MATH48181/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 8

Let X represent loss and suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. Suppose that the probability density function of X is given by

$$f(x) = \exp(a - x)$$

where $x > a > 0$.

First, we need to determine the cumulative distribution function of X as follows

$$\begin{aligned}
F(x) &= \int_a^x f(y) dy \\
&= \int_a^x \exp(a - y) dy \\
&= \exp(a) \int_a^x \exp(-y) dy \\
&= \exp(a) \left[-\exp(-y) \right]_a^x \\
&= \exp(a) \left[\exp(-a) - \exp(-x) \right] \\
&= 1 - \exp(a - x).
\end{aligned}$$

Hence, the cumulative distribution function of X is

$$F(x) = 1 - \exp(a - x).$$

Next, we invert $F(x) = p$ to find the value at risk at p. This gives

$$\mathrm{VaR}_p(X) = a - \log(1 - p).$$

Next, we find the expected shortfall at p as follows

$$\begin{aligned}
\mathrm{ES}_p(X) &= \frac{1}{p} \int_0^p \left[a - \log(1 - t) \right] dt \\
&= \frac{1}{p} \left[\int_0^p a dt - \int_0^p \log(1 - t) dt \right] \\
&= \frac{1}{p} \left[ap - \int_0^p \log(1 - t) dt \right] \\
&= a - \frac{1}{p} \int_0^p \log(1 - t) dt \\
&= a - \frac{1}{p} \left\{ \left[t \log(1 - t) \right]_0^p + \int_0^p \frac{t}{1 - t} dt \right\}
\end{aligned}$$

$$
\begin{aligned}
&= a - \frac{1}{p}\left\{[p\log(1-p) - 0] + \int_0^p \frac{t-1+1}{1-t} dt\right\} \\
&= a - \frac{1}{p}\left\{p\log(1-p) + \int_0^p \left[-1 + \frac{1}{1-t}\right] dt\right\} \\
&= a - \frac{1}{p}\left\{p\log(1-p) + [-t - \log(1-t)]_0^p\right\} \\
&= a - \frac{1}{p}\left\{p\log(1-p) + [-p - \log(1-p) - 0]\right\} \\
&= a - \frac{1}{p}\left\{p\log(1-p) - p - \log(1-p)\right\}.
\end{aligned}
$$

Finally, to find the maximum likelihood estimator of $\mathrm{ES}_p(X)$, we need to find the maximum likelihood estimator of a. The likelihood function of a is

$$
\begin{aligned}
L(a) &= \prod_{i=1}^{n} \left[\exp\left(a - x_i\right) I\left\{x_i > a\right\}\right] \\
&= \exp\left(na - \sum_{i=1}^{n} x_i\right) \prod_{i=1}^{n} \left[I\left\{x_i > a\right\}\right] \\
&= \exp\left(na - \sum_{i=1}^{n} x_i\right) I\left\{\min\left(x_1, x_2, \ldots, x_n\right) > a\right\}.
\end{aligned}
$$

We see that $L(a)$ is an increasing function of a over $(-\infty, \min\left(x_1, x_2, \ldots, x_n\right))$ and is zero elsewhere. Hence, the maximum likelihood estimator of a is $\min\left(x_1, x_2, \ldots, x_n\right)$.

Hence, the maximum likelihood estimator of $\mathrm{ES}_p(X)$ is

$$
\min\left(x_1, x_2, \ldots, x_n\right) - \frac{1}{p}\left\{p\log(1-p) - p - \log(1-p)\right\}.
$$