MATH48181/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 7

Let X represent loss and suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. Suppose that the probability density function of X is given by

$$f(x) = \frac{1}{2a} \exp\left(-\frac{\mid x \mid}{a}\right)$$

where $a > 0$ and $-\infty < x < \infty$.

First, we need to determine the cumulative distribution function of X. Suppose $x > 0$. Then

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(y) dy \\
&= 1 - \int_{x}^{\infty} f(y) dy \\
&= 1 - \frac{1}{2a} \int_{x}^{\infty} \exp\left(-\frac{\mid y \mid}{a}\right) dy \\
&= 1 - \frac{1}{2a} \int_{x}^{\infty} \exp\left(-\frac{y}{a}\right) dy \\
&= 1 - \frac{1}{2a} \left[-a \exp\left(-\frac{y}{a}\right)\right]_{x}^{\infty} \\
&= 1 - \frac{1}{2a} \left[0 + a \exp\left(-\frac{x}{a}\right)\right] \\
&= 1 - \frac{1}{2} \exp\left(-\frac{x}{a}\right).
\end{aligned}$$

Suppose now $x \leq 0$. Then

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(y) dy \\
&= \frac{1}{2a} \int_{-\infty}^{x} \exp\left(-\frac{\mid y \mid}{a}\right) dy \\
&= \frac{1}{2a} \int_{-\infty}^{x} \exp\left(\frac{y}{a}\right) dy \\
&= \frac{1}{2a} \left[a \exp\left(\frac{y}{a}\right)\right]_{-\infty}^{x} \\
&= \frac{1}{2a} \left[a \exp\left(\frac{x}{a}\right) - 0\right] \\
&= \frac{1}{2} \exp\left(\frac{x}{a}\right).
\end{aligned}$$

Hence, the cumulative distribution function of X is

$$F(x) = \begin{cases} 1 - \frac{1}{2} \exp\left(-\frac{x}{a}\right), \text{if } x > 0, \\ \frac{1}{2} \exp\left(\frac{x}{a}\right), \text{if } x \leq 0. \end{cases}$$

Next, we invert $F(x) = p$ to find the value at risk at p. This gives

$$\mathrm{VaR}_p(X) = \begin{cases} -a \log\left[2(1-p)\right], \text{if } p > 1/2, \\ a \log\left[2p\right], \text{if } p \leq 1/2. \end{cases}$$

Finally, to find the maximum likelihood estimator of $\mathrm{VaR}_p(X)$, we need to find the maximum likelihood estimator of a. The likelihood function of a is

$$\begin{aligned} L(a) &= \prod_{i=1}^{n} \left[\frac{1}{2a} \exp\left(-\frac{\mid x_i \mid}{a}\right)\right] \\ &= \frac{1}{(2a)^n} \prod_{i=1}^{n} \left[\exp\left(-\frac{\mid x_i \mid}{a}\right)\right] \\ &= \frac{1}{(2a)^n} \exp\left(-\frac{1}{a} \sum_{i=1}^{n} \mid x_i \mid\right) \end{aligned}$$

and its log is

$$\log L(a) = -n \log(2a) - \frac{1}{a} \sum_{i=1}^{n} \mid x_i \mid.$$

The derivative with respect to a is

$$\frac{d \log L(a)}{da} = -\frac{n}{a} + \frac{1}{a^2} \sum_{i=1}^{n} \mid x_i \mid.$$

Setting this derivative to zero and solving for a, we obtain

$$\widehat{a} = \frac{1}{n} \sum_{i=1}^{n} \mid x_i \mid.$$

This is a maximum likelihood estimator of a since

$$\begin{aligned} \frac{d^2 \log L(a)}{da^2} &= \frac{n}{a^2} - \frac{2}{a^3} \sum_{i=1}^{n} \mid x_i \mid \\ &= \frac{1}{a^3} \left[na - 2 \sum_{i=1}^{n} \mid x_i \mid\right] \\ &= \frac{1}{a^3} \left[n\widehat{a} - 2 \sum_{i=1}^{n} \mid x_i \mid\right] \\ &= -\frac{1}{a^3} \sum_{i=1}^{n} \mid x_i \mid \\ &< 0 \end{aligned}$$

at $a = \widehat{a}$.

Hence, the maximum likelihood estimator of $\mathrm{VaR}_p(X)$ is

$$\widehat{\mathrm{VaR}}_p(X) = \begin{cases} -\dfrac{1}{n} \log\left[2(1-p)\right] \displaystyle\sum_{i=1}^{n} \mid x_i \mid, \text{if } p > 1/2, \\ \dfrac{1}{n} \log\left[2p\right] \displaystyle\sum_{i=1}^{n} \mid x_i \mid, \text{if } p \leq 1/2. \end{cases}$$