MATH48181/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 3

Consider a class of distributions defined by the cumulative distribution function

$$F(x) = \frac{a^{G(x)} - 1}{(a-1)\left[b + \frac{1-b}{a-1}\left(a^{G(x)} - 1\right)\right]}$$

where $a > 0$, $a \neq 1$, $b > 0$ and $G(\cdot)$ is a valid cumulative distribution function. Assume that F and G have the same upper end points.

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = e^{-x}$$

for every $x > 0$. But

$$\begin{aligned}
& \lim_{t \to w(F)} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)} \\
= & \lim_{t \to w(F)} \frac{1 - \frac{a^{G(t+xh(t))}-1}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(t+xh(t))}-1\right)\right]}}{1 - \frac{a^{G(t)}-1}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(t)}-1\right)\right]}} \\
= & \lim_{t \to w(F)} \frac{\frac{b\left[a-a^{G(t+xh(t))}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(t+xh(t))}-1\right)\right]}}{\frac{b\left[a-a^{G(t)}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(t)}-1\right)\right]}} \\
= & \lim_{t \to w(F)} \frac{\frac{ab\left[1-a^{G(t+xh(t))-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(t+xh(t))}-1\right)\right]}}{\frac{ab\left[1-a^{G(t)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(t)}-1\right)\right]}} \\
= & \lim_{t \to w(F)} \frac{\frac{\left[1-a^{G(t+xh(t))-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(t+xh(t))}-1\right)\right]}}{\frac{\left[1-a^{G(t)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(t)}-1\right)\right]}} \\
= & \lim_{t \to w(G)} \frac{\frac{\left[1-a^{G(t+xh(t))-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(t+xh(t))}-1\right)\right]}}{\frac{\left[1-a^{G(t)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(t)}-1\right)\right]}}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{t \to w(G)} \frac{\frac{\left[1-a^{G(t+xh(t))-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}(a-1)\right]}}{\frac{\left[1-a^{G(t)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}(a-1)\right]}} \\
&= \lim_{t \to w(G)} \frac{1-a^{G(t+xh(t))-1}}{1-a^{G(t)-1}} \\
&= \lim_{t \to w(G)} \frac{1-\exp\left\{\log a\left[G\left(t+xh(t)\right)-1\right]\right\}}{1-\exp\left\{\log a\left[G\left(t\right)-1\right]\right\}} \\
&= \lim_{t \to w(G)} \frac{1-\left\{1+\log a\left[G\left(t+xh(t)\right)-1\right]\right\}}{1-\left\{1+\log a\left[G\left(t\right)-1\right]\right\}} \\
&= \lim_{t \to w(G)} \frac{-\log a\left[G\left(t+xh(t)\right)-1\right]}{-\log a\left[G\left(t\right)-1\right]} \\
&= \lim_{t \to w(G)} \frac{1-G\left(t+xh(t)\right)}{1-G\left(t\right)} \\
&= \exp(-x)
\end{aligned}
$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left[-\exp(-x)\right]
$$

for some suitable norming constants $a_n > 0$ and b_n.

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$
\lim_{t \to \infty} \frac{1-G\left(tx\right)}{1-G(t)} = x^{-\beta}
$$

for every $x > 0$. But

$$
\begin{aligned}
& \lim_{t \to \infty} \frac{1-F\left(tx\right)}{1-F(t)} \\
&= \lim_{t \to \infty} \frac{1-\frac{a^{G(tx)}-1}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(tx)}-1\right)\right]}}{1-\frac{a^{G(t)}-1}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(t)}-1\right)\right]}} \\
&= \lim_{t \to \infty} \frac{\frac{b\left[a-a^{G(tx)}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(tx)}-1\right)\right]}}{\frac{b\left[a-a^{G(t)}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(t)}-1\right)\right]}} \\
&= \lim_{t \to \infty} \frac{\frac{ab\left[1-a^{G(tx)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(tx)}-1\right)\right]}}{\frac{ab\left[1-a^{G(t)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(t)}-1\right)\right]}} \\
&= \lim_{t \to \infty} \frac{\frac{\left[1-a^{G(tx)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(tx)}-1\right)\right]}}{\frac{\left[1-a^{G(t)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(t)}-1\right)\right]}}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \frac{\frac{\left[1-a^{G(tx)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}(a-1)\right]}}{\frac{\left[1-a^{G(t)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}(a-1)\right]}} \\
&= \lim_{t\to\infty} \frac{1-a^{G(tx)-1}}{1-a^{G(t)-1}} \\
&= \lim_{t\to\infty} \frac{1-\exp\left\{\log a\left[G(tx)-1\right]\right\}}{1-\exp\left\{\log a\left[G(t)-1\right]\right\}} \\
&= \lim_{t\to\infty} \frac{1-\left\{1+\log a\left[G(tx)-1\right]\right\}}{1-\left\{1+\log a\left[G(t)-1\right]\right\}} \\
&= \lim_{t\to\infty} \frac{-\log a\left[G(tx)-1\right]}{-\log a\left[G(t)-1\right]} \\
&= \lim_{t\to\infty} \frac{1-G(tx)}{1-G(t)} \\
&= x^{-\beta}
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n\to\infty} P\left(\frac{M_n-b_n}{a_n} \le x\right) = \exp\left(-x^{-\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t\to 0} \frac{1-G(w(G)-tx)}{1-G(w(G)-t)} = x^{\beta}$$

for every $x > 0$. But

$$
\begin{aligned}
& \lim_{t\to 0} \frac{1-F(w(F)-tx)}{1-F(w(F)-t)} \\
&= \lim_{t\to 0} \frac{1-\frac{a^{G(w(F)-tx)}-1}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(w(F)-tx)}-1\right)\right]}}{1-\frac{a^{G(w(F)-t)}-1}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(w(F)-t)}-1\right)\right]}} \\
&= \lim_{t\to 0} \frac{\frac{b\left[a-a^{G(w(F)-tx)}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(w(F)-tx)}-1\right)\right]}}{\frac{b\left[a-a^{G(w(F)-t)}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(w(F)-t)}-1\right)\right]}} \\
&= \lim_{t\to 0} \frac{\frac{ab\left[1-a^{G(w(F)-tx)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(w(F)-tx)}-1\right)\right]}}{\frac{ab\left[1-a^{G(w(F)-t)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(w(F)-t)}-1\right)\right]}} \\
&= \lim_{t\to 0} \frac{\frac{\left[1-a^{G(w(F)-tx)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(w(F)-tx)}-1\right)\right]}}{\frac{\left[1-a^{G(w(F)-t)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(w(F)-t)}-1\right)\right]}}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{t\to 0} \frac{\frac{\left[1-a^{G(w(G)-tx)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(w(G)-tx)}-1\right)\right]}}{\frac{\left[1-a^{G(w(G)-t)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}\left(a^{G(w(G)-t)}-1\right)\right]}} \\
&= \lim_{t\to 0} \frac{\frac{\left[1-a^{G(w(G)-tx)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}(a-1)\right]}}{\frac{\left[1-a^{G(w(G)-t)-1}\right]}{(a-1)\left[b+\frac{1-b}{a-1}(a-1)\right]}} \\
&= \lim_{t\to 0} \frac{1-a^{G(w(G)-tx)-1}}{1-a^{G(w(G)-t)-1}} \\
&= \lim_{t\to 0} \frac{1-\exp\left\{\log a\left[G\left(w(G)-tx\right)-1\right]\right\}}{1-\exp\left\{\log a\left[G\left(w(G)-t\right)-1\right]\right\}} \\
&= \lim_{t\to 0} \frac{1-\left\{1+\log a\left[G\left(w(G)-tx\right)-1\right]\right\}}{1-\left\{1+\log a\left[G\left(w(G)-t\right)-1\right]\right\}} \\
&= \lim_{t\to 0} \frac{-\log a\left[G\left(w(G)-tx\right)-1\right]}{-\log a\left[G\left(w(G)-t\right)-1\right]} \\
&= \lim_{t\to 0} \frac{1-G\left(w(G)-tx\right)}{1-G\left(w(G)-t\right)} \\
&= x^{\beta}
\end{aligned}
$$

for every $x > 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$
\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \le x\right) = \exp\left(-(-x)^{\beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.