MATH48181/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 5

Suppose a portfolio is made of up of two dependent investments. Let X and Y denote the losses. Assume that X and Y have the joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \alpha_1\beta_2 \exp\left(-\beta_2 y - \gamma_2 x\right), \text{if } 0 \leq x < y, \\ \alpha_2\beta_1 \exp\left(-\beta_1 x - \gamma_1 y\right), \text{if } 0 \leq y < x. \end{cases}$$

Let $S = X + Y$. The probability density function of S is

$$\begin{aligned}
f_S(s) &= \int_0^s f_{X,Y}(x, s-x)dx \\
&= \int_0^{s/2} f_{X,Y}(x, s-x)dx + \int_{s/2}^s f_{X,Y}(x, s-x)dx \\
&= \alpha_1\beta_2 \int_0^{s/2} \exp\left[-\beta_2(s-x) - \gamma_2 x\right] dx + \alpha_2\beta_1 \int_{s/2}^s \exp\left[-\beta_1 x - \gamma_1(s-x)\right] dx \\
&= \alpha_1\beta_2 \exp\left(-\beta_2 s\right) \int_0^{s/2} \exp\left[\left(\beta_2 - \gamma_2\right) x\right] dx + \alpha_2\beta_1 \exp\left(-\gamma_1 s\right) \int_{s/2}^s \exp\left[\left(\gamma_1 - \beta_1\right) x\right] dx \\
&= \alpha_1\beta_2 \exp\left(-\beta_2 s\right) \left[\frac{\exp\left[\left(\beta_2 - \gamma_2\right) x\right]}{\beta_2 - \gamma_2}\right]_0^{s/2} + \alpha_2\beta_1 \exp\left(-\gamma_1 s\right) \left[\frac{\exp\left[\left(\gamma_1 - \beta_1\right) x\right]}{\gamma_1 - \beta_1}\right]_{s/2}^s \\
&= \frac{\alpha_1\beta_2}{\beta_2 - \gamma_2} \exp\left(-\beta_2 s\right) \left\{\exp\left[\left(\beta_2 - \gamma_2\right) s/2\right] - 1\right\} \\
&\qquad + \frac{\alpha_2\beta_1}{\gamma_1 - \beta_1} \exp\left(-\gamma_1 s\right) \left\{\exp\left[\left(\gamma_1 - \beta_1\right) s\right] - \exp\left[\left(\gamma_1 - \beta_1\right) s/2\right]\right\} \\
&= \frac{\alpha_1\beta_2}{\beta_2 - \gamma_2} \left\{\exp\left[-\left(\beta_2 + \gamma_2\right) s/2\right] - \exp\left(-\beta_2 s\right)\right\} \\
&\qquad + \frac{\alpha_2\beta_1}{\gamma_1 - \beta_1} \left\{\exp\left(-\beta_1 s\right) - \exp\left[-\left(\gamma_1 + \beta_1\right) s/2\right]\right\}.
\end{aligned}$$

The cumulative distribution function of S is

$$\begin{aligned}
F_S(s) &= \int_0^s f_S(t)dt \\
&= \frac{\alpha_1\beta_2}{\beta_2 - \gamma_2} \int_0^s \left\{\exp\left[-\left(\beta_2 + \gamma_2\right) t/2\right] - \exp\left(-\beta_2 t\right)\right\} dt \\
&\qquad + \frac{\alpha_2\beta_1}{\gamma_1 - \beta_1} \int_0^s \left\{\exp\left(-\beta_1 t\right) - \exp\left[-\left(\gamma_1 + \beta_1\right) t/2\right]\right\} dt \\
&= \frac{\alpha_1\beta_2}{\beta_2 - \gamma_2} \left\{\int_0^s \exp\left[-\left(\beta_2 + \gamma_2\right) t/2\right] dt - \int_0^s \exp\left(-\beta_2 t\right) dt\right\}
\end{aligned}$$

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\begin{aligned}
& \quad + \frac{\alpha_2 \beta_1}{\gamma_1 - \beta_1} \left\{ \int_0^s \exp\left(-\beta_1 t\right) dt - \int_0^s \exp\left[-\left(\gamma_1 + \beta_1\right) t/2\right] dt \right\} \\
= & \quad \frac{\alpha_1 \beta_2}{\beta_2 - \gamma_2} \left[\frac{2}{\beta_2 + \gamma_2} \left\{1 - \exp\left[-\left(\beta_2 + \gamma_2\right) s/2\right]\right\} - \frac{1}{\beta_2} \left[1 - \exp\left(-\beta_2 s\right)\right] \right] \\
& \quad + \frac{\alpha_2 \beta_1}{\gamma_1 - \beta_1} \left[\frac{1}{\beta_1} \left[1 - \exp\left(-\beta_1 s\right)\right] - \frac{2}{\gamma_1 + \beta_1} \left\{1 - \exp\left[-\left(\gamma_1 + \beta_1\right) s/2\right]\right\} \right].
\end{aligned}
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