MATH48181/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 9

Suppose $C_1, C_2, \ldots, C_p$ are known copulas. Let

$$C\left(u_1, u_2\right) = \left[C_1\left(u_1, u_2\right)\right]^{\alpha_1} \left[C_2\left(u_1, u_2\right)\right]^{\alpha_2} \cdots \left[C_p\left(u_1, u_2\right)\right]^{\alpha_p},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_p$ are non-negative real numbers.

Condition (i) is satisfied since

$$C(0, u) = \left[C_1(0, u)\right]^{\alpha_1} \left[C_2(0, u)\right]^{\alpha_2} \cdots \left[C_p(0, u)\right]^{\alpha_p} = 0^{\alpha_1} 0^{\alpha_2} \cdots 0^{\alpha_p} = 0.$$

Condition (ii) is satisfied since

$$C(u, 0) = \left[C_1(u, 0)\right]^{\alpha_1} \left[C_2(u, 0)\right]^{\alpha_2} \cdots \left[C_p(u, 0)\right]^{\alpha_p} = 0^{\alpha_1} 0^{\alpha_2} \cdots 0^{\alpha_p} = 0.$$

Condition (iii) is satisfied since

$$C(1, u) = \left[C_1(1, u)\right]^{\alpha_1} \left[C_2(1, u)\right]^{\alpha_2} \cdots \left[C_p(1, u)\right]^{\alpha_p} = u^{\alpha_1} u^{\alpha_2} \cdots u^{\alpha_p} = u.$$

Condition (iv) is satisfied since

$$C(u, 1) = \left[C_1(u, 1)\right]^{\alpha_1} \left[C_2(u, 1)\right]^{\alpha_2} \cdots \left[C_p(u, 1)\right]^{\alpha_p} = u^{\alpha_1} u^{\alpha_2} \cdots u^{\alpha_p} = u.$$

Condition (v) is satisfied since

$$\begin{aligned}
\frac{\partial}{\partial u_1} C\left(u_1, u_2\right) &= \frac{\partial}{\partial u_1} \left\{\left[C_1\left(u_1, u_2\right)\right]^{\alpha_1} \left[C_2\left(u_1, u_2\right)\right]^{\alpha_2} \cdots \left[C_p\left(u_1, u_2\right)\right]^{\alpha_p}\right\} \\
&= \frac{\partial}{\partial u_1} \left\{\prod_{i=1}^{p} \left[C_i\left(u_1, u_2\right)\right]^{\alpha_i}\right\} \\
&= \sum_{i=1}^{p} \frac{\partial}{\partial u_1} \left[C_i\left(u_1, u_2\right)\right]^{\alpha_i} \prod_{j=1, j \neq i}^{p} \left[C_j\left(u_1, u_2\right)\right]^{\alpha_j} \\
&= \sum_{i=1}^{p} \alpha_i \left[C_i\left(u_1, u_2\right)\right]^{\alpha_i - 1} \frac{\partial}{\partial u_1} C_i\left(u_1, u_2\right) \prod_{j=1, j \neq i}^{p} \left[C_j\left(u_1, u_2\right)\right]^{\alpha_j} \\
&\geq 0.
\end{aligned}$$

Condition (vi) is satisfied since

$$\frac{\partial}{\partial u_2} C\left(u_1, u_2\right) = \frac{\partial}{\partial u_2} \left\{\left[C_1\left(u_1, u_2\right)\right]^{\alpha_1} \left[C_2\left(u_1, u_2\right)\right]^{\alpha_2} \cdots \left[C_p\left(u_1, u_2\right)\right]^{\alpha_p}\right\}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial u_2}\left\{\prod_{i=1}^{p}\left[C_i\left(u_1,u_2\right)\right]^{\alpha_i}\right\} \\
&= \sum_{i=1}^{p}\frac{\partial}{\partial u_2}\left[C_i\left(u_1,u_2\right)\right]^{\alpha_i}\prod_{j=1,j\neq i}^{p}\left[C_j\left(u_1,u_2\right)\right]^{\alpha_j} \\
&= \sum_{i=1}^{p}\alpha_i\left[C_i\left(u_1,u_2\right)\right]^{\alpha_i-1}\frac{\partial}{\partial u_2}C_i\left(u_1,u_2\right)\prod_{j=1,j\neq i}^{p}\left[C_j\left(u_1,u_2\right)\right]^{\alpha_j} \\
&\geq 0.
\end{aligned}
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