MATH48181/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 10

Suppose $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ is a random sample from the joint probability density function

$$f_{X,Y}(x, y) = x + y$$

for $0 \leq x, y \leq 1$.

We follow the four-step procedure to find the limiting bivariate extreme value distribution of $(M_{n,1}, M_{n,2})$, where $M_{n,1} = \max(X_1, X_2, \ldots, X_n)$ and $M_{n,2} = \max(Y_1, Y_2, \ldots, Y_n)$. The first step is to find the marginal cumulative distribution functions of X and Y. The marginal probability density functions of X and Y are

$$f_X(x) = \int_0^1 (x + y) dy = \left[xy + \frac{y^2}{2} \right]_0^1 = x + \frac{1}{2}$$

and

$$f_Y(y) = \int_0^1 (x + y) dx = \left[\frac{x^2}{2} + xy \right]_0^1 = y + \frac{1}{2},$$

respectively. So, the marginal cumulative distribution functions of X and Y are

$$F_X(x) = \int_0^x \left(t + \frac{1}{2} \right) dt = \left[\frac{t^2}{2} + \frac{t}{2} \right]_0^x = \frac{x^2}{2} + \frac{x}{2}$$

and

$$F_Y(y) = \int_0^y \left(t + \frac{1}{2} \right) dt = \left[\frac{t^2}{2} + \frac{t}{2} \right]_0^y = \frac{y^2}{2} + \frac{y}{2},$$

respectively.

The second step is to find the max domain of attractions of X and Y. Clearly, $w(F_X) = 1$ and $w(F_Y) = 1$. Since

$$\begin{aligned}
\lim_{t \to 0} \frac{1 - F_X(1 - tx)}{1 - F_X(1 - t)} &= \lim_{t \to 0} \frac{1 - \frac{(1-tx)^2}{2} - \frac{(1-tx)}{2}}{1 - \frac{(1-t)^2}{2} - \frac{(1-t)}{2}} \\
&= \lim_{t \to 0} \frac{\frac{3tx}{2} - \frac{(tx)^2}{2}}{\frac{3t}{2} - \frac{t^2}{2}} \\
&= \lim_{t \to 0} \frac{\frac{3x}{2} - \frac{tx^2}{2}}{\frac{3}{2} - \frac{t}{2}} \\
&= x,
\end{aligned}$$

F_X belongs to the Weibull limit. Also since

$$\begin{aligned}
\lim_{t\to 0}\frac{1-F_Y(1-tx)}{1-F_Y(1-t)} &= \lim_{t\to 0}\frac{1-\frac{(1-tx)^2}{2}-\frac{(1-tx)}{2}}{1-\frac{(1-t)^2}{2}-\frac{(1-t)}{2}} \\
&= \lim_{t\to 0}\frac{\frac{3tx}{2}-\frac{(tx)^2}{2}}{\frac{3t}{2}-\frac{t^2}{2}} \\
&= \lim_{t\to 0}\frac{\frac{3x}{2}-\frac{tx^2}{2}}{\frac{3}{2}-\frac{t}{2}} \\
&= x,
\end{aligned}$$

F_Y belongs to the Weibull limit too.

The third step is to find the norming constants. The formulas are

$$\begin{aligned}
a_n &= w\left(F_X\right)-F_X^{-1}\left(1-\frac{1}{n}\right)=1-F_X^{-1}\left(1-\frac{1}{n}\right), \\
b_n &= w\left(F_X\right)=1, \\
c_n &= w\left(F_Y\right)-F_Y^{-1}\left(1-\frac{1}{n}\right)=1-F_Y^{-1}\left(1-\frac{1}{n}\right), \\
d_n &= w\left(F_Y\right)=1.
\end{aligned}$$

To find $F_X^{-1}\left(1-\frac{1}{n}\right)$, we set

$$F_X(x)=\frac{x^2}{2}+\frac{x}{2}=1-\frac{1}{n}$$

which implies

$$x^2+x-2\left(1-\frac{1}{n}\right)=0$$

which implies

$$x=\frac{-1\pm\sqrt{1+8\left(1-\frac{1}{n}\right)}}{2}.$$

The valid root is

$$x=\frac{-1+\sqrt{9-\frac{8}{n}}}{2}.$$

So,

$$a_n=\frac{3-\sqrt{9-\frac{8}{n}}}{2}.$$

Similarly, to find $F_Y^{-1}\left(1-\frac{1}{n}\right)$, we set

$$F_Y(y)=\frac{y^2}{2}+\frac{y}{2}=1-\frac{1}{n}$$

which implies

$$y^2 + y - 2\left(1 - \frac{1}{n}\right) = 0$$

which implies

$$y = \frac{-1 \pm \sqrt{1 + 8\left(1 - \frac{1}{n}\right)}}{2}.$$

The valid root is

$$y = \frac{-1 + \sqrt{9 - \frac{8}{n}}}{2}.$$

So,

$$c_n = \frac{3 - \sqrt{9 - \frac{8}{n}}}{2}.$$

The fourth and the final step is to determine the limit of $\left[F_{X,Y}\left(a_n x + b_n, c_n y + d_n\right)\right]^n$. The joint cumulative distribution function of (X, Y) is

$$\begin{aligned}
F_{X,Y}(x, y) &= \int_0^x \int_0^y (u + v) dv du \\
&= \int_0^x \left[uv + \frac{v^2}{2}\right]_0^y du \\
&= \int_0^x \left[uy + \frac{y^2}{2}\right] du \\
&= \left[\frac{u^2}{2} y + u \frac{y^2}{2}\right]_0^x \\
&= \frac{x^2}{2} y + x \frac{y^2}{2} \\
&= xy \frac{x + y}{2}.
\end{aligned}$$

Hence, the limiting bivariate extreme value distribution is

$$\begin{aligned}
G(x, y) &= \lim_{n \to \infty} \left[F_{X,Y}\left(a_n x + b_n, c_n y + d_n\right)\right]^n \\
&= \lim_{n \to \infty} 2^{-n} \left(a_n x + b_n\right)^n \left(c_n y + d_n\right)^n \left(a_n x + c_n y + b_n + d_n\right)^n \\
&= \lim_{n \to \infty} 2^{-n} \left(\frac{3 - \sqrt{9 - \frac{8}{n}}}{2} x + 1\right)^n \left(\frac{3 - \sqrt{9 - \frac{8}{n}}}{2} y + 1\right)^n \left[\frac{3 - \sqrt{9 - \frac{8}{n}}}{2} (x + y) + 2\right]^n \\
&= \lim_{n \to \infty} 2^{-n} \left[\frac{3}{2}\left(1 - \sqrt{1 - \frac{8}{9n}}\right) x + 1\right]^n \left[\frac{3}{2}\left(1 - \sqrt{1 - \frac{8}{9n}}\right) y + 1\right]^n \\
&\quad \cdot \left[\frac{3}{2}\left(1 - \sqrt{1 - \frac{8}{9n}}\right) x + \frac{3}{2}\left(1 - \sqrt{1 - \frac{8}{9n}}\right) y + 2\right]^n
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} 2^{-n} \left[\frac{3}{2}\left(1-\left(1-\frac{4}{9n}+\cdots\right)\right)x+1\right]^n \left[\frac{3}{2}\left(1-\left(1-\frac{4}{9n}+\cdots\right)\right)y+1\right]^n \\
&\qquad \cdot \left[\frac{3}{2}\left(1-\left(1-\frac{4}{9n}+\cdots\right)\right)x+\frac{3}{2}\left(1-\left(1-\frac{4}{9n}+\cdots\right)\right)y+2\right]^n \\
&= \lim_{n\to\infty} 2^{-n} \left[\frac{2}{3n}x+1\right]^n \left[\frac{2}{3n}y+1\right]^n \left[\frac{2}{3n}x+\frac{2}{3n}y+2\right]^n \\
&= \lim_{n\to\infty} \left[\frac{2}{3n}x+1\right]^n \left[\frac{2}{3n}y+1\right]^n \left[\frac{1}{3n}x+\frac{1}{3n}y+1\right]^n \\
&= \exp\left[\frac{2x}{3}\right]\exp\left[\frac{2y}{3}\right]\exp\left[\frac{x}{3}+\frac{y}{3}\right] \\
&= \exp(x+y).
\end{aligned}
$$