

$A(1) = \theta \cdot 1^2 - \theta \cdot 1 + 1 = 1$ ✓

(b) $A(w) \ge w \Leftrightarrow \theta w^2 - \theta w + 1 \ge w$

$\Leftrightarrow \theta w^2 - \theta w + 1 - w \ge 0$

$\Leftrightarrow \theta w (w-1) + 1 - w \ge 0$

$\Leftrightarrow \underbrace{(-\theta w + 1)}_{\ge 0} \underbrace{(1-w)}_{\ge 0} \ge 0$ ✓

$$A(w) \geq 1-w \iff \theta w^2 - \theta w + 1 \geq 1 - w$$

$$\iff \underbrace{\theta w^2}_{\geq 0} + \underbrace{(1-\theta) w}_{\geq 0} \geq 0$$

$$A(w) \leq 1 \iff \theta w^2 - \theta w + 1 \leq 1$$

$$\iff \theta w^2 - \theta w \leq 0$$

$$\iff \underbrace{\theta w}_{\geq 0} \underbrace{(w-1)}_{\leq 0} \leq 0$$

(c) $$A'(w) = 2\theta w - \theta$$

$$A''(w) = 2\theta > 0$$

$\Rightarrow$ A is convex.

Hence, $\bar{G}(x, y)$ is an bivariate extreme value distribution.

(b) $G(x, y) = 1 - \bar{G}(x, 0)$

$- \bar{G}(0, y)$

$+ \bar{G}(x, y)$

(c)

$$\frac{\frac{\partial G(x, y)}{\partial x}}{e^{-x}}$$

(d)

$$\frac{\frac{\partial G(x, y)}{\partial y}}{e^{-y}}$$

(e) $$\frac{\partial^2}{\partial x \partial y} G(x, y)$$

(f)

$$\frac{\frac{\partial^2}{\partial x \partial y} G(x, y)}{e^{-x}}$$

(g)

$$\frac{\frac{\partial^2}{\partial x \partial y} G(x, y)}{e^{-y}} .$$

Q2, Prob Sheet 12

$$\bar{G}(x,y) = e^{\frac{\alpha x y}{x+y} - x - y}$$

$$\left.\begin{array}{l} P(Y>y) = \bar{G}(0,y) = e^{-y} \\ P(X>x) = \bar{G}(x,0) = e^{-x} \end{array}\right\} \Rightarrow \begin{array}{l} P(Y\leq y) = 1-e^{-y} \\ P(X\leq x) = 1-e^{-x} \end{array}$$

$$\bar{G}(x,y) = \boxed{e^{-(x+y)\left[1 - \frac{\alpha x y}{(x+y)^2}\right]}}.$$

$$= e^{-(x+y) A\left(\frac{y}{x+y}\right)},$$

$$e^{-(x+y)\left[1-\alpha \cdot \frac{y}{x+y}\left(1-\frac{y}{x+y}\right)\right]}$$

so $A(w) = 1 - \alpha w(1-w)$

(a) $A(0) = 1 - \alpha \cdot 0 \cdot (1-0) = 1$ ✓

$A(1) = 1 - \alpha \cdot 1 \cdot (1-1) = 1$ ✓

(b) $A(w) \geq w \Leftrightarrow 1 - \alpha w(1-w) \geq w$

$\Leftrightarrow 1 - w - \alpha \cdot w \cdot (1-w) \geq 0$

$\Leftrightarrow \underbrace{(1-\alpha w)}_{\geq 0} \underbrace{(1-w)}_{\geq 0} \geq 0$ ✓

$A(w) \geq 1-w \Leftrightarrow 1 - \alpha w(1-w) \geq 1-w$

$\Leftrightarrow w - \alpha w(1-w) \geq 0$

$\Leftrightarrow \underbrace{w}_{\geq 0} \underbrace{[1-\alpha(1-w)]}_{\geq 0} \geq 0$ ✓

$$A(w) \leq 1 \iff 1 - \alpha w(1-w) \leq 1$$

$$\iff -\underbrace{\alpha}_{\geq 0} \underbrace{w(1-w)}_{\geq 0} \leq 0$$

(c) $A'(w) = -\alpha + 2\alpha w$

$A''(w) = 2\alpha > 0$

$\Rightarrow A$ is convex

Hence $\bar{G}$ is a biv extreme value distn.

LECTURE

18 DECEMBER

9:00-10:00AM

MATH3/4/68181

Q4, 2014/2015 Exam

(a) $F(x) = 1 - (1 + x^c)^{-k}, \quad x > 0$

$F(x) = 1$

$\Rightarrow \quad 1 - (1 + x^c)^{-k} = 1$

$\Rightarrow \quad (1 + x^c)^{-k} = 0$

$\Rightarrow \quad 1 + x^c = \infty \Rightarrow x = +\infty \Rightarrow \omega(F) = \infty.$

$$\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t \uparrow \infty} \frac{\cancel{1} - (\cancel{1} - (1 + (tx)^c)^{-k})}{\cancel{1} - (\cancel{1} - (1 + t^c)^{-k})}$$

$$= \lim_{t \uparrow \infty} \left[\frac{1 + (tx)^c}{1 + t^c} \right]^{-k} = \lim_{t \uparrow \infty} \left[\frac{\underbrace{t^{-c}}_{\to 0} + x^c}{\underbrace{t^{-c}}_{\to 0} + 1} \right]^{-k}$$

$= x^{-ck} \Rightarrow$ Condition $\underline{II}$ is satisfied.

So, F belongs to the Frechet domain.

(b) $F(x) = 1 - (1 - x^b)^a, \quad 0 < x < 1$

$F(x) = 1$

$\Rightarrow \quad 1 - (1 - x^b)^a = 1$

$\Rightarrow \quad (1 - x^b)^a = 0$

$\Rightarrow \quad 1 - x^b = 0 \Rightarrow x^b = 1 \Rightarrow x = 1$

$\Rightarrow \omega(F) = 1$

$$\lim_{t \downarrow 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)} = \lim_{t \downarrow 0} \frac{\cancel{1} - [\cancel{1} - (1 - (1 - tx)^b)^a]}{\cancel{1} - [\cancel{1} - (1 - (1 - t)^b)^a]}$$

$$= \lim_{t \downarrow 0} \left[\frac{1 - (1 - tx)^b}{1 - (1 - t)^b} \right]^a = \lim_{t \downarrow 0} \left[\frac{\cancel{1} - (\cancel{1} - btx)}{\cancel{1} - (\cancel{1} - bt)} \right]^a$$

$\boxed{(1 - y)^b \approx 1 - by}$

$$= \lim_{t \downarrow 0} \left[\frac{btx}{bt} \right]^a = x^a$$

$\Rightarrow$ Condition $\overline{III}$ satisfied

$\Rightarrow$ So, F belongs to the Weibull domain.

(e) $F(x) = e^{-e^{-x}}$

$F(x) = 1 \Rightarrow e^{-e^{-x}} = 1$

$\Rightarrow -e^{-x} = 0 \Rightarrow e^{-x} = 0$

$\Rightarrow -x = -\infty \Rightarrow x = +\infty \Rightarrow w(F) = +\infty$,

$$\lim_{t \uparrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \lim_{t \uparrow \infty} \frac{1 - e^{-e^{-t - x\gamma(t)}}}{1 - e^{-e^{-t}}}$$

$$= \lim_{t \uparrow \infty} \frac{\cancel{1} - (\cancel{1} - e^{-t - x\gamma(t)})}{\cancel{1} - (\cancel{1} - e^{-t})} \qquad [e^{-y} \approx 1 - y]$$

$$= \lim_{t \uparrow \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}} = \lim_{t \uparrow \infty} e^{-x\gamma(t)} = e^{-x}$$

if $\gamma(t) = 1$

$\Rightarrow$ Condition I is satisfied

$\Rightarrow$ So, F belongs to the Gumbel domain.

Q2, 2014 / 2015 Exam

X = Stock Return

$X/\theta \sim \text{Uni}[-\theta, \theta]$

θ has PDF $\frac{\lambda}{\theta^2} e^{-\frac{\lambda}{\theta}}, \quad \theta > 0$

(a)

$$F_X(x) = \int_0^\infty \underbrace{F_{X/\theta}(x/\theta)}\,.\,\underbrace{f(\theta)}\,d\theta$$

$$= \int_0^\infty \frac{X+\theta}{2\theta} \cdot \frac{\lambda}{\theta^2} \cdot e^{-\frac{\lambda}{\theta}}\,d\theta$$

$$= \frac{X\lambda}{2}\int_0^\infty \frac{1}{\theta^3} e^{-\frac{\lambda}{\theta}}\,d\theta + \frac{\lambda}{2}\int_0^\infty \frac{1}{\theta^2} \cdot e^{-\frac{\lambda}{\theta}}\,d\theta$$

$$\boxed{\text{Set } y = \frac{\lambda}{\theta} \Rightarrow \theta = \frac{\lambda}{y} \Rightarrow \frac{d\theta}{dy} = -\frac{\lambda}{y^2}}$$

$$= \frac{X\lambda}{2} \cdot \int_\infty^0 \left(\frac{y}{\lambda}\right)^3 e^{-y} \cdot \left(-\frac{\lambda}{y^2}\right) dy + \frac{\lambda}{2}\int_\infty^0 \left(\frac{y}{\lambda}\right)^2 \cdot e^{-y} \cdot \left(-\frac{\lambda}{y^2}\right) dy$$

$$= \frac{X}{2\lambda} \underbrace{\int_0^\infty y e^{-y}\,dy}_{=\Gamma(2)=1} + \frac{1}{2} \underbrace{\int_0^\infty e^{-y}\,dy}_{\Gamma(1)=1}$$

$$= \frac{X}{2\lambda} + \frac{1}{2}.$$

(b) $$f_X(x) = \frac{d}{dx} F_X(x) = \frac{1}{2\lambda}$$

(c) $$E(X) = \int_{-\lambda}^{\lambda} x \cdot \frac{1}{2\lambda}\, dx$$

$$= \frac{1}{2\lambda}\left[\frac{x^2}{2}\right]_{-\lambda}^{\lambda}$$

$$= \frac{1}{2\lambda}\left(\frac{\lambda^2}{2} - \frac{\lambda^2}{2}\right) = 0$$

(d) $$E(X^2) = \int_{-\lambda}^{\lambda} x^2 \cdot \frac{1}{2\lambda}\, dx$$

$$= \frac{1}{2\lambda}\left[\frac{x^3}{3}\right]_{-\lambda}^{\lambda}$$

$$= \frac{1}{2\lambda}\left[\frac{\lambda^3}{3} - \frac{(-\lambda)^3}{3}\right] = \frac{\lambda^2}{3}$$

$$Var(X) = \frac{\lambda^2}{3}$$

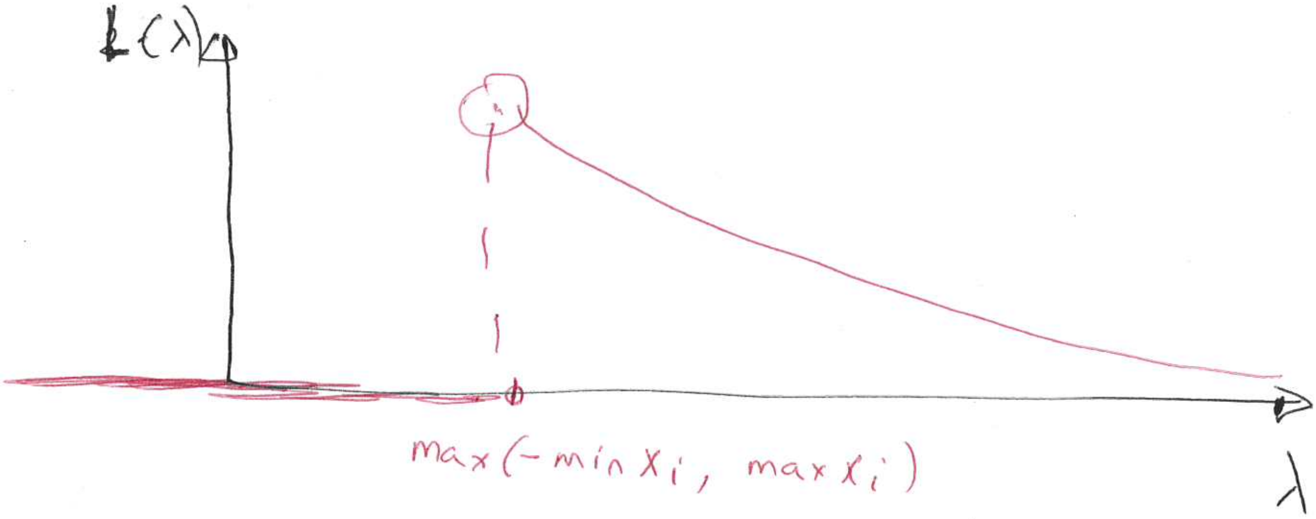
(e) $$X \sim Uni[-\lambda, \lambda]$$

$$L(\lambda) = \prod_{i=1}^{n}\left[\frac{1}{2\lambda} I\{-\lambda < X_i < \lambda\}\right]$$

$$= \frac{1}{(2\lambda)^n} \cdot I\{-\lambda < \min X_i \ \& \ \lambda > \max X_i\}$$

$$= \frac{1}{(2\lambda)^n} \cdot I\{\lambda > -\min X_i \ \& \ \lambda > \max X_i\}$$

$$= \frac{1}{(2\lambda)^n} \cdot I\{\lambda > \max(-\min X_i, \max X_i)\}$$



$$\boxed{\hat{\lambda} = \max(-\min X_i,\ \max X_i)}$$

GOOD

LUCK!