MATH48181/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 8

Suppose X represents loss and has the following probability density function

$$f_X(x) = \begin{cases} \frac{abx^{b-1}}{\delta^b(a+b)}, & \text{if } x \le \delta, \\ \frac{ab\delta^a}{x^{a+1}(a+b)}, & \text{if } x > \delta \end{cases}$$

for $x > 0$, $a > 0$, $b > 0$ and $\delta > 0$.

We know from the quiz 7 that

$$\mathrm{VaR}_p(X) = \begin{cases} \delta \left[\frac{p(a+b)}{a}\right]^{1/b}, & \text{if } p \le a/(a+b), \\ \delta \left[\frac{(1-p)(a+b)}{b}\right]^{-1/a}, & \text{if } p > a/(a+b) \end{cases}$$

for $0 < p < 1$, $a > 0$, $b > 0$ and $\delta > 0$.

If $p \le a/(a+b)$ then

$$\begin{aligned} \frac{1}{p}\int_0^p \mathrm{VaR}_t(X)dt &= \frac{1}{p}\delta\left[\frac{a+b}{a}\right]^{1/b}\int_0^p t^{1/b}dt \\ &= \frac{1}{p}\delta\left[\frac{a+b}{a}\right]^{1/b}\left[\frac{t^{1/b+1}}{1/b+1}\right]_0^p \\ &= \frac{1}{p}\delta\left[\frac{a+b}{a}\right]^{1/b}\left[\frac{p^{1/b+1}}{1/b+1} - 0\right] \\ &= \delta\left[\frac{a+b}{a}\right]^{1/b}\frac{p^{1/b}}{1/b+1}. \end{aligned}$$

If $p > a/(a+b)$ then

$$\begin{aligned} \frac{1}{p}\int_0^p \mathrm{VaR}_t(X)dt &= \frac{1}{p}\int_0^{a/(a+b)} \mathrm{VaR}_t(X)dt + \frac{1}{p}\int_{a/(a+b)}^p \mathrm{VaR}_t(X)dt \\ &= \frac{ab\delta}{p(a+b)(1+b)} + \frac{\delta}{p}\left[\frac{a+b}{b}\right]^{-1/a}\int_{a/(a+b)}^p (1-t)^{-1/a}dt \\ &= \frac{ab\delta}{p(a+b)(1+b)} + \frac{\delta}{p}\left[\frac{a+b}{b}\right]^{-1/a}\left[\frac{(1-t)^{1-1/a}}{1/a-1}\right]_{a/(a+b)}^p \\ &= \frac{ab\delta}{p(a+b)(1+b)} + \frac{\delta}{p(1/a-1)}\left[\frac{a+b}{b}\right]^{-1/a}\left[(1-p)^{1-1/a} - \left(\frac{b}{a+b}\right)^{1-1/a}\right]. \end{aligned}$$

Hence,

$$\mathrm{ES}_p(X) = \begin{cases} \delta \left[\frac{a+b}{a}\right]^{1/b} \frac{p^{1/b}}{1/b+1}, & \text{if } p \leq a/(a+b), \\ \frac{ab\delta}{p(a+b)(1+b)} + \frac{\delta}{p(1/a-1)} \left[\frac{a+b}{b}\right]^{-1/a} \left[(1-p)^{1-1/a} - \left(\frac{b}{a+b}\right)^{1-1/a}\right], & \text{if } p > a/(a+b) \end{cases}$$

for $0 < p < 1$, $a > 0$, $b > 0$ and $\delta > 0$.