MATH48181/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 10

Consider a bivariate distribution specified by the joint survival function

$$\overline{G}(x,y) = \exp\left\{ -(x+y) \sum_{i=1}^{p} w_i A_i \left(\frac{y}{x+y} \right) \right\}$$

for $x > 0$ and $y > 0$, where $w_1, \ldots, w_p$ are non-negative numbers summing to 1, and $A_i(\cdot)$ is a convex function satisfying $A_i(0) = A_i(1) = 1$ and $\max(w, 1-w) \leq A_i(w) \leq 1$.

First note that

$$\overline{G}(x,0) = \exp\left\{ -(x+0) \sum_{i=1}^{p} w_i A_i\left(0\right) \right\} = \exp\left\{ -x \sum_{i=1}^{p} w_i \right\} = \exp(-x)$$

and

$$\overline{G}(0,y) = \exp\left\{ -(0+y) \sum_{i=1}^{p} w_i A_i\left(1\right) \right\} = \exp\left\{ -y \sum_{i=1}^{p} w_i \right\} = \exp(-y).$$

So, the marginals follow the unit exponential distribution.

Setting

$$\overline{G}(x,y) = \exp\left\{ -(x+y) \sum_{i=1}^{p} w_i A_i \left(\frac{y}{x+y} \right) \right\} = \exp\left\{ -(x+y) A \left(\frac{y}{x+y} \right) \right\},$$

we see that

$$A(w) = \sum_{i=1}^{p} w_i A\left(w\right).$$

We need to prove that $A(\cdot)$ is a convex function and it satisfies $A(0) = A(1) = 1$ and $\max(w, 1-w) \leq A(w) \leq 1$.

Firstly,

$$A(0) = \sum_{i=1}^{p} w_i A_i\left(0\right) = \sum_{i=1}^{p} w_i = 1.$$

Secondly,

$$A(1) = \sum_{i=1}^{p} w_i A_i\left(1\right) = \sum_{i=1}^{p} w_i = 1.$$

Thirdly,

$$A(w) = \sum_{i=1}^{p} w_i A_i\,(w) \leq \sum_{i=1}^{p} w_i = 1.$$

Fourthly,

$$A(w) = \sum_{i=1}^{p} w_i A_i\,(w) \geq \sum_{i=1}^{p} w_i w = w.$$

Fifthly,

$$A(w) = \sum_{i=1}^{p} w_i A_i\,(w) \geq \sum_{i=1}^{p} w_i (1 - w) = 1 - w.$$

Finally, $A(\cdot)$ is convex since

$$A^{''}(w) = \sum_{i=1}^{p} w_i A_i^{''}\,(w) > 0.$$