MATH48181/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 4

Suppose a portfolio is made of up of k independent investments. Let $X_1, X_2, \ldots, X_k$ denote the losses. Assume that X_i are Uniform $[-\theta, \theta]$ random variables.

Let $U = \min(X_1, \ldots, X_k)$. The cumulative distribution function of U is

$$\begin{aligned}
F_U(u) &= \Pr(U \leq u) \\
&= 1 - \Pr(U > u) \\
&= 1 - \Pr\left(\min(X_1, \ldots, X_k) > u\right) \\
&= 1 - \Pr\left(X_1 > u, \ldots, X_k > u\right) \\
&= 1 - \Pr\left(X_1 > u\right) \cdots \Pr\left(X_k > u\right) \\
&= 1 - \left[\Pr\left(X_1 > u\right)\right]^k \\
&= 1 - \left[1 - \Pr\left(X_1 \leq u\right)\right]^k \\
&= 1 - \left(1 - \frac{u+\theta}{2\theta}\right)^k \\
&= 1 - \left(\frac{\theta - u}{2\theta}\right)^k
\end{aligned}$$

So, the probability density function of U is

$$f_U(u) = \frac{k}{(2\theta)^k} (\theta - u)^{k-1}$$

for $-\theta < u < \theta$. Hence, the expected minimum portfolio loss is

$$\begin{aligned}
E(U) &= \frac{k}{(2\theta)^k} \int_{-\theta}^{\theta} u\,(\theta - u)^{k-1}\,du \\
&= \frac{k}{(2\theta)^k} \int_{-\theta}^{\theta} \left[\theta - (\theta - u)\right](\theta - u)^{k-1}\,du \\
&= \frac{k}{(2\theta)^k} \int_{-\theta}^{\theta} \left[\theta\,(\theta - u)^{k-1} - (\theta - u)^k\right] du \\
&= \frac{k}{(2\theta)^k} \left\{ \left[\theta \frac{(\theta - u)^k}{-k}\right]_{-\theta}^{\theta} + \left[\frac{(\theta - u)^{k+1}}{k+1}\right]_{-\theta}^{\theta} \right\} \\
&= \frac{k}{(2\theta)^k} \left\{ \theta \frac{(2\theta)^k}{k} - \frac{(2\theta)^{k+1}}{k+1} \right\} \\
&= \frac{k}{(2\theta)^k} \left\{ \frac{\theta}{k} - \frac{2\theta}{k+1} \right\} \\
&= \frac{\theta(1-k)}{k+1}.
\end{aligned}$$