MATH48181/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 7

Suppose X represents loss and has the following probability density function

$$f_X(x) = \begin{cases} \frac{abx^{b-1}}{\delta^b(a+b)}, & \text{if } x \leq \delta, \\ \frac{ab\delta^a}{x^{a+1}(a+b)}, & \text{if } x > \delta \end{cases}$$

for $x > 0$, $a > 0$, $b > 0$ and $\delta > 0$.

First, we derive the cumulative distribution function of X. If $x \leq \delta$ then

$$\begin{aligned} F_X(x) &= \frac{ab}{\delta^b(a+b)} \int_0^x y^{b-1} dy \\ &= \frac{ab}{\delta^b(a+b)} \left[\frac{y^b}{b} \right]_0^x \\ &= \frac{ab}{\delta^b(a+b)} \left[\frac{x^b}{b} - 0 \right] \\ &= \frac{ax^b}{\delta^b(a+b)}. \end{aligned}$$

If $x > \delta$ then

$$\begin{aligned} F_X(x) &= \Pr(X \leq x) \\ &= 1 - \Pr(X > x) \\ &= 1 - \frac{ab\delta^a}{a+b} \int_x^\infty y^{-a-1} dy \\ &= 1 - \frac{ab\delta^a}{a+b} \left[\frac{y^{-a}}{-a} \right]_x^\infty \\ &= 1 - \frac{ab\delta^a}{a+b} \left[0 - \frac{x^{-a}}{-a} \right] \\ &= 1 - \frac{bx^{-a}\delta^a}{a+b}. \end{aligned}$$

So,

$$F_X(x) = \begin{cases} \frac{ax^b}{\delta^b(a+b)}, & \text{if } x \leq \delta, \\ 1 - \frac{bx^{-a}\delta^a}{a+b}, & \text{if } x > \delta \end{cases}$$

for $x > 0$, $a > 0$, $b > 0$ and $\delta > 0$.

Now we invert the cumulative distribution function of X to find VaR. Setting

$$\frac{ax^b}{\delta^b(a+b)} = p$$

gives

$$x = \delta \left[\frac{p(a+b)}{a} \right]^{1/b}.$$

Setting

$$1 - \frac{bx^{-a}\delta^a}{a+b} = p$$

gives

$$x = \delta \left[\frac{(1-p)(a+b)}{b} \right]^{-1/a}.$$

Hence,

$$\mathrm{VaR}_p(X) = \begin{cases} \delta \left[\frac{p(a+b)}{a} \right]^{1/b}, & \text{if } p \leq a/(a+b), \\ \delta \left[\frac{(1-p)(a+b)}{b} \right]^{-1/a}, & \text{if } p > a/(a+b) \end{cases}$$

for $0 < p < 1$, $a > 0$, $b > 0$ and $\delta > 0$.