MATH48181/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 9

Suppose $C_1, C_2, \ldots, C_p$ are known copulas. Let

$$C(u_1, u_2) = \left\{ w_1 \left[C_1(u_1, u_2) \right]^\beta + \cdots + \alpha_p \left[C_p(u_1, u_2) \right]^\beta \right\}^{1/\beta}$$

where $\beta > 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_p$ are non-negative real numbers summing to 1.

Condition (i) is satisfied since

$$C(0, u) = \left\{ w_1 \left[C_1(0, u) \right]^\beta + \cdots + \alpha_p \left[C_p(0, u) \right]^\beta \right\}^{1/\beta} = [0 + \cdots + 0]^{1/\beta} = 0.$$

Condition (ii) is satisfied since

$$C(u, 0) = \left\{ w_1 \left[C_1(u, 0) \right]^\beta + \cdots + \alpha_p \left[C_p(u, 0) \right]^\beta \right\}^{1/\beta} = [0 + \cdots + 0]^{1/\beta} = 0.$$

Condition (iii) is satisfied since

$$C(1, u) = \left\{ w_1 \left[C_1(1, u) \right]^\beta + \cdots + \alpha_p \left[C_p(1, u) \right]^\beta \right\}^{1/\beta} = \left[w_1 u^\beta + \cdots + w_p u^\beta \right]^{1/\beta} = \left[u^\beta \right]^{1/\beta} = u.$$

Condition (iv) is satisfied since

$$C(u, 1) = \left\{ w_1 \left[C_1(u, 1) \right]^\beta + \cdots + \alpha_p \left[C_p(u, 1) \right]^\beta \right\}^{1/\beta} = \left[w_1 u^\beta + \cdots + w_p u^\beta \right]^{1/\beta} = \left[u^\beta \right]^{1/\beta} = u.$$

Condition (v) is satisfied since

$$\begin{aligned}
& \frac{\partial}{\partial u_1} C(u_1, u_2) \\
= & \frac{\partial}{\partial u_1} \left\{ w_1 \left[C_1(u_1, u_2) \right]^\beta + \cdots + \alpha_p \left[C_p(u_1, u_2) \right]^\beta \right\}^{1/\beta} \\
= & \frac{1}{\beta} \left\{ w_1 \left[C_1(u_1, u_2) \right]^\beta + \cdots + \alpha_p \left[C_p(u_1, u_2) \right]^\beta \right\}^{1/\beta - 1} \frac{\partial}{\partial u_1} \left\{ w_1 \left[C_1(u_1, u_2) \right]^\beta + \cdots + \alpha_p \left[C_p(u_1, u_2) \right]^\beta \right\} \\
= & \frac{1}{\beta} \left\{ w_1 \left[C_1(u_1, u_2) \right]^\beta + \cdots + \alpha_p \left[C_p(u_1, u_2) \right]^\beta \right\}^{1/\beta - 1} \sum_{i=1}^{p} w_i \frac{\partial}{\partial u_1} \left[C_i(u_1, u_2) \right]^\beta \\
= & \left\{ w_1 \left[C_1(u_1, u_2) \right]^\beta + \cdots + \alpha_p \left[C_p(u_1, u_2) \right]^\beta \right\}^{1/\beta - 1} \sum_{i=1}^{p} w_i \left[C_i(u_1, u_2) \right]^{\beta - 1} \frac{\partial}{\partial u_1} C_i(u_1, u_2) \\
\geq & \ 0.
\end{aligned}$$

Condition (vi) is satisfied since

$$
\begin{aligned}
& \frac{\partial}{\partial u_2} C\left(u_1, u_2\right) \\
= & \frac{\partial}{\partial u_2}\left\{w_1\left[C_1\left(u_1, u_2\right)\right]^\beta+\cdots+\alpha_p\left[C_p\left(u_1, u_2\right)\right]^\beta\right\}^{1 / \beta} \\
= & \frac{1}{\beta}\left\{w_1\left[C_1\left(u_1, u_2\right)\right]^\beta+\cdots+\alpha_p\left[C_p\left(u_1, u_2\right)\right]^\beta\right\}^{1 / \beta-1} \frac{\partial}{\partial u_2}\left\{w_1\left[C_1\left(u_1, u_2\right)\right]^\beta+\cdots+\alpha_p\left[C_p\left(u_1, u_2\right)\right]^\beta\right\} \\
= & \frac{1}{\beta}\left\{w_1\left[C_1\left(u_1, u_2\right)\right]^\beta+\cdots+\alpha_p\left[C_p\left(u_1, u_2\right)\right]^\beta\right\}^{1 / \beta-1} \sum_{i=1}^p w_i \frac{\partial}{\partial u_2}\left[C_i\left(u_1, u_2\right)\right]^\beta \\
= & \left\{w_1\left[C_1\left(u_1, u_2\right)\right]^\beta+\cdots+\alpha_p\left[C_p\left(u_1, u_2\right)\right]^\beta\right\}^{1 / \beta-1} \sum_{i=1}^p w_i\left[C_i\left(u_1, u_2\right)\right]^{\beta-1} \frac{\partial}{\partial u_2} C_i\left(u_1, u_2\right) \\
\geq & \; 0 .
\end{aligned}
$$