MATH4/68181: EXTREME VALUES AND FINANCIAL RISK
SEMESTER 1
SOLUTIONS TO QUIZ PROBLEM 5

Let X be a normal random variable with mean μ and standard deviation σ. The cdf of X is

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Inverting $F_X(x) = p$, we obtain

$$\begin{aligned}
&\Phi\left(\frac{x-\mu}{\sigma}\right) = p \\
&\Leftrightarrow \frac{x-\mu}{\sigma} = \Phi^{-1}(p) \\
&\Leftrightarrow x-\mu = \sigma\Phi^{-1}(p) \\
&\Leftrightarrow x = \mu + \sigma\Phi^{-1}(p).
\end{aligned}$$

So,

$$\mathrm{VaR}_p(X) = \mu + \sigma\Phi^{-1}(p)$$

and

$$\begin{aligned}
\mathrm{ES}_p(X) &= \frac{1}{p}\int_0^p \left[\mu + \sigma\Phi^{-1}(t)\right] dt \\
&= \frac{1}{p}\left[\mu p + \sigma\int_0^p \Phi^{-1}(t)dt\right] \\
&= \mu + \frac{\sigma}{p}\int_0^p \Phi^{-1}(t)dt. \qquad (1)
\end{aligned}$$

Now set $u = \Phi^{-1}(t)$, so $t = \Phi(u)$ and $dt/du = \phi(u)$. Then, (1) can be reduced to

$$\begin{aligned}
\mathrm{ES}_p(X) &= \mu + \frac{\sigma}{p}\int_{-\infty}^{\Phi^{-1}(p)} u\phi(u)du \\
&= \mu + \frac{\sigma}{p}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\Phi^{-1}(p)} u\exp\left(-\frac{u^2}{2}\right) du \\
&= \mu + \frac{\sigma}{p}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\Phi^{-1}(p)} \exp\left(-\frac{u^2}{2}\right) d\left(u^2/2\right) \\
&= \mu + \frac{\sigma}{p}\frac{1}{\sqrt{2\pi}}\left[-\exp\left(-\frac{u^2}{2}\right)\right]_{-\infty}^{\Phi^{-1}(p)} \\
&= \mu + \frac{\sigma}{p}\frac{1}{\sqrt{2\pi}}\left[-\exp\left(-\frac{\left[\Phi^{-1}(p)\right]^2}{2}\right) + 0\right]
\end{aligned}$$

$$= \quad \mu - \frac{\sigma}{p} \frac{1}{\sqrt{2\pi}} \exp\left(- \frac{\left[\Phi^{-1}(p)\right]^2}{2} \right)$$

$$\mu - \frac{\sigma}{p} \phi\left(\Phi^{-1}(p)\right).$$