MATH48181/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 1

Suppose X is a random variable with probability density function

$$f(x) = \frac{1}{2}\text{sech}\left(\frac{\pi x}{2}\right)$$

for $-\infty < x < \infty$.

Clearly, $w(F) = \infty$. Note that

$$\begin{aligned}
\lim_{t\to\infty} \frac{1-F\left(t+\gamma(t)x\right)}{1-F(t)} &= \lim_{t\to\infty} \frac{-\left(1+\gamma^{'}(t)x\right) f\left(t+\gamma(t)x\right)}{-f(t)} \\
&= \lim_{t\to\infty} \frac{\left(1+\gamma^{'}(t)x\right) f\left(t+\gamma(t)x\right)}{f(t)} \\
&= \lim_{t\to\infty} \frac{\left(1+\gamma^{'}(t)x\right) \text{sech}\left(\frac{\pi(t+\gamma(t)x)}{2}\right)}{\text{sech}\left(\frac{\pi t}{2}\right)} \\
&= \lim_{t\to\infty} \frac{\left(1+\gamma^{'}(t)x\right) \frac{1}{\exp\left(\frac{\pi(t+\gamma(t)x)}{2}\right)+\exp\left(-\frac{\pi(t+\gamma(t)x)}{2}\right)}}{\frac{1}{\exp\left(\frac{\pi t}{2}\right)+\exp\left(-\frac{\pi t}{2}\right)}} \\
&= \lim_{t\to\infty} \left(1+\gamma^{'}(t)x\right) \frac{\exp\left(\frac{\pi t}{2}\right)+\exp\left(-\frac{\pi t}{2}\right)}{\exp\left(\frac{\pi(t+\gamma(t)x)}{2}\right)+\exp\left(-\frac{\pi(t+\gamma(t)x)}{2}\right)} \\
&= \lim_{t\to\infty} \left(1+\gamma^{'}(t)x\right) \frac{\exp\left(\frac{\pi t}{2}\right)\left[1+\exp\left(-\pi t\right)\right]}{\exp\left(\frac{\pi(t+\gamma(t)x)}{2}\right)\left[1+\exp\left(-\pi\left(t+\gamma(t)x\right)\right)\right]} \\
&= \lim_{t\to\infty} \left(1+\gamma^{'}(t)x\right) \frac{\exp\left(\frac{\pi t}{2}\right)}{\exp\left(\frac{\pi(t+\gamma(t)x)}{2}\right)} \frac{1+\exp\left(-\pi t\right)}{1+\exp\left(-\pi\left(t+\gamma(t)x\right)\right)} \\
&= \lim_{t\to\infty} \left(1+\gamma^{'}(t)x\right) \exp\left(-\frac{\pi\gamma(t)x}{2}\right) \frac{1+\exp\left(-\pi t\right)}{1+\exp\left(-\pi\left(t+\gamma(t)x\right)\right)} \\
&= \lim_{t\to\infty} \left(1+\gamma^{'}(t)x\right) \exp\left(-\frac{\pi\gamma(t)x}{2}\right) \frac{1+0}{1+0} \\
&= \lim_{t\to\infty} \left(1+\gamma^{'}(t)x\right) \exp\left(-\frac{\pi\gamma(t)x}{2}\right) \\
&= \exp(-x)
\end{aligned}$$

if $\gamma(t) = 2/\pi$. Hence, F belongs to Gumbel domain of attraction.