MATH48181/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 3

Consider a class of distributions defined by the cumulative distribution function

$$F(x) = \frac{[G(x)]^{ab}}{[G(x)]^{ab} + \left\{1 - [G(x)]^b\right\}^a}$$

where $a > 0$ $b > 0$ and $G(\cdot)$ is a valid cumulative distribution function. You may assume that F and G have the same upper end points.

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = e^{-x}$$

for every $x > 0$. But

$$\begin{aligned}
& \lim_{t \to w(F)} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)} \\
= & \lim_{t \to w(F)} \frac{1 - \frac{[G(t+xh(t))]^{ab}}{[G(t+xh(t))]^{ab} + \left\{1 - [G(t+xh(t))]^b\right\}^a}}{1 - \frac{[G(t)]^{ab}}{[G(t)]^{ab} + \left\{1 - [G(t)]^b\right\}^a}} \\
= & \lim_{t \to w(F)} \frac{\frac{\left\{1 - [G(t+xh(t))]^b\right\}^a}{[G(t+xh(t))]^{ab} + \left\{1 - [G(t+xh(t))]^b\right\}^a}}{\frac{\left\{1 - [G(t)]^b\right\}^a}{[G(t)]^{ab} + \left\{1 - [G(t)]^b\right\}^a}} \\
= & \lim_{t \to w(G)} \frac{\frac{\left\{1 - [G(t+xh(t))]^b\right\}^a}{[G(t+xh(t))]^{ab} + \left\{1 - [G(t+xh(t))]^b\right\}^a}}{\frac{\left\{1 - [G(t)]^b\right\}^a}{[G(t)]^{ab} + \left\{1 - [G(t)]^b\right\}^a}} \\
= & \lim_{t \to w(G)} \frac{\frac{\left\{1 - [G(t+xh(t))]^b\right\}^a}{[1]^{ab} + \left\{1 - 1^b\right\}^a}}{\frac{\left\{1 - [G(t)]^b\right\}^a}{1^{ab} + \left\{1 - 1^b\right\}^a}} \\
= & \lim_{t \to w(G)} \left\{\frac{1 - \left[G\left(t + xh(t)\right)\right]^b}{1 - \left[G\left(t\right)\right]^b}\right\}^a \\
= & \lim_{t \to w(G)} \left[\frac{1 - \left\{1 - \left[1 - G\left(t + xh(t)\right)\right]\right\}^b}{1 - \left\{1 - \left[1 - G\left(t\right)\right]\right\}^b}\right]^a
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to w(G)} \left[\frac{1-\{1-b\left[1-G\left(t+xh(t)\right)\right]\}}{1-\{1-b\left[1-G\left(t\right)\right]\}} \right]^a \\
&= \lim_{t\to w(G)} \left\{ \frac{b\left[1-G\left(t+xh(t)\right)\right]}{b\left[1-G\left(t\right)\right]} \right\}^a \\
&= \lim_{t\to w(G)} \left\{ \frac{1-G\left(t+xh(t)\right)}{1-G\left(t\right)} \right\}^a \\
&= \{\exp(-x)\}^a \\
&= \exp(-ax)
\end{aligned}
$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$
\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \le x\right) = \exp\left[-\exp(-ax)\right]
$$

for some suitable norming constants $a_n > 0$ and b_n.

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$
\lim_{t\to\infty} \frac{1-G\left(tx\right)}{1-G(t)} = x^{-\beta}
$$

for every $x > 0$. But

$$
\begin{aligned}
& \lim_{t\to\infty} \frac{1-F\left(tx\right)}{1-F(t)} \\
&= \lim_{t\to\infty} \frac{1-\frac{[G(tx)]^{ab}}{[G(tx)]^{ab}+\left\{1-[G(tx)]^b\right\}^a}}{1-\frac{[G(t)]^{ab}}{[G(t)]^{ab}+\left\{1-[G(t)]^b\right\}^a}} \\
&= \lim_{t\to\infty} \frac{\frac{\left\{1-[G(tx)]^b\right\}^a}{[G(tx)]^{ab}+\left\{1-[G(tx)]^b\right\}^a}}{\frac{\left\{1-[G(t)]^b\right\}^a}{[G(t)]^{ab}+\left\{1-[G(t)]^b\right\}^a}} \\
&= \lim_{t\to\infty} \frac{\frac{\left\{1-[G(tx)]^b\right\}^a}{[1]^{ab}+\left\{1-1^b\right\}^a}}{\frac{\left\{1-[G(t)]^b\right\}^a}{1^{ab}+\left\{1-1^b\right\}^a}} \\
&= \lim_{t\to\infty} \left\{ \frac{1-\left[G\left(tx\right)\right]^b}{1-\left[G\left(t\right)\right]^b} \right\}^a \\
&= \lim_{t\to\infty} \left[\frac{1-\{1-\left[1-G\left(tx\right)\right]\}^b}{1-\{1-\left[1-G\left(t\right)\right]\}^b} \right]^a \\
&= \lim_{t\to\infty} \left[\frac{1-\{1-b\left[1-G\left(tx\right)\right]\}}{1-\{1-b\left[1-G\left(t\right)\right]\}} \right]^a \\
&= \lim_{t\to\infty} \left\{ \frac{b\left[1-G\left(tx\right)\right]}{b\left[1-G\left(t\right)\right]} \right\}^a \\
&= \lim_{t\to\infty} \left\{ \frac{1-G\left(tx\right)}{1-G\left(t\right)} \right\}^a
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{x^{-\beta}\right\}^a \\
&= x^{-a\beta}
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$
\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left(-x^{-a\beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$
\lim_{t\to 0} \frac{1 - G\left(w(G) - tx\right)}{1 - G\left(w(G) - t\right)} = x^{\beta}
$$

for every $x > 0$. But

$$
\begin{aligned}
& \lim_{t\to 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F(w(F) - t)} \\
&= \lim_{t\to 0} \frac{1 - \frac{[G(w(F)-tx)]^{ab}}{[G(w(F)-tx)]^{ab}+\left\{1-[G(w(F)-tx)]^b\right\}^a}}{1 - \frac{[G(w(F)-t)]^{ab}}{[G(w(F)-t)]^{ab}+\left\{1-[G(w(F)-t)]^b\right\}^a}} \\
&= \lim_{t\to 0} \frac{\frac{\left\{1-[G(w(F)-tx)]^b\right\}^a}{[G(w(F)-tx)]^{ab}+\left\{1-[G(w(F)-tx)]^b\right\}^a}}{\frac{\left\{1-[G(w(F)-t)]^b\right\}^a}{[G(w(F)-t)]^{ab}+\left\{1-[G(w(F)-t)]^b\right\}^a}} \\
&= \lim_{t\to 0} \frac{\frac{\left\{1-[G(w(G)-tx)]^b\right\}^a}{[G(w(G)-tx)]^{ab}+\left\{1-[G(w(G)-tx)]^b\right\}^a}}{\frac{\left\{1-[G(w(G)-t)]^b\right\}^a}{[G(w(G)-t)]^{ab}+\left\{1-[G(w(G)-t)]^b\right\}^a}} \\
&= \lim_{t\to 0} \frac{\frac{\left\{1-[G(w(G)-tx)]^b\right\}^a}{[1]^{ab}+\left\{1-1^b\right\}^a}}{\frac{\left\{1-[G(w(G)-t)]^b\right\}^a}{1^{ab}+\left\{1-1^b\right\}^a}} \\
&= \lim_{t\to 0} \left\{\frac{1 - \left[G\left(w(G) - tx\right)\right]^b}{1 - \left[G\left(w(G) - t\right)\right]^b}\right\}^a \\
&= \lim_{t\to 0} \left[\frac{1 - \left\{1 - \left[1 - G\left(w(G) - tx\right)\right]\right\}^b}{1 - \left\{1 - \left[1 - G\left(w(G) - t\right)\right]\right\}^b}\right]^a \\
&= \lim_{t\to 0} \left[\frac{1 - \left\{1 - b\left[1 - G\left(w(G) - tx\right)\right]\right\}}{1 - \left\{1 - b\left[1 - G\left(w(G) - t\right)\right]\right\}}\right]^a \\
&= \lim_{t\to 0} \left\{\frac{b\left[1 - G\left(w(G) - tx\right)\right]}{b\left[1 - G\left(w(G) - t\right)\right]}\right\}^a \\
&= \lim_{t\to 0} \left\{\frac{1 - G\left(w(G) - tx\right)}{1 - G\left(w(G) - t\right)}\right\}^a \\
&= \lim_{t\to w(G)} \left\{x^{\beta}\right\}^a \\
&= x^{a\beta}
\end{aligned}
$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \le x\right) = \exp\left(-(-x)^{a\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.