MATH48181/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 6

Suppose X represents loss and has the following cumulative distribution function

$$F_X(x) = \frac{\theta \exp(-\lambda x)}{1 - (1-\theta) \exp(-\lambda x)}$$

for $x > 0$, $\theta > 0$ and $\lambda > 0$.

To find $\mathrm{VaR}_p(X)$, set

$$F_X(x) = \frac{\theta \exp(-\lambda x)}{1 - (1-\theta) \exp(-\lambda x)} = p$$

which implies

$$\theta \exp(-\lambda x) = p - p(1-\theta) \exp(-\lambda x)$$

which implies

$$[\theta + p(1-\theta)] \exp(-\lambda x) = p$$

which implies

$$\exp(-\lambda x) = \frac{p}{\theta + p(1-\theta)}$$

which implies

$$x = -\frac{1}{\lambda} \log \frac{p}{\theta + p(1-\theta)}$$

which implies

$$x = -\frac{1}{\lambda} \log p + \frac{1}{\lambda} \log \left[\theta + p(1-\theta)\right].$$

Hence,

$$\mathrm{VaR}_p(X) = -\frac{1}{\lambda} \log p + \frac{1}{\lambda} \log \left[\theta + p(1-\theta)\right].$$

The $\mathrm{ES}_p(X)$ can be derived as

$$\mathrm{ES}_p(X) \quad = \quad \frac{1}{p} \int_0^p \mathrm{VaR}_t(X) dt$$

$$
\begin{aligned}
&= -\frac{1}{\lambda p}\int_0^p \log t dt + \frac{1}{\lambda p}\int_0^p \log\left[\theta + t(1-\theta)\right] dt \\
&= -\frac{1}{\lambda p}\left\{[t\log t]_0^p - \int_0^p t\frac{1}{t}dt\right\} \\
&\qquad +\frac{1}{\lambda p}\left\{[t\log\left[\theta + t(1-\theta)\right]]_0^p - \int_0^p \frac{t(1-\theta)}{\theta + t(1-\theta)}dt\right\} \\
&= -\frac{1}{\lambda p}\left\{p\log p - p\right\} \\
&\qquad +\frac{1}{\lambda p}\left\{p\log\left[\theta + p(1-\theta)\right] - \int_0^p \frac{\theta + t(1-\theta) - \theta}{\theta + t(1-\theta)}dt\right\} \\
&= -\frac{\log p - 1}{\lambda} + \frac{1}{\lambda p}\left\{p\log\left[\theta + p(1-\theta)\right] - \int_0^p \left[1 - \frac{\theta}{\theta + t(1-\theta)}\right] dt\right\} \\
&= -\frac{\log p - 1}{\lambda} + \frac{1}{\lambda p}\left\{p\log\left[\theta + p(1-\theta)\right] - \left\{t - \frac{\theta}{1-\theta}\log\left[\theta + t(1-\theta)\right]\right\}_0^p\right\} \\
&= -\frac{\log p - 1}{\lambda} + \frac{1}{\lambda p}\left\{p\log\left[\theta + p(1-\theta)\right] - p + \frac{\theta}{1-\theta}\log\frac{\theta + p(1-\theta)}{\theta}\right\}.
\end{aligned}
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